www.combinatorics.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 8 No. 3 (2019), pp. 23-28.

www.ui.ac.ir

A GENERALIZATION OF HALL'S THEOREM FOR k-UNIFORM k-PARTITE HYPERGRAPHS

REZA JAFARPOUR-GOLZARI

Communicated by Manouchehr Zaker

Abstract. In this paper we prove a generalized version of Hall's theorem in graphs, for hypergraphs. More precisely, let $\mathcal{H}$ be a k-uniform k-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching. In this case, we give a necessary and sufficient condition for having a matching of size $t = |V_1|$ in $\mathcal{H}$. Some relevant results and counterexamples are given as well.

1. Introduction

We refer to [7] and [6] for elementary backgrounds in graph theory and hypergraph theory, respectively.

Let G be a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. A matching in G, is a set M of pairwise disjoint edges of G. A matching M is said to be a perfect matching, if every $x \in V(G)$, lies in one of elements of M. A matching M in G, is maximum whenever for every matching M', $|M'| \leq |M|$.

For every set of vertices A, $N(A)$ which is called the neighborhood of A, is the set of vertices which are adjacent with at least one element of A. The following theorem is known as Hall's theorem in bipartite graphs.

MSC(2010): Primary: 05E40; Secondary: 05C65, 05D15.

Keywords: k-uniform k-partite hypergraph, matching, perfect matching, vertex cover, Hall's theorem.

Received: 27 June 2017, Accepted: 13 May 2019.

DOI: http://dx.doi.org/10.22108/toc.2019.105022.1506

.

Theorem 1.1. *([7, Theorem 5.2]) Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X, if and only if*

$$|N(S)| \geq |S| \quad for\ all\ S \subseteq X.$$

A vertex cover in G, is a subset C of $V(G)$ such that for every edge e of G, e intersects C. A vertex cover C is called a minimum vertex cover, if for every vertex cover $C^{'}$, $|C| \leq |C^{'}|$. The following theorem is known as König's theorem in graph theory.

Theorem 1.2. *([7, Theorem 5.3]) In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

Let V be a finite nonempty set. A hypergraph $\mathcal{H}$ on V is the pair (V, E), where E is a collection of nonempty subsets of V. Also here we assume that $\bigcup_{e \in \mathcal{H}} e = V$. Each subset is said to be a hyperedge and each element of V is called a vertex. We denote the set of vertices and hyperedges of $\mathcal{H}$, by $V(\mathcal{H})$ and $E(\mathcal{H})$, respectively. Two vertices x and y of a hypergraph are said to be adjacent whenever they lie in a hyperedge.

A matching in the hypergraph $\mathcal{H}$ is a set M of pairwise disjoint hyperedges of $\mathcal{H}$. A perfect matching is a matching such that every $x \in V(\mathcal{H})$ lies in one of its elements. A matching M in $\mathcal{H}$ is called a maximum matching whenever for every matching $M^{'}$, we have $|M^{'}| \leq |M|$.

In a hypergraph $\mathcal{H}$, a subset C of $V(\mathcal{H})$ is called a vertex cover if every hyperedge of $\mathcal{H}$ intersects C. A vertex cover C is said to be minimum if for every vertex cover $C^{'}$, $|C| \leq |C^{'}|$. We denote the number of hyperedges in a maximum matching of the hypergraph $\mathcal{H}$, by $\alpha^{'}(\mathcal{H})$ and the number of vertices in a minimum vertex cover of $\mathcal{H}$, by $\beta(\mathcal{H})$.

A hypergraph $\mathcal{H}$ is said to be simple or a clutter if non of its two distinct hyperedges contains another. A hypergraph is called t-uniform (or t-graph), if all its hyperedges have the same size t. A t-uniform ($t \geq 2$) hypergraph $\mathcal{H}$ is said to be r-partite ($r \geq t$), whenever $V(\mathcal{H})$ can be partitioned into r subsets such that for every two vertices x, y in one part, x and y are not adjacent. If $r = 2, 3$, the hypergraph is said to be bipartite and tripartite, respectively.

Meny researches have been done about matchings and existence of perfect matchings in hypergraphs (see for instance [1], [9], [12]). Also some attemps have been produced in generalization of Hall's theorem and König's theorem to hypergraphs (see [2], [3], [4], [5], [10], [11]).

Definition 1.3. *Let $\mathcal{H}$ be a k-uniform hypergraph with $k \geq 2$. A subset $\mathfrak{e} \subseteq V(\mathcal{H})$ of size $k-1$ is called a submaximal hyperedge if there is a hyperedge containing $\mathfrak{e}$. For a submaximal hyperedge $\mathfrak{e}$, define the neighborhood of $\mathfrak{e}$ as the set $N(\mathfrak{e}) := \{v \in V(\mathcal{H}) |\ \mathfrak{e} \cup \{v\} \in E(\mathcal{H})\}$.*

For a set A, consisting of submaximal hyperedges of $\mathcal{H}$, $\{v \in V(\mathcal{H}) |\ \exists \mathfrak{e} \in A, v \in N(\mathfrak{e})\}$ is denoted by $N(A)$.

DOI: http://dx.doi.org/10.22108/toc.2019.105022.1506

Definition 1.4. *Let $\mathcal{H}$ be a hypergraph and $\emptyset \neq V' \subseteq V(\mathcal{H})$. The subhypergraph generated on V' is*

$$< V' >:= (V', \{e \cap V' |\ e \in E(\mathcal{H}), e \cap V' \neq \emptyset\}).$$

If $k \geq 3$ and $\mathcal{H}$ be a k-uniform k-partite hypergraph with parts $V_1, V_2, \ldots, V_k$, it is clear that the subhypergraph generated on the union of every $k-1$ distinct parts is a $(k-1)$-uniform $(k-1)$-partite hypergraph.

Let $\mathfrak{A} = (A_1, \ldots, A_n)$ be a family of subsets of a set E. A subset $\{x_1, \ldots, x_n\}$ of E, where $x_i \neq x_j$, is said to be a transversal (or SDR) for $\mathfrak{A}$, if for every i $(1 \leq i \leq n)$, $x_i \in A_i$. A partial transversal (partial SDR) of length l $(1 \leq l \leq n-1)$ for $\mathfrak{A}$, is a transversal for a subfamily of $\mathfrak{A}$ with l sets.[8]

The following theorem is known as Hall's theorem in combinatorics.

Theorem 1.5. *([8, Theorem 4.1]) The family $\mathfrak{A} = (A_1, \ldots, A_n)$ of subsets of a set E has a transversal if and only if*

$$|\bigcup_{i \in I'} A_i| \geq |I'|, \quad \forall I' \subseteq \{1, \ldots, n\}.$$

Corollary 1.6. *([8, Corollary 4.3]) The family $\mathfrak{A} = (A_1, \ldots, A_n)$ of subsets of a set E has a partial transversal of length $l(>0)$ if and only if*

$$|\bigcup_{i \in I'} A_i| \geq |I'| - n + l, \quad \forall I' \subseteq \{1, \ldots, n\}.$$

2. The main results

Now we are ready to present our first theorem.

Theorem 2.1. *Let $\mathcal{H}$ be a k-uniform k-partite hypergraph with some ordering on parts, as $V_1, V_2, \ldots, V_k$, such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching M. Then $\mathcal{H}$ has a matching of size $t = |V_1|$, if and only if for every subset A of M, $|N(A)| \geq |A|$.*

Proof. Let $t = |V_1|$ and let the elements of M are $\mathfrak{e}_1, \ldots, \mathfrak{e}_t$. Assume $\mathcal{H}$ has a matching of size t with elements $e_1, \ldots, e_t$. Upon uniqueness of M, $M = \{e_1 - V_k, \ldots, e_t - V_k\}$. Therefore

$$(N(\mathfrak{e}_1), \ldots, N(\mathfrak{e}_t)) = (N(e_{i_1} - V_k), \ldots, N(e_{i_t} - V_k)).$$

Then the family $(N(\mathfrak{e}_1), \ldots, N(\mathfrak{e}_t))$ has an SDR. Thus by Theorem 1.1,

$$|\bigcup_{i \in I} N(\mathfrak{e}_i)| \geq |I|, \quad \forall I \subseteq \{1, \ldots, t\},$$

and therefore for every subset A of M, $|N(A)| \geq |A|$.

Conversely, let for every subset A of M, we have $|N(A)| \geq |A|$. Now, $(N(\mathfrak{e}_1), \ldots, N(\mathfrak{e}_t))$ is a family such that

$$|\bigcup_{i \in I} N(\mathfrak{e}_i)| \geq |I|, \quad \forall I \subseteq \{1, \ldots, t\}.$$

DOI: http://dx.doi.org/10.22108/toc.2019.105022.1506

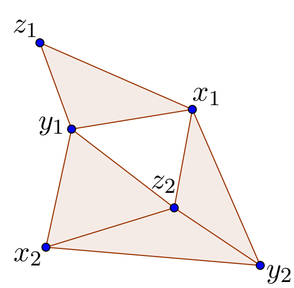
Therefore by Theorem 1.3, the mentioned family has an SDR. That is, there are distinct elements $x_1, \ldots, x_t$ of V_k such that $x_j \in N(\mathfrak{e}_j)$. Now, for every $1 \leq j \leq t$, $\mathfrak{e}_j \cup \{x_j\}$ is a hyperedge of $\mathcal{H}$ and these hyperedges are pairwise disjoint. Then they form a matching of size t for $\mathcal{H}$. □

Corollary 2.2. *Let $\mathcal{H}$ be a k-uniform k-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ where $|V_1| = |V_2| = \cdots = |V_k|$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a unique perfect matching M. Then $\mathcal{H}$ has a perfect matching if and only if for every subset A of M, $|N(A)| \geqslant |A|$.*

Remark 2.3. *Theorem 2.1 implies Theorem 1.1 (Hall's theorem) in case $k = 2$.*

Remark 2.4. *In Theorem 2.1, if the hypothesis of uniqueness of perfect matching of subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ is removed, only one side of theorem will remains correct. That is, from this fact that for every subset A of M, $|N(A)| \geqslant |A|$, we conclude that $\mathcal{H}$ has a matching of size $t = |V_1|$. The following example shows that the inverse case is not true in general.*

Example 2.5. *Assume the 3-uniform 3-partite hypergraph $\mathcal{H}$ with the following presentation.*



$\mathcal{H}$

Indeed, $E(\mathcal{H}) = \{\{x_1, y_1, z_1\}, \{x_1, y_2, z_2\}, \{x_2, y_2, z_2\}, \{x_2, y_1, z_2\}\}$, where the parts of $\mathcal{H}$ are

$$V_1 = \{x_1, x_2\},\ V_2 = \{y_1, y_2\},\ V_3 = \{z_1, z_2\}.$$

In this case, there is a perfect matching $M_1 = \{\{x_2, y_1\}, \{x_1, y_2\}\}$ for subhypergraph generated on $V_1 \cup V_2$. Although the hypergraph $\mathcal{H}$ has a matching $M' = \{\{x_1, y_1, z_1\}, \{x_2, y_2, z_2\}\}$ of size 2, if $A = M_1$, we have $N(A) = \{z_2\}$. Therefore $|N(A)| \ngeqslant |A|$. Note that M_1 is not the unique perfect matching of subhypergraph generated on $V_1 \cup V_2$ because $M_2 = \{\{x_1, y_1\}, \{x_2, y_2\}\}$ is also yet.

Theorem 2.6. *Let $\mathcal{H}$ be a k-uniform k-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ such that the subhypergraph generated on $\bigcup_{i=1}^{k-1} V_i$ has a perfect matching M. If for every subset A of M, we have $|N(A)| \geqslant |A| - p$, where p is a fixed integer and $1 \leq p \leq t-1$, then $\mathcal{H}$ has a matching of size $t - p$, where t is the size of V_1.*

Proof. Let the elements of M are $\mathfrak{e}_1, \ldots, \mathfrak{e}_t$. $(N(\mathfrak{e}_1), \ldots, N(\mathfrak{e}_t))$ is a family such that the cardinality of the union of each s terms is greater than or equal to $s - t + (t - p)$. Then by Corollary 1.6, the family $(N(\mathfrak{e}_1), \ldots, N(\mathfrak{e}_t))$ has a partial SDR of size $t - p$. That is, there are distinct elements $y_1, \ldots, y_{t-p}$

DOI: http://dx.doi.org/10.22108/toc.2019.105022.1506

of V_k such that $y_j \in N(\mathfrak{e}_{i_j})$. Now, for every $1 \leq j \leq t-p$, $\mathfrak{e}_{i_j} \cup \{y_j\}$ is a hyperedge of $\mathcal{H}$ and these hyperedges are pairwise disjoint. Thus they form a matching of size $t-p$ for $\mathcal{H}$. □

Theorem 2.7. *Let $\mathcal{H}$ be a k-uniform k-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$, and let $t = |V_1|$. Then $\mathcal{H}$ has a matching of size t if and only if $\alpha^{'} = \beta = t$, where $\alpha^{'}$ and β denotes the number of hyperedges in a maximum matching, and the number of vertices in a minimum vertex cover of $\mathcal{H}$, respectively.*

Proof. Let $\mathcal{H}$ has a matching of size t. We show that $\alpha^{'} = \beta = t$. Clearly $\beta \geq \alpha^{'}$ because for covering each hyperedge of maximum matching, one vertex is needed. But since there is a matching of size t, then $\alpha^{'} \geq t$. Now, V_1 is a minimal vertex cover of $\mathcal{H}$ because each hyperedge has only one vertex in V_1 and each vertex of V_1 lies in a hyperedge. Therefore $t \geq \beta$, which implies that $\alpha^{'} \geq \beta$. Then $\alpha^{'} = \beta$. The matching of size t is the maximum matching because it covers all vertices of V_1.

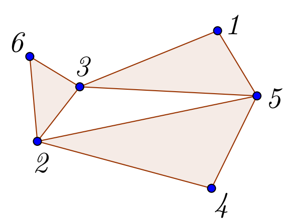
Conversely, if $\alpha^{'} = \beta = t$, it is clear that $\mathcal{H}$ has a matching of size t. □

The following example shows that removing the condition $t = |V_1|$ in Theorem 2.7 is not possible even if the subhypergraph generated on union of every $k-1$ parts, has a perfect matching.

Example 2.8. *Assume the 3-uniform 3-partite hypergraph $\mathcal{H}$ with the following presentation, where the parts of $\mathcal{H}$ are*

$$V_1 = \{1,2\},\ V_2 = \{3,4\},\ V_3 = \{5,6\}.$$

Indeed, $E(\mathcal{H}) = \{\{1,3,5\},\{2,3,6\},\{2,4,5\}\}$.



$\mathcal{H}$

In this hypergraph, we have the matching $\{\{1,3,5\}\}$ *of size 1. But* $\alpha^{'} \neq \beta$ *because* $\alpha^{'} = 1$ *and* $\beta = 2$. *Note that each one of subhypergraph generated on* $V_1 \cup V_2$, $V_2 \cup V_3$ *and* $V_1 \cup V_3$, *have a perfect matching.*

Acknowledgments

The author wish to thank Rashid Zaare-Nahandi for his helpful suggestions for the improvement of this work. I would also like to thank the referee for his or her useful and valuable comments.

References

[1] R. Aharoni and A. Georgakopoulos, P. Sprüssel, Perfect matching in r-partite r-graphs, *European J. Combin.*, **30** (2009) 39–42.

[2] R. Aharoni and O. Kessler, On a possible extension of Hall's theorem to bipartite hypergraphs, *Discrete. Math.*, **84** (1990) 309–313.

[3] R. Aharoni, Matchings in n-partite n-graphs, *Graphs Combin.*, **1** (1985) 303–304.

[4] R. Aharoni, On a criterion for matching in hypergraphs, *Graphs Combin.*, **9** (1993) 209–212.

[5] R. Aharoni, Ryser's conjecture for 3-partite 3-graphs, *Combinatorica*, **21** (2001) 1–4.

[6] C. Berge, *Hypergraphs*, Combinatorics of finite sets, Translated from the French, North-Holland Mathematical Library, **45**, North-Holland Publishing Co., Amsterdam, 1989.

[7] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, 1976.

[8] V. W. Bryant and H. Perfect, *Independence Theory in Combinatorics*, Chapman & Hall, London, 1980.

[9] Z. Füredi, Matchings and covers in hypergraphs, *Graphs Combin.*, **4** (1988) 115–206.

[10] P. E. Haxell, A condition for matchability in hypergraphs, *Graphs Combin.*, **11** (1995) 245–248.

[11] P. E. Haxell and A. D. Scott, On Ryser's conjecture, *Electron. J. Combin.*, **19** (2012) ♯R23.

[12] D. Khün, D. Osthus, Matchings in hypergraphs of large minimum degree, *J. Graph Theory*, **51** (2006) 269–280.

Reza jafarpour-Golzari

Department of Mathematics, Institute for Advanced Studies in Basic Science (IASBS), P.O.Box 45195-1159, Zanjan, Iran

Email: r.golzary@iasbs.ac.ir

DOI: http://dx.doi.org/10.22108/toc.2019.105022.1506