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# COLORING PROBLEM OF SIGNED INTERVAL GRAPHS

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## 1. Introduction

We consider simple graphs $G = (V, E)$, i.e graphs without loops and multiple edges. A graph $G$ together with a function $s : E \longrightarrow \{+, -\}$ on the edge set of $G$ is called a *signed graph*. If $\sigma$ is the set of edges whose image under $s$ is "$-$", then we denote the signed graph by $\Sigma = (G, \sigma)$. The graph $G$ is called the ground of $\Sigma$ and the set $\sigma$ is called the signature of it. For any edge $e$ of $\Sigma$, we call it a positive or negative edge if $s(e)$ has positive or negative sign respectively. By the edge and vertex set of $\Sigma$ we mean those of the ground graph that are $V, E$ respectively. For a signed graph $\Sigma = (G, \sigma)$ by the *positive* (*negative* ) *subgraph* we mean the spanning subgraph of $G$ where the edge set is the set of positive (negative) edges of $\Sigma$ and is denoted by $\Sigma^+$ ($\Sigma^-$).

For a cycle $C$ of $G$, the signature of $C$ in $(G, \sigma)$ is the product of signs of its edges. A cycle is called *balanced* if it has positive signature, otherwise we call it *unbalanced*. An unbalanced cycle of length $n$ will be denoted by $UC_n$. A signed graph all of whose cycles are balanced is called *balanced* otherwise we call it *unbalanced*. By *resigning* at a vertex $v$ of $\Sigma$ we mean to change signs of all the edges incident with $v$. Two signed graphs $(G, \sigma)$ and $(G, \sigma')$ are called *switching equivalent* if one is obtained from the other by a sequence of resignings. A well-known theorem of Zaslavsky states that two signed graphs $(G, \sigma)$ and $(G, \sigma')$ are switching equivalent if and only if they have the same set of unbalanced cycles, see [11]. A signed graph $(H, \sigma')$ is called a *signed subgraph* of $(G, \sigma)$ if $H$ is a subgraph of $G$ and $\sigma' = \sigma \cap E(H)$. For a subset $S$ of $V$, by $\langle S\rangle_\Sigma$ we denote the subgraph of $\Sigma$ whose

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ground is the induced subgraph of $G$ on $S$. If there is no doubt about $\Sigma$ then we simply write $\langle S \rangle$. For two subsets $X, Y$ of $V$, by $\langle X, Y \rangle_\Sigma$ we denote the subgraph of $\Sigma$ whose ground is the bipartite subgraph of $G$ on the partition $X, Y$ with all possible edges between $X$ and $Y$.

There are few variants of definitions for coloring problem of signed graphs. An initial definition due to Zaslavsky, is stated in [13]. There a proper vertex colouring of $\Sigma = (G, \sigma)$ is defined to be a mapping $\phi : V(G) \longrightarrow \mathbb{Z}$ such that for each edge $e = uv$ of $G$ the colour $\phi(u)$ is distinct from the colour $s(e)\phi(v)$, where $s(e)$ is sign of $e$. In other words, the colours of nodes of a positive edge must not be the same while those of a negative edge must not be opposite. A $k$-colouring of a signed graph $\Sigma = (G, \sigma)$, is defined to be a colouring $V(G) \longrightarrow \{-k, -(k-1), \ldots, 0, \ldots, k-1, k\}$. Therefore the chromatic number $\gamma(\Sigma)$ of $\Sigma$, according to Zaslavsky [12], is the smallest nonnegative integer $k$ for which $\Sigma$ admits a proper $k$-coloring. With respect to Zaslavsky's proper coloring definition, Macajova et al in [7] have proposed a new definition of chromatic number of signed graphs to be the whole number of signed colours used in an optimal colouring. This modifies Zaslavsky's definition to a natural extension of ordinary graphs chromatic number. See [7] for more details.

Naserasr et. al. in [8] has introduced the idea of homomorphism of signed graphs. Their definition of homomorphism of signed graphs follows. Let $\Sigma = (G, \sigma)$ and $\Pi = (H, \pi)$ be given. A function $\Phi : V(\Sigma) \rightarrow V(\Pi)$ is called a homomorphism of $\Sigma$ to $\Pi$ if there is a switching equivalent pair of $\Sigma$ say $\Sigma' = (G, \sigma')$ such that for any pair $u, v$ of vertices of $\Sigma'$ if there is a positive (negative) edge between $u, v$ in $\Sigma'$ then there is a positive (negative) edge between $\Phi(u), \Phi(v)$ in $\Pi$. Based on this definition the *signed chromatic number* of a signed graph $\Sigma$, denoted by $\chi_s(\Sigma)$, is the smallest order of a homomorphic image of it. Naserasr et al in section 3.4 of their paper proposed an equivalent definition of signed chromatic number based on the following proper coloring notion. An assignment of $k$ colors to the vertices of $(G, \sigma)$ is called a *proper coloring* if it satisfies the following properties.

- Colors of adjacent vertices must be different,
- Edges with different signs must not have the same set of colors on their end nodes.

Based on the previous definition, for a signed graph $(G, \sigma)$ the smallest number $k$ for which there is a switching equivalent pair of $(G, \sigma)$ say $(G, \sigma_1)$, which admits a proper coloring with $k$ colors, is its *signed chromatic number*(denoted by $\chi_s(G, \sigma)$ or $\chi_s(\Sigma)$). A signed graph $(H, \sigma_0)$ on $n$ vertices is called a *signed clique* or *S-clique* for short if its signed chromatic number equals $n$. The signed graph $UC_4$ is an example of a S-clique. Given a signed graph $(G, \sigma)$ its *S-clique number* is the maximum size of a signed subgraph of it which is a S-clique. The S-clique number of $(G, \sigma)$ will be denoted by $w_s(G, \sigma)$. Both the coloring and homomorphism problems of signed graphs are studied widely. In [1] it is proved that deciding whether there exists a signed graph homomorphism to any fixed signed graph, except some special cases, is NP-complete. It is also discussed that the coloring problem in the sense of Zaslavsky can be reformulated as signed graph homomorphism of some specified signed graphs, this implies that Zaslavsky coloring problem is NP-complete. In [8] it is also proved that the coloring problem defined by Naserasr et al is NP-complete.

For a vertex $v$ of a graph $G$, by $N_G(v)$ (or simply $N(v)$) we denote the set of neighbors of $v$ in $G$ and by $N_G[v]$ (or simply $N[v]$) we denote the closed neighborhood of $v$ in $G$, that is, $N(v) \cup \{v\}$.

An interval graph $I$ is the intersection graph of a set of real intervals. There exist some handy characterizations of them which are useful in this paper. For instance the following lemma from [3] provides a useful classification of these graphs.

**Lemma 1.1.** [3] *The graph $I$ is an interval graph if and only if the maximal cliques of it can be ordered $M_1, M_2, \ldots, M_l$ such that for any $i, j, s$, where $i \leqslant j \leqslant s$, it has the property $M_i \cap M_s \subseteq M_j$.*

In this paper we consider signed interval graphs, i.e signed graphs with an interval graph as their ground. The coloring problem INTSCOL of signed interval graphs is based on the definition by Naserasr et al as already mentioned and is defined in the following.

**INTSCOL**

Input: A signed interval graph $\mathcal{I}$ and a positive integer $k$.

Question: Is $\chi_s(\mathcal{I}) \leq k$?

The S-clique problem of signed interval graphs called INTSCLIQ is the following.

**INTSCLIQ**

Input: A signed interval graph $\mathcal{I}$ and a positive integer $k$.

Question: Is $w_s(\mathcal{I}) \geqslant k$?

We prove the following main theorems beyond other related results.

**Theorem 1.2.** *The INTSCLIQ problem is in P.*

**Theorem 1.3.** *The INTSCOL problem is NP-complete.*

## 2. Preliminaries

We say that the vertex set of a graph $G$ has a *perfect elimination ordering* (abbreviated P.E.O) if there is an ordering $v_1, v_2, \ldots, v_n$ of vertices of $G$, such that the subgraph of $G$ induced on the vertex set $\{v_i, \ldots, v_n\} \cap N_G[v_i]$ forms a clique. It is well known that any interval graph $I$ admits a P. E. O., see for instance [10]. Using this ordering, the vertices of an interval graph $I$ can be colored with $\omega$ (clique number of $I$) colors by a greedy algorithm efficiently. Hence the coloring problem of interval graphs is in P, see [6] for more details. A perfect elimination ordering of vertices of an interval graph can be obtained by the property already mentioned in Lemma 1.1.

In the following lemma from [8] an equivalent statement for a signed graph to be a S-clique is established.

**Lemma 2.1.** *A signed graph is a S-clique if and only if for each pair $u$ and $v$ of vertices either $uv$ is an edge or $u$ and $v$ are vertices of an unbalanced cycle of length 4.*

For more convenience, we define a neighborhood vector of a vertex in a signed graph as follows. Let $\Sigma$ be a signed graph on the ordered vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant n$ be an increasing sequence of integers. For $S = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\}$ and a vertex $v$ of $\Sigma$ we define the $S-$neighborhood vector of $v$, denoted by $\overrightarrow{N}_S(v)$, as below.

$$\overrightarrow{N}_S(v)_j = \begin{cases} 1 & \text{if there is a positive edge between } v \text{ and } v_{i_j}, \\ -1 & \text{if there is a negative edge between } v \text{ and } v_{i_j}, \\ 0 & \text{otherwise.} \end{cases}$$

The $V-$neighborhood vector of a vertex $v$ is simply denoted by $\overrightarrow{N}(v)$. By the definition just mentioned we can restate Lemma 2.1. A signed graph on an ordered set $V$ is a S-clique if and only if for each pair $u$ and $v$ of vertices either $u$ is adjacent to $v$ or for $S = N(u) \cap N(v)$, $\overrightarrow{N}_S(u) \neq \pm\overrightarrow{N}_S(v)$.

An induced complete bipartite subgraph of a graph is called a *biclique*. Here by *size* of a biclique we mean its number of vertices. Note that a biclique may consist only of an independent set of vertices. The maximum biclique problem of a graph abbreviated to MBC, is defined below.

**MBC**

Input: A graph $G$ and a positive integer $k$.

Question: Does $G$ contain a biclique of size more than or equal to $k$?

A set $L$ of edges of a graph $G$ which, when removed increases the number of components of the graph is an edge cut of $G$. The maximum edge cut problem is defined below.

**MEC**

Input: A graph $G$ and a positive integer $k$.

Question: Does $G$ contain an edge cut with more than or equal to $k$ edges?

A *clique* in a graph $G$ is a subgraph of it all of whose vertices are adjacent. The maximum size of a clique in $G$ is called the clique number of it and is denoted by $\omega(G)$. The definition of maximum clique problem in graphs, abbreviated to MC follows.

**MC**

Input: A graph $G$ and a positive integer $k$.

Question: Is $\omega(G) > k$?

A set $U$ of vertices of a graph is called *independent* if there is no edge between them. The maximum size of an independent set in a graph is called the *independence number* of graph. In what follows, we preview the complexity of the above problems from the literature which will be useful later.

**Fact 1.** [9] *The maximum independence number of a bipartite graph can be found in polynomial time.*

**Fact 2.** [5] *The MC problem in graphs is NP-complete.*

**Fact 3.** [5] *The MBC problem in graphs is NP-complete.*

**Remark 2.2.** *We need to clarify validity of Fact 2 and 3 by using the method mentioned in [5]. A graph property is called hereditary on induced subgraphs if for any graph which satisfies the property, any vertex induced subgraphs of it also satisfy the property. A property is called nontrivial if it is true for infinitely many graphs and also false for infinitely many graphs. For a fixed graph and property $\pi$, the node-deletion problem is finding the minimum number of vertices which must be deleted from the graph so that the result satisfies $\pi$. Lewis and Yannakakis [5] prove that the node-deletion problem for any hereditary (on induced subgraphs), and nontrivial graph property is NP-complete. Note that this applies to the MBC (and MC) problem since the property "being a biclique" ("being a clique") is hereditary (on induced subgraphs) and nontrivial, and maximizing the size of the biclique (clique) is the same as minimizing the number of vertices to be deleted so that the obtained subgraph satisfies the property.*

**Fact 4.** [9] *The MEC problem in a graph is NP-complete.*

We will make advantage of the following theorem by Harary, [4].

**Theorem 2.3.** *Two signed graphs $(G, \sigma)$ and $(G, \sigma')$ are switching equivalent if and only if the symmetric difference of $\sigma$ and $\sigma'$ is an edge cut of $G$.*

As a consequence of the above theorem note that a balanced signed graph $(G, \sigma)$ is switching equivalent to $(G, \emptyset)$ in which $\sigma$ is an edge cut of $G$.

## 3. Main Results

In this section we present our main results on the complexity of the INTSCOL and INTSCLIQ problems. There will be also some other related results. It is known that the coloring and the clique problems of interval graphs both belong to P. In this section these problems for signed interval graphs are studied.

For a signed graph $\mathcal{I} = (I, \sigma)$, we define a graph $\mathcal{I}^*$ as follows. The vertex set of $\mathcal{I}^*$ is $V(I)$ in which there is an edge between two vertices if they are adjacent in $\mathcal{I}$ or they belong to an unbalanced 4-cycle of $\mathcal{I}$. By Lemma 2.1, a set $A$ of vertices is a maximum signed clique in $\mathcal{I}$ if and only if it is a maximum clique of $\mathcal{I}^*$. Hence instead of considering the INTSCLIQ problem, it suffices to consider the maximum clique problem in $\mathcal{I}^*$ or equivalently the maximum independent set problem for its complement, $\mathcal{I}^{*c}$. For an interval graph $I$ whose maximal cliques have perfect elimination ordering $M_1, M_2, \ldots, M_l$, set ${M_1}^* = M_1 \setminus \bigcup_{i=2}^{l} M_i$ and ${M_l}^* = M_l \setminus \bigcup_{i=1}^{l-1} M_i$. We prove the following lemma about the graph $\mathcal{I}^{*c}$ which enables us to prove Theorem 1.2.

**Lemma 3.1.** *Let $H$ be a component of $\mathcal{I}^{*c}$, and $H_1$ be the induced subgraph of $H$ on the subset ${M_1}^* \cup {M_l}^*$. Then $H_1$ is a complete bipartite graph.*

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*Proof.* The graph $H_1$ is bipartite since all vertices in ${M_1}^*$ as well as ${M_l}^*$ are adjacent in $\mathcal{I}^*$ and so non-adjacent in $\mathcal{I}^{*c}$. Suppose that $A, B$ are bipartition of the vertices of $H_1$. For any $u \in A$ and $v \in B$ we prove there is an edge between them in $\mathcal{I}^{*c}$ and so in $H_1$. Note that the graph $H$ is connected so there exists a path between $u$ and $v$ in $H$. Let $u = u_1, u_2, \dots, u_s = v$ be a shortest $(u, v)$ path in $H$. Note that if we prove $s$ is equal to 2 then we are done. Now for $i = 1, 2, \dots, s$, let $M_{l_i}$ be the maximal clique of $I$ which contains $u_i$ so that $l_i$ is minimum between indices of maximal cliques containing $u_i$. Note that $l_1$ must be 1 and $l_s$ must be $l$. Since there is an edge between $u_i$ and $u_{i+1}$ and as $M_{l_i} \subseteq N_I(u_i)$, for each $i$, we have the following equalities, by definition of $\mathcal{I}^*$:

$$\overrightarrow{N}_{M_{l_i} \cap M_{l_{i+1}}}(u_i) = \pm \overrightarrow{N}_{M_{l_i} \cap M_{l_{i+1}}}(u_{i+1}), \qquad i = 1, \dots, l-1. \qquad (*)$$

On the other hand Lemma 1.1, implies that $M_1 \cap M_l \subseteq M_i$ for each $i = 1, \dots, l$ hence $M_1 \cap M_l \subseteq M_{l_i} \cap M_{l_{i+1}}$, for any $i = 1, \dots, s$. Thus $(*)$ implies the following equalities:

$$\overrightarrow{N}_{M_1 \cap M_l}(u_1) = \pm \overrightarrow{N}_{M_1 \cap M_l}(u_2) = \cdots = \pm \overrightarrow{N}_{M_1 \cap M_l}(u_s).$$

Note that $u$ and $v$ are not adjacent in $I$. We claim they are also non-adjacent in $\mathcal{I}^*$, since otherwise they would belong to some $UC_4$, and the two other vertices of this $UC_4$ must be in $M_1 \cap M_l$, which in view of the above equalities imply that $u, v$ do not belong to a $UC_4$, which is impossible. So $u, v$ are adjacent in $\mathcal{I}^{*c}$ and so in $H_1$. Hence the assertion follows. □

**Theorem 3.2.** *The INTSCLIQ problem is in P.*

*Proof.* As already discussed, the INTSCLIQ problem is equivalent to the maximum independent set problem in $\mathcal{I}^{*c}$. To find a maximum independent set of $\mathcal{I}^{*c}$, it suffices to compute the maximum independent sets of all $O(n)$ connected components of $\mathcal{I}^{*c}$ and then taking the union of all the solutions (we call this process the UNION MAKING). Let the graphs $H$ and $H_1$ be as above. By Lemma 3.1, $H_1$ is a complete bipartite graph. Then any maximum independent set in each component of $\mathcal{I}^{*c}$ has empty intersection either with ${M_1}^*$ or ${M_l}^*$. Hence a maximum independent set of any of $O(n)$ components of $\mathcal{I}^{*c}$, can be found by looking at one of the subgraphs induced on $M_2 \cup M_3 \cup \cdots \cup M_l$ or $M_1 \cup M_2 \cup \cdots \cup M_{l-1}$, say ${\mathcal{I}_1}^*$ and ${\mathcal{I}_2}^*$. Actually we find the maximum independent sets of ${\mathcal{I}_1}^*$ and ${\mathcal{I}_2}^*$ then choose the one with maximum cardinality between these two independent sets (we call this process the COMPARISON). Note that Lemma 3.1 applies for ${\mathcal{I}_1}^*$ and ${\mathcal{I}_2}^*$. Thus a similar approach as $\mathcal{I}{*}^c$ can be applied for each of ${\mathcal{I}_1}^*$ and ${\mathcal{I}_2}^*$ then we end up with induced subgraphs on the sets $M_1 \cup M_2 \cup \cdots \cup M_{l-2}$, or $M_2 \cup M_3 \cup \cdots \cup M_{l-1}$, or $M_3 \cup M_4 \cup \cdots \cup M_l$. By continuing the procedure we finally end up with the subgraphs of $\mathcal{I}^{*c}$ on the union of two consecutive $M_i$'s, that is a bipartite graph and so by Fact 1 their maximum independence number can be found in polynomial time. We should note that at each step which we reduce the number of $M_i$'s, we have a COMPARISON and at each obtained subgraph we have to do UNION MAKING for finding a maximum independent set. Note that at the end there are at most $\binom{l}{2}$ different subgraphs to be considered (this is the number of sets of consecutive maximal cliques). So, in total the number of inductive steps of this process is quadratic in $l$, and each step can be done in polynomial time. Hence the independence number of $\mathcal{I}^{*c}$ can be found in a polynomial time. Consequently, the maximum clique problem of $\mathcal{I}^*$ and thus the INSCLIQ problem can be solved in polynomial time. □

Now we consider the coloring problem of signed interval graphs. We prove that this problem is NP-complete, which is in contrast with the existing results for the complexity of ordinary coloring problem in interval graphs.

**Theorem 3.3.** *The INTSCOL problem is NP-complete.*

*Proof.* We make use of Fact 3 to prove the assertion. We reduce an instance of the MBC problem to an instance of the INTSCOL problem. Let $(G, k)$ be an instance of the MBC problem, where $G$ is a graph with $n$ vertices. Consider the graph $I$ to be equal to $K_n \wedge K_n$ which is the disjoint union of two copies of the complete graph $K_n$, say $I_1$ and $I_2$. The graph $I$ is clearly an interval graph. Let $\mathcal{I} = (I, \sigma)$ be a signed graph in which the edges of the subgraph $I_1$ are all positive and in $I_2$, the edges of a subgraph $G'$ isomorphic to $G$ be assigned $-$ and $+$ elsewhere. We prove that the MBC problem for the instance $(G, k)$ is correct if and only if the INTSCOL problem for $(\mathcal{I}, 2n-k)$ is correct. On the other hand $G$ has a biclique on more than $k$ vertices if and only if $\mathcal{I}$ has a proper coloring with $2n-k$ colors. Suppose that a switching mate of $\mathcal{I}$ say $\mathcal{I}'$ admits a feasible coloring $c$ with $2n-k$ colors. Let $\mathcal{I}_1$, $\mathcal{I}_2$ (${\mathcal{I}'}_1$, ${\mathcal{I}'}_2$) be the subgraphs of $\mathcal{I}$ ($\mathcal{I}'$) with ground $I_1$ and $I_2$ respectively. With no loss of generality we may assume the vertices of $\mathcal{I}'_1$ have been assigned the colors $1, 2, \ldots, n$ in the coloring $c$ and some colors from $[n] = \{1, 2, \ldots, n\}$, say $i_1, i_2, \ldots, i_r$ are used for coloring some vertices of the subgraph $\mathcal{I}'_2$. Set $W$ be the set of such vertices in $\mathcal{I}'_2$ sharing the colors with some of vertices in $\mathcal{I}'_1$. As it has been used $2n-k$ colors in the coloring of $\mathcal{I}'$ then the number of common colors between vertices of ${\mathcal{I}'}_1$ and ${\mathcal{I}'}_2$ must be equal to $k$. On the other hand the subgraph ${\mathcal{I}'}_1$ can be considered all positive hence the subgraph of ${\mathcal{I}'}_2$ induced on $W$, must be all positive as well. By Theorem 2.3, the subgraph of $\mathcal{I}^-$, induced on $W$ must be a biclique which is of order $k$. Therefore the subgraph induced on $W$ is a biclique of $G(G')$ on $k$ vertices. Thus the MBC problem of instance $(G, k)$ is correct. The converse also holds as if $G$ has a biclique on more than $k$ vertices, say $U$, the signed graph $\mathcal{I}$ can be resigned to a signed graph $\mathcal{I}'$ in which the subgraph induced on $U$ becomes all positive. Now one can assign a proper coloring of $\mathcal{I}'$ with $2n-k$ colors. Such that all vertices in $\mathcal{I}'_1$ be colored with $n$ distinct colors, say $1, 2, \ldots, n$, vertices of $U$ in $\mathcal{I}'_2$ to be colored with $1, 2, \ldots, k$ and the rest of vertices of $\mathcal{I}'_2$ be given the colors $n+1, \ldots, 2n-k$. Hence the assertion follows by Fact 3. □

## 4. Miscellaneous

In this section, we present two minor results. We consider the minimum signature problem abbreviated to MS which is defined below.

**MS** problem

Input: A signed graph $(G, \sigma_0)$ and a positive integer $k$

Question: Is there an edge set $\sigma$ where $|\sigma| \leq k$ and $(G, \sigma_0)$ is switching equivalent to $(G, \sigma)$?

**Proposition 4.1.** *The MS problem for signed graphs is NP-complete.*

*Proof.* The MS problem belongs to NP indeed. Let $(G, k)$ be an instance of the MEC problem. We reduce the problem to the instance $((G, E(G)), |E(G)| - k)$ of the MS problem. Consider the signed graph $(G, E(G))$ and let $(G, \sigma)$ be a switching equivalent signed graph of it where $\sigma$ is of size less than $|E(G)| - k$. Suppose that $(G, \sigma)$ is obtained from $(G, E(G))$ by resigning at the vertex set $S$. Resigning of the vertices in $S$ does not change the sign of edges in the subgraphs $\langle S \rangle$ and $\langle \overline{S} \rangle$ of $G$. But it changes the sign of all the edges in the edge cut $[S, \overline{S}]$. But $|\sigma| \leq |E(G)| - k$ if and only if $[S, \overline{S}]$ has size more than or equal to $k$. Thus the instance $(G, k)$ of the MEC problem is correct if and only if the instance $((G, E(G)), |E(G)| - k)$ of the MS problem is correct, hence the assertion follows by Fact 4. □

**Proposition 4.2.** *Let $\mathcal{I} = [I, \sigma]$ be a signed interval graph with connected ground. The following statements hold:*

(a) *$\chi_s(\mathcal{I}) = 2$ if and only if $I$ is a caterpillar tree that is a tree in which removing leaves produces a path.*

(b) *If $\chi_s(\mathcal{I}) = 3$, then $I$ is an interval graph with clique number 3 and all the triangles have the same signature.*

*Proof.* For the case (a), it is well known that the interval graphs do not have an induced cycle of length greater than three, therefore any connected signed interval graph with chromatic number two, say $I$ is a tree. It is proved in [2] that an interval graph is triangle free if and only if it is a caterpillar tree. On the other hand all signed trees are switching equivalent therefore the assertion holds.

For case (b), the graph $I$ contains a cycle and then a triangle. The graph does not contain both balanced and unbalanced triangles since if there is a balanced triangle then all possible pairs of three colors are used in coloring the vertices of it therefore by definition the vertices of an unbalanced triangle can not be colored with the same three colors. On the other hand it is known that the chromatic number of an interval graph is equal to its clique number, thus the assertion follows. □

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