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A GENERALIZATION OF HALL'S THEOREM FOR k -UNIFORM k -PARTITE HYPERGRAPHS

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ABSTRACT. In this paper we prove a generalized version of Hall's theorem in graphs, for hypergraphs. More precisely, let \mathcal{H} be a k -uniform k -partite hypergraph with some ordering on parts as V_1, V_2, \dots, V_k such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching. In this case, we give a necessary and sufficient condition for having a matching of size $t = |V_1|$ in \mathcal{H} . Some relevant results and counterexamples are given as well.

1. Introduction

We refer to [7] and [6] for elementary backgrounds in graph theory and hypergraph theory, respectively.

Let G be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A matching in G , is a set M of pairwise disjoint edges of G . A matching M is said to be a perfect matching, if every $x \in V(G)$, lies in one of elements of M . A matching M in G , is maximum whenever for every matching M' , $|M'| \leq |M|$.

For every set of vertices A , $N(A)$ which is called the neighborhood of A , is the set of vertices which are adjacent with at least one element of A . The following theorem is known as Hall's theorem in bipartite graphs.

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Theorem 1.1. ([7, Theorem 5.2]) *Let G be a bipartite graph with bipartition (X, Y) . Then G contains a matching that saturates every vertex in X , if and only if*

$$|N(S)| \geq |S| \quad \text{for all } S \subseteq X.$$

A vertex cover in G , is a subset C of $V(G)$ such that for every edge e of G , e intersects C . A vertex cover C is called a minimum vertex cover, if for every vertex cover C' , $|C| \leq |C'|$. The following theorem is known as König's theorem in graph theory.

Theorem 1.2. ([7, Theorem 5.3]) *In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

Let V be a finite nonempty set. A hypergraph \mathcal{H} on V is the pair (V, E) , where E is a collection of nonempty subsets of V . Also here we assume that $\bigcup_{e \in \mathcal{H}} e = V$. Each subset is said to be a hyperedge and each element of V is called a vertex. We denote the set of vertices and hyperedges of \mathcal{H} , by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. Two vertices x and y of a hypergraph are said to be adjacent whenever they lie in a hyperedge.

A matching in the hypergraph \mathcal{H} is a set M of pairwise disjoint hyperedges of \mathcal{H} . A perfect matching is a matching such that every $x \in V(\mathcal{H})$ lies in one of its elements. A matching M in \mathcal{H} is called a maximum matching whenever for every matching M' , we have $|M'| \leq |M|$.

In a hypergraph \mathcal{H} , a subset C of $V(\mathcal{H})$ is called a vertex cover if every hyperedge of \mathcal{H} intersects C . A vertex cover C is said to be minimum if for every vertex cover C' , $|C| \leq |C'|$. We denote the number of hyperedges in a maximum matching of the hypergraph \mathcal{H} , by $\alpha'(\mathcal{H})$ and the number of vertices in a minimum vertex cover of \mathcal{H} , by $\beta(\mathcal{H})$.

A hypergraph \mathcal{H} is said to be simple or a clutter if non of its two distinct hyperedges contains another. A hypergraph is called t -uniform (or t -graph), if all its hyperedges have the same size t . A t -uniform ($t \geq 2$) hypergraph \mathcal{H} is said to be r -partite ($r \geq t$), whenever $V(\mathcal{H})$ can be partitioned into r subsets such that for every two vertices x, y in one part, x and y are not adjacent. If $r = 2, 3$, the hypergraph is said to be bipartite and tripartite, respectively.

Meny researches have been done about matchings and existence of perfect matchings in hypergraphs (see for instance [1], [9], [12]). Also some attempts have been produced in generalization of Hall's theorem and König's theorem to hypergraphs (see [2], [3], [4], [5], [10], [11]).

Definition 1.3. *Let \mathcal{H} be a k -uniform hypergraph with $k \geq 2$. A subset $\epsilon \subseteq V(\mathcal{H})$ of size $k - 1$ is called a submaximal hyperedge if there is a hyperedge containing ϵ . For a submaximal hyperedge ϵ , define the neighborhood of ϵ as the set $N(\epsilon) := \{v \in V(\mathcal{H}) \mid \epsilon \cup \{v\} \in E(\mathcal{H})\}$.*

For a set A , consisting of submaximal hyperedges of \mathcal{H} , $\{v \in V(\mathcal{H}) \mid \exists \epsilon \in A, v \in N(\epsilon)\}$ is denoted by $N(A)$.

Definition 1.4. Let \mathcal{H} be a hypergraph and $\emptyset \neq V' \subseteq V(\mathcal{H})$. The subhypergraph generated on V' is

$$\langle V' \rangle := (V', \{e \cap V' \mid e \in E(\mathcal{H}), e \cap V' \neq \emptyset\}).$$

If $k \geq 3$ and \mathcal{H} be a k -uniform k -partite hypergraph with parts V_1, V_2, \dots, V_k , it is clear that the subhypergraph generated on the union of every $k - 1$ distinct parts is a $(k - 1)$ -uniform $(k - 1)$ -partite hypergraph.

Let $\mathfrak{A} = (A_1, \dots, A_n)$ be a family of subsets of a set E . A subset $\{x_1, \dots, x_n\}$ of E , where $x_i \neq x_j$, is said to be a transversal (or SDR) for \mathfrak{A} , if for every i ($1 \leq i \leq n$), $x_i \in A_i$. A partial transversal (partial SDR) of length l ($1 \leq l \leq n - 1$) for \mathfrak{A} , is a transversal for a subfamily of \mathfrak{A} with l sets.[8]

The following theorem is known as Hall’s theorem in combinatorics.

Theorem 1.5. ([8, Theorem 4.1]) *The family $\mathfrak{A} = (A_1, \dots, A_n)$ of subsets of a set E has a transversal if and only if*

$$\left| \bigcup_{i \in I'} A_i \right| \geq |I'|, \quad \forall I' \subseteq \{1, \dots, n\}.$$

Corollary 1.6. ([8, Corollary 4.3]) *The family $\mathfrak{A} = (A_1, \dots, A_n)$ of subsets of a set E has a partial transversal of length $l (> 0)$ if and only if*

$$\left| \bigcup_{i \in I'} A_i \right| \geq |I'| - n + l, \quad \forall I' \subseteq \{1, \dots, n\}.$$

2. The main results

Now we are ready to present our first theorem.

Theorem 2.1. *Let \mathcal{H} be a k -uniform k -partite hypergraph with some ordering on parts, as V_1, V_2, \dots, V_k , such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching M . Then \mathcal{H} has a matching of size $t = |V_1|$, if and only if for every subset A of M , $|N(A)| \geq |A|$.*

Proof. Let $t = |V_1|$ and let the elements of M are $\epsilon_1, \dots, \epsilon_t$. Assume \mathcal{H} has a matching of size t with elements e_1, \dots, e_t . Upon uniqueness of M , $M = \{e_1 - V_k, \dots, e_t - V_k\}$. Therefore

$$(N(\epsilon_1), \dots, N(\epsilon_t)) = (N(e_{i_1} - V_k), \dots, N(e_{i_t} - V_k)).$$

Then the family $(N(\epsilon_1), \dots, N(\epsilon_t))$ has an SDR. Thus by Theorem 1.1,

$$\left| \bigcup_{i \in I} N(\epsilon_i) \right| \geq |I|, \quad \forall I \subseteq \{1, \dots, t\},$$

and therefore for every subset A of M , $|N(A)| \geq |A|$.

Conversely, let for every subset A of M , we have $|N(A)| \geq |A|$. Now, $(N(\epsilon_1), \dots, N(\epsilon_t))$ is a family such that

$$\left| \bigcup_{i \in I} N(\epsilon_i) \right| \geq |I|, \quad \forall I \subseteq \{1, \dots, t\}.$$

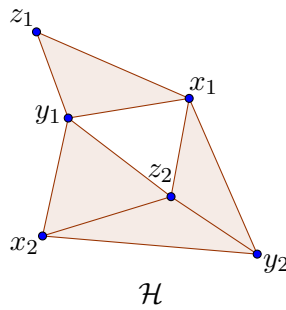
Therefore by Theorem 1.3, the mentioned family has an SDR. That is, there are distinct elements x_1, \dots, x_t of V_k such that $x_j \in N(\epsilon_j)$. Now, for every $1 \leq j \leq t$, $\epsilon_j \cup \{x_j\}$ is a hyperedge of \mathcal{H} and these hyperedges are pairwise disjoint. Then they form a matching of size t for \mathcal{H} . \square

Corollary 2.2. *Let \mathcal{H} be a k -uniform k -partite hypergraph with some ordering on parts as V_1, V_2, \dots, V_k where $|V_1| = |V_2| = \dots = |V_k|$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching M . Then \mathcal{H} has a perfect matching if and only if for every subset A of M , $|N(A)| \geq |A|$.*

Remark 2.3. *Theorem 2.1 implies Theorem 1.1 (Hall's theorem) in case $k = 2$.*

Remark 2.4. *In Theorem 2.1, if the hypothesis of uniqueness of perfect matching of subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ is removed, only one side of theorem will remain correct. That is, from this fact that for every subset A of M , $|N(A)| \geq |A|$, we conclude that \mathcal{H} has a matching of size $t = |V_1|$. The following example shows that the inverse case is not true in general.*

Example 2.5. *Assume the 3-uniform 3-partite hypergraph \mathcal{H} with the following presentation.*



Indeed, $E(\mathcal{H}) = \{\{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{x_2, y_2, z_2\}, \{x_2, y_1, z_2\}\}$, where the parts of \mathcal{H} are

$$V_1 = \{x_1, x_2\}, V_2 = \{y_1, y_2\}, V_3 = \{z_1, z_2\}.$$

In this case, there is a perfect matching $M_1 = \{\{x_2, y_1\}, \{x_1, y_2\}\}$ for subhypergraph generated on $V_1 \cup V_2$. Although the hypergraph \mathcal{H} has a matching $M' = \{\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$ of size 2, if $A = M_1$, we have $N(A) = \{z_2\}$. Therefore $|N(A)| \not\geq |A|$. Note that M_1 is not the unique perfect matching of subhypergraph generated on $V_1 \cup V_2$ because $M_2 = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ is also yet.

Theorem 2.6. *Let \mathcal{H} be a k -uniform k -partite hypergraph with some ordering on parts as V_1, V_2, \dots, V_k such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a perfect matching M . If for every subset A of M , we have $|N(A)| \geq |A| - p$, where p is a fixed integer and $1 \leq p \leq t - 1$, then \mathcal{H} has a matching of size $t - p$, where t is the size of V_1 .*

Proof. Let the elements of M are $\epsilon_1, \dots, \epsilon_t$. $(N(\epsilon_1), \dots, N(\epsilon_t))$ is a family such that the cardinality of the union of each s terms is greater than or equal to $s - t + (t - p)$. Then by Corollary 1.6, the family $(N(\epsilon_1), \dots, N(\epsilon_t))$ has a partial SDR of size $t - p$. That is, there are distinct elements y_1, \dots, y_{t-p}

of V_k such that $y_j \in N(\epsilon_{i_j})$. Now, for every $1 \leq j \leq t - p$, $\epsilon_{i_j} \cup \{y_j\}$ is a hyperedge of \mathcal{H} and these hyperedges are pairwise disjoint. Thus they form a matching of size $t - p$ for \mathcal{H} . \square

Theorem 2.7. *Let \mathcal{H} be a k -uniform k -partite hypergraph with some ordering on parts as V_1, V_2, \dots, V_k , and let $t = |V_1|$. Then \mathcal{H} has a matching of size t if and only if $\alpha' = \beta = t$, where α' and β denotes the number of hyperedges in a maximum matching, and the number of vertices in a minimum vertex cover of \mathcal{H} , respectively.*

Proof. Let \mathcal{H} has a matching of size t . We show that $\alpha' = \beta = t$. Clearly $\beta \geq \alpha'$ because for covering each hyperedge of maximum matching, one vertex is needed. But since there is a matching of size t , then $\alpha' \geq t$. Now, V_1 is a minimal vertex cover of \mathcal{H} because each hyperedge has only one vertex in V_1 and each vertex of V_1 lies in a hyperedge. Therefore $t \geq \beta$, which implies that $\alpha' \geq \beta$. Then $\alpha' = \beta$. The matching of size t is the maximum matching because it covers all vertices of V_1 .

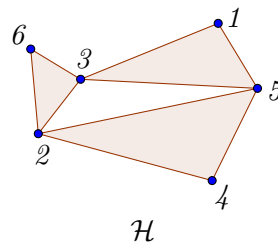
Conversely, if $\alpha' = \beta = t$, it is clear that \mathcal{H} has a matching of size t . \square

The following example shows that removing the condition $t = |V_1|$ in Theorem 2.7 is not possible even if the subhypergraph generated on union of every $k - 1$ parts, has a perfect matching.

Example 2.8. *Assume the 3-uniform 3-partite hypergraph \mathcal{H} with the following presentation, where the parts of \mathcal{H} are*

$$V_1 = \{1, 2\}, V_2 = \{3, 4\}, V_3 = \{5, 6\}.$$

Indeed, $E(\mathcal{H}) = \{\{1, 3, 5\}, \{2, 3, 6\}, \{2, 4, 5\}\}$.



In this hypergraph, we have the matching $\{\{1, 3, 5\}\}$ of size 1. But $\alpha' \neq \beta$ because $\alpha' = 1$ and $\beta = 2$. Note that each one of subhypergraph generated on $V_1 \cup V_2$, $V_2 \cup V_3$ and $V_1 \cup V_3$, have a perfect matching.

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