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ELLIPTIC ROOT SYSTEMS OF TYPE A_1, A COMBINATORIAL STUDY

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Abstract. We consider some combinatorics of elliptic root systems of type A_1. In particular, with respect to a fixed reflectable base, we give a precise description of the positive roots in terms of a "positivity" theorem. Also the set of reduced words of the corresponding Weyl group is precisely described. These then lead to a new characterization of the core of the corresponding Lie algebra, namely we show that the core is generated by positive root spaces.

1. Introduction

The class of extended affine root systems (EARS) as a natural generalization of the class of finite and affine irreducible root systems was introduced in 1985 by K. Saito [8]. Extended affine root systems of nullity 1 and 2 are called affine root systems and elliptic root systems, respectively. Elliptic root systems have some applications in algebraic geometry, in fact they are used in the description of the lattices generated by vanishing cycles for simply elliptic singularities [7].

In 1990 a new class of Lie algebras was introduced, this class was later studied systematically in [1] under the name of extended affine Lie algebras. Extended affine root systems appear as the root systems of extended affine Lie algebras. Extended affine Lie algebras of nullity 1 are just affine Lie algebras, and extended affine Lie algebras of nullity 2 are called elliptic Lie algebras. There are different realizations for elliptic Lie algebras, for example in [9], the authors give a finite presentation of elliptic Lie algebras by generalizing the Serre relations in terms of an elliptic Dynkin diagram.

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The core of an extended affine Lie algebra, which is the subalgebra generated by non-isotropic root spaces is classified in [5] and [2]. In [10], the author determines coordinate algebras of extended affine Lie algebras of type A_1, and gives a classification of this kind of algebras by using the concept of Jordan torus. In fact he shows that the core of any extended affine Lie algebra of type A_1 is a central extension of the Lie algebra obtained from Tits-Kantor-Koecher construction of a Jordan algebra J, where J is a torus over $\mathbb{C}$ isomorphic to one of the following four tori

$$\mathbb{C}_{q^+}, H(\mathbb{C}_\varepsilon, *), J_{S^m}(a_{\epsilon\{\epsilon\in I\}}), \text{or } \mathbb{A}_t,$$

for details see [10, Theorem 7.1, Corollary 7.2].

In [3], the authors examine some combinatorics of affine theory for A_1-type extended affine root systems. In particular they show that if R is any such root system and Π is a reflectable base for R, then the set of non-isotropic roots admits a natural decomposition into positive and negative roots. Here by "natural" we mean a prototype decomposition in terms of finite and affine theory. It is therefore of interest to rewrite these results for elliptic root systems and apply them to the corresponding Lie algebras. In the present work, we only consider elliptic Lie algebras where the corresponding Jordan algebra is $J_{S^m}(a_{\epsilon\{\epsilon\in I\}})$.

The organization of the paper is as follows. Some necessary preliminaries and definitions are given in Section 2. In Section 3, we give a precise description of the reduced words of the Weyl group associated to an elliptic root system in terms of a fixed reflectable base, Proposition 3.1. Also, positive roots are precisely determined, Theorem 3.4. In Section 4, using the Tits-Kantor-Koecher (TKK) construction, we describe the construction of an extended affine Lie algebra of type A_1, starting from a specific Jordan algebra. We then apply our concept of positive and negative roots to these Lie algebras. The main interesting result is a new characterization of the core in terms of positive roots, namely we show that the core is generated by positive root spaces, Theorem 4.2. In the last section, the above results are examined for a different reflectable base of an A_1-type elliptic root system.

2. **Preliminaries**

In this section we collect some notations and facts which are needed throughout the paper. Let A be an abelian group equipped with a positive semi-definite symmetric bilinear form and $A^0 = \{\alpha \in A \mid (\alpha, A) = 0\}$ be the radical of the form. Let $(A, (\cdot,\cdot), R)$ be a tame irreducible reduced affine reflection system, namely R is a subset of A satisfying axioms (R1)-(R6) of [4, Definition 1.3]. If A^0 is a lattice, then its rank is called the *nullity* of R and is denoted by ν. An affine reflection system of finite nullity is called an *extended affine root system* and an extended affine root system of nullity 2 is called an *elliptic root system*. Let $R^\times = \{\alpha \in R \mid (\alpha,\alpha) \neq 0\}$ and $R^0 = \{\alpha \in R \mid (\alpha,\alpha) = 0\}$ be the sets of non-isotropic and isotropic roots of R and $\mathcal{W}$ be the Weyl group of R defined by $\mathcal{W} = \langle r_\alpha \mid \alpha \in R^\times\rangle$ where $r_\alpha \in \mathrm{Aut}(A)$ and $r_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$, for any $\beta \in A$. A subset $\Pi \subseteq R^\times$ is called a *reflectable base* for R, see [4, Definition 1.19], if $R^\times = \mathcal{W}_\Pi \Pi$ and Π has no proper subset satisfying this property.

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Here $\mathcal{W}_\Pi$ is the subgroup of $\mathcal{W}$ generated by reflections r_α, $\alpha \in \Pi$. It follows easily that if Π is a reflectable base, then $\mathcal{W} = \mathcal{W}_\Pi$.

For $\overline{\beta} = (\beta_1, \ldots, \beta_n) \in \Pi^n$ and $\overline{\gamma} = (\gamma_1, \ldots, \gamma_m) \in \Pi^m$, we set

$$\begin{aligned}
&\overline{\beta}\overline{\gamma} := (\beta_1, \ldots, \beta_n, \gamma_1, \ldots, \gamma_m),\\
&\overline{\beta}^{-1} := (\beta_n, \ldots, \beta_1),\\
&|\overline{\beta}| := n,\\
&w(\overline{\beta}) := r_{\beta_1} \cdots r_{\beta_n} \in \mathcal{W}.
\end{aligned}$$

Let Π be a reflectable base for R. For $w \in \mathcal{W}$, we denote by $\ell(w)$, the length of w with respect to reflections based on elements of Π. Recall that for $\overline{\beta} = (\beta_1, \ldots, \beta_n) \in \Pi^n$, $w(\overline{\beta})$ is called *reduced* if $\ell(w(\overline{\beta})) = n$. By convention, we say $\overline{\beta}$ is *reduced in* Π if $w(\overline{\beta})$ is reduced. For an n-tuple $\overline{\beta} \in \Pi^n$, we set

$$S(\overline{\beta}) := \{\beta_1, r_{\beta_1}(\beta_2), \ldots, r_{\beta_1} \cdots r_{\beta_{n-1}}(\beta_n)\},$$

and

$$\langle S(\overline{\beta})\rangle := \sum_{j=1}^{n} r_{\beta_1} \cdots r_{\beta_{j-1}}(\beta_j).$$

From now on we assume that R is of type A_1. Then R is of the form

$$R = (S + S) \cup (\pm\dot{\alpha} + S) \tag{2.1}$$

where S is a pointed reflection space in A^0, $(\dot{\alpha}, \dot{\alpha}) = 2$ and $A = \dot{A} \oplus A^0$, where $\dot{A} = \mathbb{Z}\dot{\alpha}$, $\langle R^\circ\rangle = \langle S\rangle = A^\circ$ ([4, §1]).

Consider the projection and sign maps

$$\begin{aligned} p : A &\longrightarrow A^0 \\ r\dot{\alpha} + \sigma &\longmapsto \sigma \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{sgn} : A &\longrightarrow \mathbb{Z} \\ r\dot{\alpha} + \sigma &\longmapsto r. \end{aligned}$$

Let Π be a reflectable base for R. By a base change of $\dot{A}$ if necessary we may assume that $\dot{\alpha} \in \Pi$.

3. **Elliptic root systems of type A_1**

Throughout this section R is an elliptic root system of type A_1. Let $\{\sigma_1, \sigma_2\}$ be a basis of A^0, namely $A^0 = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$. Up to isomorphism, there exist two non-isomorphic elliptic root systems of type A_1, namely

$$\tilde{R} := A^0 \cup (\pm\dot{\alpha} + A^0), \tag{3.1}$$

and

$$R := (S + S) \cup (\pm\dot{\alpha} + S), \text{ where } S = \{n_1\sigma_1 + n_2\sigma_2 \mid n_1 n_2 = 0 \pmod 2\}. \tag{3.2}$$

In this work we consider the elliptic roost system R given in (3.2). We fix the reflectable base $\Pi = \{\dot{\alpha}, \sigma_1 - \dot{\alpha}, \sigma_2 - \dot{\alpha}\}$ of R. According to Theorem 5.2 of [3] which is called the positivity theorem,

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we have

$$R^{\times} = R^{+} \uplus R^{-} \quad \text{where} \quad R^{+} := \bigcup_{\overline{\beta}} S(\overline{\beta}) \tag{3.3}$$

and the union is over all reduced words $\overline{\beta}$ in Π. Moreover, $R^{-} = -R^{+}$.

In this section we give a precise description of elements of R^{+} given in (3.3), for the root system and the reflectable base under consideration.

Proposition 3.1. *Each element of $\mathcal{W}$ has only one of the following reduced forms:*
$(xy)^m(xz)^n x^k$,
$(xy)^m(zy)^n z^k$,
where x, y, z are distinct elements of $\{r_\beta \mid \beta \in \Pi\}$, and $m, n \in \mathbb{Z}_{\geq 0}$, $k \in \{0, 1\}$.

Proof. By [3, Theorem 4.1], $\mathcal{W} \cong G = (s_\alpha \mid {s_\alpha}^2 = 1,\ (s_{\alpha_i} s_{\alpha_j} s_{\alpha_k})^2 = 1,\ \alpha, \alpha_t \in \Pi)$ and note from [3, Theorem 4.6] that all the expressions given in the statement are reduced in $\mathcal{W}$. Let $w \in \mathcal{W}$, we have $w = w(\overline{\beta})$, where $\overline{\beta} = (\beta_1, \ldots, \beta_t) \in \Pi^t$ is reduced. We use induction on t where the case $t = 1$ clearly holds. Assume now that $t > 1$. Then $\overline{\beta}' := (\beta_1, \ldots, \beta_{t-1})$ is also reduced and so by induction $w(\overline{\beta}')$ has one of the forms in the statement. Assume first that $w(\overline{\beta}') = (xy)^m(xz)^n x^k$ with $k = 0$. Now $r_{\beta_t} \in \{x, y, z\}$. If $n \geq 1$ and $r_{\beta_t} = z$, then we get $\ell(w(\overline{\beta})) < \ell(w(\overline{\beta}'))$ which is absurd, and if $n = 0$ and $r_{\beta_t} = z$, then $w(\overline{\beta}) = (xy)^m z$ which is one of the required forms. If $r_{\beta_t} = x$, then $w(\overline{\beta})$ is clearly one of the desired forms. If $r_{\beta_t} = y$ and $m = 0$, $w(\overline{\beta})$ is again one of the required forms. If $m \geq 1$, then $r_{\beta_t} \neq y$ since otherwise

$$w(\overline{\beta}) = (xy)^m(xz)^n y = (xy)^{m-1}(xy)(xz)^n y = (xy)^{m-1}(yy)(xz)^n x = (xy)^{m-1}(xz)^n x,$$

where the term in the right hand side of the third equality is obtained by interchanging x and y which are both in odd positions (using the parity relations [3, Lemma 4.2(i)]). This shows that $\ell(w(\overline{\beta})) < \ell(w(\overline{\beta}'))$ which is absurd.

Assume now that $k = 1$. Then clearly $r_{\beta_t} \neq x$ by a length argument. If $r_{\beta_t} = z$, then $w(\overline{\beta})$ has one of the required forms. If $m = 0$ and $r_{\beta_t} = y$, then $w(\overline{\beta})$ is in desired from again. If $m \geq 1$ and $r_{\beta_t} = y$, then $w(\overline{\beta}) = (xy)^m(xz)^n xy = (xy)^{m+1}(xz)^n$ which is again one of the forms in the statement, where the latter expression is obtained by interchanging the last right hand side y with one of the z's (both are in even positions).

Next let $w(\overline{\beta}') = (xy)^m(zy)^n z^k$ with $k = 0$. Then the case $r_{\beta_t} = y$ can not happen by a trivial length argument, and the case $r_{\beta_t} = z$ puts $w(\overline{\beta})$ in one of the required forms. If $r_{\beta_t} = x$ and either m or n is equal to zero, we are clearly done. Otherwise $w(\overline{\beta}) = (xy)^m(zy)^n x = (xy)^{m+1}(zy)^{n-1} z$ which is one of the assumed forms, where here we have used the parity relations (interchanging the last right hand side x with one of the z's as both are on odd positions).

Finally assume that $w(\overline{\beta}') = (xy)^m(zy)^n z$. Clearly we are done of $r_{\beta_t} = y$. The case $r_{\beta_t} = z$ can not happen by a length argument. If $r_{\beta_t} = x$ and $m = 0$ we are clearly done, if $r_{\beta_t} = x$ and $m \geq 1$,

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then $w(\overline{\beta}) = (xy)^m(zy)^n zx = (xy)^{m-1}(zy)^{n+1}$ which is absurd as this gives $\ell(w(\overline{\beta})) < \ell(w(\overline{\beta}'))$. This completes the proof. □

For $0 \neq n \in \mathbb{Z}$ and $x, y \in \Pi$, we set by convention

$$(x,y)^n := \begin{cases} \overbrace{(x,y,\ldots,x,y)}^{2n} & \text{if } n>0 \\ \overbrace{(y,x,\ldots,y,x)}^{2n} & \text{if } n<0. \end{cases}$$

By Proposition 3.1, the following words which will be used in the sequel are reduced.

$$\begin{aligned} \overline{\beta}_j^n &:= (\dot{\alpha}, \sigma_j - \dot{\alpha})^n, \\ \overline{\eta}^n &:= (\sigma_1 - \dot{\alpha}, \sigma_2 - \dot{\alpha})^n. \end{aligned} \tag{3.4}$$

By a straightforward computation we get the following lemma.

Lemma 3.2. *Let $n \in \mathbb{Z}$ and $j = 1, 2$. Then we have*
(i) $S(\overline{\beta}_j^n) = \{\dot{\alpha} + (i-1)\sigma_j \mid 1 \le i \le 2n\}$, *if* $n > 0$,
(ii) $S(\overline{\beta}_j^n) = \{-\dot{\alpha} + i\sigma_j \mid 1 \le i \le 2|n|\}$, *if* $n < 0$,
(iii) $S(\overline{\eta}^n) = \{(-1)^i(\dot{\alpha} - i\sigma_1 + (i-1)\sigma_2) \mid 1 \le i \le 2n\}$, *if* $n > 0$,
(iv) $S(\overline{\eta}^n) = \{(-1)^i(\dot{\alpha} - i\sigma_2 + (i-1)\sigma_1) \mid 1 \le i \le 2|n|\}$, *if* $n < 0$.

For $\alpha \in A \setminus A^0$ and $\sigma \in A^0$, we define $t_\alpha^\sigma : A \longrightarrow A$, by $t_\alpha^\sigma(\beta) := \beta + (\alpha, \beta)\sigma$. Then

$$t_\alpha^\sigma = r_{\alpha+\sigma} r_\alpha = r_\alpha r_{\alpha-\sigma}, \quad (t_\alpha^\sigma)^n = t_\alpha^{n\sigma} \quad \text{and} \quad (t_\alpha^\sigma)^n(\alpha) = \alpha + 2n\sigma \quad (n \in \mathbb{Z}). \tag{3.5}$$

In particular,

$$\alpha + m\sigma_1 + n\sigma_2 = \begin{cases} (t_\alpha^{\sigma_1})^{m/2}(t_\alpha^{\sigma_2})^{n/2}(\alpha) & m, n \text{ even} \\ (t_\alpha^{\sigma_1})^{m/2}(t_\alpha^{\sigma_2})^{[n/2]}(\alpha + \sigma_2) & m \text{ even and } n \text{ odd.} \end{cases} \tag{3.6}$$

Lemma 3.3. *We have for $k, k' > 0$,*
(i) $S(\overline{\beta}_1^{-k}\overline{\beta}_2^{-k'}) = S(\overline{\beta}_1^{-k}) \cup \{-\dot{\alpha} + j\sigma_2 + 2k\sigma_1 \mid 1 \le j \le 2k'\}$,
(ii) $S(\overline{\beta}_1^{k}\overline{\beta}_2^{k'}) = S(\overline{\beta}_1^{k}) \cup \{\dot{\alpha} + (j-1)\sigma_2 + 2k\sigma_1 \mid 1 \le j \le 2k'\}$,
(iii) $S(\overline{\eta}^{k}\overline{\beta}_1^{-k'}) = S(\overline{\eta}^{k}) \cup \{-\dot{\alpha} + j\sigma_1 + 2k(\sigma_1 - \sigma_2) \mid 1 \le j \le 2k'\}$,
(iv) $S(\overline{\eta}^{-k}\overline{\beta}_1^{k'}) = S(\overline{\eta}^{-k}) \cup \{\dot{\alpha} + (j-1)\sigma_1 + 2k(\sigma_1 - \sigma_2) \mid 1 \le j \le 2k'\}$,
(v) $S(\overline{\eta}^{k}\overline{\beta}_2^{k'}) = S(\overline{\eta}^{k}) \cup \{\dot{\alpha} + (j-1)\sigma_2 + 2k(\sigma_2 - \sigma_1) \mid 1 \le j \le 2k'\}$,
(vi) $S(\overline{\eta}^{-k}\overline{\beta}_2^{-k'}) = S(\overline{\eta}^{-k}) \cup \{-\dot{\alpha} + (2k+j)\sigma_2 - 2k\sigma_1 \mid 1 \le j \le 2k'\}$,
(vii) $S(\overline{\beta}_1^{-k}\overline{\eta}^{k'}) = S(\overline{\beta}_1^{-k}) \cup \{(-1)^j(\dot{\alpha} - (2k+j)\sigma_1 + (j-1)\sigma_2) \mid 1 \le j \le 2k'\}$,
(viii) $S(\overline{\beta}_1^{k}\overline{\eta}^{-k'}) = S(\overline{\beta}_1^{k}) \cup \{(-1)^j(\dot{\alpha} + (2k+j-1)\sigma_1 - j\sigma_2) \mid 1 \le j \le 2k'\}$,
(ix) $S(\overline{\beta}_2^{k}\overline{\eta}^{k'}) = S(\overline{\beta}_2^{k}) \cup \{(-1)^j(\dot{\alpha} - j\sigma_1 + (2k+j-1)\sigma_2) \mid 1 \le j \le 2k'\}$,
(x) $S(\overline{\beta}_2^{-k}\overline{\eta}^{-k'}) = S(\overline{\beta}_2^{-k}) \cup \{(-1)^j(\dot{\alpha} + (j-1)\sigma_1 - (2k+j)\sigma_2) \mid 1 \le j \le 2k'\}$.

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Proof. From [3, Lemma 2.3(ii)], Lemma 3.2(ii) and (3.5), we have for $\overline{\beta}=\overline{\beta}_1^{-k}\overline{\beta}_2^{-k'}$,

$$\begin{aligned}S(\overline{\beta}) &= S(\overline{\beta}_1^{-k})\cup w(\overline{\beta}_1^{-k})S(\overline{\beta}_2^{-k'})\\ &= S(\overline{\beta}_1^{-k})\cup t_{\dot\alpha}^{-k\sigma_1}\{-\dot\alpha+j\sigma_2\mid 1\leq j\leq 2k'\}\\ &= S(\overline{\beta}_1^{-k})\cup\{-\dot\alpha+j\sigma_2+2k\sigma_1\mid 1\leq j\leq 2k'\}.\end{aligned}$$

This gives the first equality. Other equalities are obtained in a similar manner. □

Theorem 3.4. *Let $\gamma=\varepsilon\dot\alpha+m_1\sigma_1+m_2\sigma_2\in R^\times=R^+\uplus R^-$. Then $\gamma\in R^+$ if and only if γ has one of the following forms:*

ϵ	m_1	m_2
±1	*even*	*odd* >0
	odd>0	*even*
	$0\neq$*even*	$0\neq$*even*$\geq -m_1$
1	*even*≥ 0	*even*≥ 0
-1	*even*≥ 0	*even*>0
-1	*even*>0	*even*≥ 0

Proof. Let $\gamma=\dot\alpha+m_1\sigma_1+m_2\sigma_2\in R^\times$ and $m_1,m_2\in\mathbb{Z}$ with $m_1m_2\in 2\mathbb{Z}$. A direct computation shows that for each given m_i, we have $\gamma\in\pm S(\overline{\beta})$ where $\overline{\beta}$ is given explicitly in the third column of the table below; in the table k,k' are nonnegative integers. Note also that by Proposition 3.1, all $\overline{\beta}$'s appearing in the table are reduced. To make the table short, we have excluded from the table those cases which follow by symmetry between σ_1 and σ_2.

m_1	m_2	$\overline{\beta}$	$\Rightarrow$	$\gamma\in$
$2k$	$2k'$	$\overline{\beta}_1^{k}\overline{\beta}_2^{(k'+1)}$	Lemma 3.3(ii) with $j=2k'+1$	$S(\overline{\beta})$
$2k$	$2k'+1$	$\overline{\beta}_1^{k}\overline{\beta}_2^{(k'+1)}$	Lemma 3.3(ii) with $j=2k'+2$	$S(\overline{\beta})$
$2k$	-$2k'$	$\overline{\eta}^{-k}\overline{\beta}_2^{-k'}$ $(k<k')$	Lemma 3.3(vi) with $j=2(k'-k)$	-$S(\overline{\beta})$
$2k$	-$2k'$	$\overline{\eta}^{-k'}\overline{\beta}_1^{k}$ $(k'\leq k)$	Lemma 3.3(iv) with $j=2(k-k')+1$	$S(\overline{\beta})$
-$2k$	$2k'$	$\overline{\eta}^{k'}\overline{\beta}_1^{-k}$ $(k'<k)$	Lemma 3.3(iii) with $j=2(k-k')$	-$S(\overline{\beta})$
-$2k$	$2k'$	$\overline{\eta}^{k}\overline{\beta}_2^{k'}$ $(k\leq k')$	Lemma 3.3(v) with $j=2(k'-k)+1$	$S(\overline{\beta})$
-$2k$	-$2k'$	$\overline{\beta}_1^{-k}\overline{\beta}_2^{-k'}$	Lemma 3.3(i)	-$S(\overline{\beta})$
$2k$	-$(2k'+1)$	$\overline{\eta}^{-k}\overline{\beta}_2^{-k'}$ $(k\leq k')$	Lemma 3.3(vi) with $j=2(k'-k)+1$	-$S(\overline{\beta})$
$2k$	-$(2k'+1)$	$\overline{\beta}_1^{(k-k')}\overline{\eta}^{-(k'+1)}$ $(k'<k)$	Lemma 3.3(viii) with $j=2k'+1$	-$S(\overline{\beta})$
-$2k$	$(2k'+1)$	$\overline{\eta}^{k}\overline{\beta}_2^{k'}$ $(k\leq k')$	Lemma 3.3(v) with $j=2(k'-k)+2$	$S(\overline{\beta})$
-$2k$	$(2k'+1)$	$\overline{\beta}_1^{(k-k'-1)}\overline{\eta}^{(k'+1)}$ $(k'<k)$	Lemma 3.3(vii) with $j=2k'+2$	$S(\overline{\beta})$
-$2k$	-$(2k'+1)$	$\overline{\beta}_1^{-k}\overline{\beta}_2^{-(k'+1)}$	Lemma 3.3(i) with $j=2k'+1$	-$S(\overline{\beta})$

□

Corollary 3.5. *We have $(R^++R^+)\cap R=R^0\setminus\{-2k\sigma_1-2k'\sigma_2\mid k,k'\geq 0\}$ and $(R^-+R^-)\cap R=R^0\setminus\{2k\sigma_1+2k'\sigma_2\mid k,k'\geq 0\}$.*

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Proof. Since $R^0 = S + S = \{m\sigma_1 + n\sigma_2 \mid m, n \in \mathbb{Z}\} = A^0$, the result follows by examining the table given in the statement of Theorem 3.4. □

Lemma 3.6. *Let α be an arbitrary positive root given in Theorem 3.4. Then we can write $\alpha = \alpha' + k\delta$, where $\alpha' \in R^+$, $k \in 2\mathbb{Z}+1$ and $\delta \in \{\sigma_1, \sigma_2\}$.*

Proof. Let $\alpha \in R^+$. We assume $\epsilon, i, j \in \{-1, +1\}$ $(i \neq j)$ and $k_i, k_j \in \mathbb{Z}_{\geq 0}$. By Theorem 3.4, we consider the following cases and for each case we obtain $\alpha = \alpha' + k\sigma_i$ where $\alpha' \in R^+$ and $k \in 2\mathbb{Z}+1$.

1) Let $\alpha = \epsilon\dot{\alpha} + k_i\sigma_i + k_j\sigma_j$. We can write $\alpha = \epsilon\dot{\alpha} + k_i\sigma_i + (k_j+1)\sigma_j - \sigma_j$. If we set $\alpha' = \epsilon\dot{\alpha} + k_i\sigma_i + (k_j+1)\sigma_j$, then $\alpha' \in R^+$ and $\alpha = \alpha' - \sigma_j$.

2) If $\alpha = \epsilon\dot{\alpha} - 2k_i\sigma_i + 2k_j\sigma_j$, it implies that $\alpha = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_j+1)\sigma_j - \sigma_j$. If we set $\alpha' = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_j+1)\sigma_j$, then $\alpha' \in R^+$ and $\alpha = \alpha' - \sigma_j$.

3) Let $\alpha = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_j+1)\sigma_j$. If $2k_i \leq 2k_j$, we have $\alpha = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_j+2)\sigma_j - \sigma_j$. In this case we set $\alpha' = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_j+2)\sigma_j$, so $\alpha' \in R^+$ and we have $\alpha = \alpha' - \sigma_j$. If $2k_j + 1 < 2k_i$, hence $\alpha = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_i+2)\sigma_j - (2(k_i - k_j)+1)\sigma_j$. If we set $\alpha' = \epsilon\dot{\alpha} - 2k_i\sigma_i + (2k_i+2)\sigma_j$, then $\alpha' \in R^+$ and we have $\alpha = \alpha' - (2(k_i - k_j)+1)\sigma_j$. □

4. Extended affine Lie algebras of type A_1

In this section, we apply our decomposition of (non-isotropic) roots into positive and negative roots to a particular extended affine Lie algebra $\mathcal{L}$ of type A_1. We specify the root spaces associated to positive roots. As a by product we obtain a new description of the core, by showing that the core of $\mathcal{L}$ is in fact the subalgebra generated by positive root spaces.

We begin by recalling the construction of an extended affine Lie algebra of type A_1, starting from a semilattice, given in [1, III, §1].

For more details concerning extended affine Lie algebras, we refer the reader to [1], in particular we use definitions and concepts there without further reference.

Let $(\mathcal{L}, \mathcal{H}, (\cdot,\cdot))$ be a tame extended affine Lie algebra of type A_1 with root system R. One knows that R is an extended affine root system. Let $\mathcal{L}_c$ be the core of $\mathcal{L}$, the subalgebra generated by non-isotropic root spaces $\mathcal{L}_\alpha$, $\alpha \in R^\times$. Then $\mathcal{G} := \mathcal{L}_c/Z(\mathcal{L}_c)$, where $Z(\mathcal{L}_c)$ is the center of $\mathcal{L}_c$, is called the centerless core of $\mathcal{L}$. Let $\Lambda := \langle R^0 \rangle$.

By [10, Theorem 1.6], $\mathcal{G}$ is isomorphic to TKK($\mathcal{J}$), called the Tits-Kantor-Koecher Lie algebra associated to a unital Λ-graded Jordan algebra $\mathcal{J}$ over $\mathbb{C}$, where $\mathcal{J}$ satisfies

- $\{\alpha \in \Lambda \mid \mathcal{J}_\alpha \neq 0\}$ generates Λ,
- all nonzero homogeneous elements of $\mathcal{J}$ are invertible,
- $\dim_{\mathbb{C}} \mathcal{J}_\alpha \leq 1$ for all $\alpha \in \Lambda$.

Such a Jordan algebra is called a *Jordan torus*. One knows that $\mathcal{G}$ satisfies conditions (1.1)-(1.11) of [1, III, §1], see [2].

By [10, Theorem 7.1], a Jordan torus $\mathcal{J}$ is one of the following:

$$F_q^+, H(F_\varepsilon, *), \mathcal{J}_{S^m}(a_{\epsilon\{\epsilon \in I\}}), \text{or } \mathbb{A}_t.$$

The Jordan algebra $\mathcal{J}$ that we consider here is of the form $\mathcal{J} := \mathcal{J}_{S^m}(a_{\epsilon\{\epsilon\in I\}})$ which can be constructed over $\mathbb{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ of a certain symmetric bilinear form which depends on a semilattice S and a family of nonzero elements $a_\epsilon \in \mathbb{F}$, $\epsilon \in I$ for some index set I. It turns out that $\mathcal{J}$ is a Jordan n-torus called a *Clifford torus*. When $a_\epsilon = 1$ for all $\epsilon \in I$, we call $\mathcal{J} = \mathcal{J}_{S^m}$ the *standard Clifford torus*.

In what follows we recall the construction of an extended affine Lie algebra $\mathcal{L}$ whose root system is an extended affine root system of type A_1 considered in Section 3. We apply our positive roots (3.3) obtained from the positivity theorem [3, Theorem 5.2] to this particular algebra. To do so, we set $\Lambda = \mathbb{Z}^2 = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$ and consider the semilattice

$$S = S_0 \cup S_1 \cup S_2 \text{ with } S_0 = 2\Lambda,\ S_1 = \sigma_1 + 2\Lambda \text{ and } S_2 = \sigma_2 + 2\Lambda. \tag{4.1}$$

For $\sigma \in S$, let x^σ be a symbol. Then by [1, III, Proposition 2.1] the formal vector space

$$\mathcal{J} = \mathcal{J}(S) = \bigoplus_{\sigma \in S} \mathbb{C}x^\sigma \tag{4.2}$$

turns into a Jordan algebra with the multiplication given by

$$x^\sigma \cdot x^\tau = \begin{cases} x^{\sigma+\tau} & \text{if } \sigma, \tau \in S_0 \cup S_i, 0 \le i \le 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Let

$$\mathrm{TKK}(\mathcal{J}) = \mathcal{J} \oplus \mathrm{Instrl}(\mathcal{J}) \oplus \overline{\mathcal{J}}, \tag{4.4}$$

where $\mathrm{Instrl}(\mathcal{J}) := L_{\mathcal{J}} \oplus \mathrm{Inder}\mathcal{J} = L_{\mathcal{J}} \oplus \{\sum_i [L_{b_i}, L_{c_i}],\ b_i, c_i \in \mathcal{J}\}$, the operator L_x acts as $L_x y = x \cdot y$, for any $x, y \in \mathcal{J}$, and $\overline{\mathcal{J}}$ is a copy of $\mathcal{J}$. We see that $\mathrm{TKK}(\mathcal{J})$ is a Lie algebra under the product

$$[a_1 + \bar{b}_1 + E_1, a_2 + \bar{b}_2 + E_2] = -E_2 a_1 + E_1 a_2 - \overline{\overline{E_2} b_1} + \overline{\overline{E_1} b_2} + a_1 \triangle b_2 - a_2 \triangle b_1 + [E1, E2], \tag{4.5}$$

where $^-$ is the automorphism of $\mathrm{Instrl}(\mathcal{J})$ of order two defined by $\overline{L_a + D} = -L_a + D$, for $a_i \in \mathcal{J}, \bar{b}_i \in \overline{\mathcal{J}}$, $a_i \triangle b_j = L_{a_i \cdot b_j} + [L_{a_i}, L_{b_j}]$, and $E_i \in \mathrm{Instrl}(\mathcal{J})$. Now let $\mathcal{G} := \mathrm{TKK}(\mathcal{J})$, then with respect to the subalgebra $\dot{\mathcal{H}} = \mathbb{C}L_1$, $\mathcal{G}$ has a root space decomposition

$$\mathcal{G} = \mathcal{G}_{\dot\alpha} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{-\dot\alpha} \tag{4.6}$$

where $\mathcal{G}_{\dot\alpha} = \mathcal{J}, \mathcal{G}_0 = \mathrm{Instrl}(\mathcal{J})$ and $\mathcal{G}_{-\dot\alpha} = \overline{\mathcal{J}}$, for $\dot\alpha \in \dot{\mathcal{H}}^\star$ satisfying $\dot\alpha(L_1) = 1$.

Next, one enlarges the Lie algebra $(\mathcal{G}, [\cdot,\cdot])$ by setting

$$\mathcal{L} := \mathcal{G} \oplus \mathcal{C} \oplus \mathcal{D} \tag{4.7}$$

where $\mathcal{C} = \mathbb{C}c_1 \oplus \mathbb{C}c_2$ and $\mathcal{D} = \mathbb{C}d_1 \oplus \mathbb{C}d_2$ are two 2-dimensional vector spaces such that $d_1, d_2 \in Der(\mathcal{G})$, with anti-commutative product $[\cdot,\cdot]'$ as follows:

$$[\mathcal{L}, \mathcal{C}]' = 0, [d_i, x]' = d_i x, \text{ for all } x \in \mathcal{G}$$

and

$$[x, y]' = [x, y] + \sum_{i=1}^{2} (d_i x, y) c_i \text{ for all } x, y \in \mathcal{G}.$$

We consider the Jordan algebra $\mathcal{J} = \sum_{\sigma \in \Lambda} J^\sigma$, where

$$\mathcal{J}^\sigma = \begin{cases} \mathbb{C}x^\sigma & \text{if } \sigma \in S \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

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So

$$\mathrm{Instrl}(\mathcal{J})^{\sigma} = L_{\mathcal{J}^{\sigma}} \oplus \sum_{\tau,\xi\in\Lambda,\tau+\xi=\sigma} [L_{\mathcal{J}^{\tau}}, L_{\mathcal{J}^{\xi}}]. \tag{4.9}$$

This in turn gives a Λ-grading on $\mathcal{G}$ by setting

$$\mathcal{G}^{\sigma} := \mathcal{J}^{\sigma} \oplus \mathrm{Instrl}(\mathcal{J})^{\sigma} \oplus \overline{\mathcal{J}^{\sigma}}. \tag{4.10}$$

We note that $\mathcal{G}^0 \cap \mathcal{G}_0 = \dot{\mathcal{H}}$. Let $\mathcal{H} = \dot{\mathcal{H}} \oplus \mathcal{C} \oplus \mathcal{D}$. Hence $\mathcal{H}$ is an abelian subalgebra of $\mathcal{L}$ and we can identify $\mathcal{H}^* = \dot{\mathcal{H}}^* \oplus \mathcal{C}^* \oplus \mathcal{D}^*$.

Then $\mathcal{L}$ can be written as

$$\mathcal{L} = \sum_{\alpha\in\mathcal{H}^*} \mathcal{L}_{\alpha} = \sum_{\sigma\in\Lambda}\sum_{\dot{\alpha}\in\dot{R}} \mathcal{L}_{\dot{\alpha}+\sigma} \tag{4.11}$$

where $\mathcal{L}_0 = \mathcal{H}$ and $\mathcal{L}_{\dot{\alpha}+\sigma} = \mathcal{G}_{\dot{\alpha}} \cap \mathcal{G}^{\sigma}$ for $\sigma \in \Lambda, \dot{\alpha} \in \dot{R}$ with $\dot{\alpha}+\sigma \neq 0$. Also

$$\mathcal{L}_{\sigma} = \mathrm{Instrl}(\mathcal{J})^{\sigma} \text{ for } \sigma \in \Lambda \setminus \{0\}, \quad \mathcal{L}_{\dot{\alpha}+\sigma} = \mathcal{J}^{\sigma} \text{ and } \mathcal{L}_{-\dot{\alpha}+\sigma} = \overline{\mathcal{J}^{\sigma}} \text{ for } \sigma \in \Lambda. \tag{4.12}$$

Now, we are ready to describe the root spaces corresponding to positive roots given in Theorem **??**. Let $\alpha = \dot{\alpha} + \sigma$ and $\beta = -\dot{\alpha} + \delta$ be positive roots with respect to Π, given in Theorem 3.4 with $\sigma, \delta \in R^0$. By (4.5), (4.8) and (4.12), we have

$$[\mathcal{L}_{\alpha}, \mathcal{L}_{\beta}] = L_{x^{\sigma}\cdot x^{\delta}} + [L_{x^{\sigma}}, L_{x^{\delta}}]. \tag{4.13}$$

On the other hand by (4.8), (4.9) and (4.12) for any $\sigma \in R^0$, we have

$$\mathcal{L}_{\sigma} = L_{x^{\sigma}} \oplus \sum_{\tau,\xi\in S,\tau+\xi=\sigma} [L_{x^{\tau}}, L_{x^{\xi}}]. \tag{4.14}$$

Lemma 4.1. *Let ε be an arbitrary element of $R^0 \setminus \{-2k\sigma_1 - 2k'\sigma_2 \mid k, k' \geq 0\}$. Then $\mathcal{L}_{\varepsilon}$ is generated by $[\mathcal{L}_{\alpha}, \mathcal{L}_{\beta}]$ for some $\alpha, \beta \in R^+$. Similarly for any $\varepsilon \in R^0 \setminus \{2k\sigma_1 + 2k'\sigma_2 \mid k, k' \geq 0\}$, there are some negative roots α, β such that $\mathcal{L}_{\varepsilon}$ is generated by $[\mathcal{L}_{\alpha}, \mathcal{L}_{\beta}]$.*

Proof. Let $\varepsilon \in R^0 \setminus \{-2k\sigma_1 - 2k'\sigma_2 \mid k, k' \geq 0\}$. Since $R^0 = S + S = \{m\sigma_1 + n\sigma_2 \mid m, n \in \mathbb{Z}\}$, ε can be written as $\varepsilon = \tau + \xi$ for some $\tau, \xi \in S$. We consider the following cases:

1) Let $\tau, \xi \in S_0 \cup S_i$ for $0 \leq i \leq 2$. We show that $[L_{x^{\tau}}, L_{x^{\xi}}] = 0$. Consider an arbitrary homogeneous element $x^{\rho} \in \mathcal{J}$, where $\rho \in S_j$, $j = 0, 1, 2$. It follows that if $\tau + (\xi + \rho), \xi + (\tau + \rho) \in S$, then $\tau + (\xi + \rho), \xi + (\tau + \rho) \in S_i$, $1 \leq i \leq 2$ or $\tau + (\xi + \rho), \xi + (\tau + \rho) \in S_0$. So by (4.8), we obtain that $[L_{x^{\tau}}, L_{x^{\xi}}](x^{\rho}) = x^{\tau}(x^{\xi} \cdot x^{\rho}) - x^{\xi}(x^{\tau} \cdot x^{\rho}) = x^{\tau+\xi+\rho} - x^{\xi+\tau+\rho} = 0$. Thus by (4.8) and (4.14), $\mathcal{L}_{\varepsilon} = L_{x^{\varepsilon}}$. Now by Corollary 3.5, there are some positive roots α, β such that $\alpha + \beta = \varepsilon$. Without loss of generality, we assume that $\alpha = \dot{\alpha} + \sigma, \beta = -\dot{\alpha} + \delta$ for some $\sigma, \delta \in S$ where $\sigma + \delta = \varepsilon$. By (4.1), we can assume that $\sigma, \delta \in S_j$ for $j = 0, 1, 2$. Therefore by (4.3) and (4.13), we have $[\mathcal{L}_{\alpha}, \mathcal{L}_{\beta}] = L_{x^{\sigma+\delta}} = L_{x^{\varepsilon}} = \mathcal{L}_{\varepsilon}$.

2) Let $\tau \in S_1, \xi \in S_2$. Since $\tau + \xi \notin S$, by (4.3) and (4.8), we get $L_{x^{\tau}\cdot x^{\xi}} = L_{x^{\varepsilon}} = 0$. Now we consider $[L_{x^{\tau}}, L_{x^{\xi}}]$ and let $x^{\rho} \in \mathcal{J}$ with $\rho \in S_0$. Since $\tau + (\xi + \rho), \xi + (\tau + \rho) \notin S$, we see that $[L_{x^{\tau}}, L_{x^{\xi}}](x^{\rho}) = 0$. Let $\rho \in S_1$, so $\xi + \rho \notin S$ and $\xi + (\tau + \rho) \in S_2$. Hence $[L_{x^{\tau}}, L_{x^{\xi}}](x^{\rho}) = -x^{\xi+\tau+\rho}$. Similarly if $\tau \in S_2$, we get that $[L_{x^{\tau}}, L_{x^{\xi}}](x^{\tau}) = x^{\tau+\xi+\tau}$. Therefore $\mathcal{L}_{\varepsilon} = \sum_{\zeta+\iota=\varepsilon}[L_{x^{\zeta}}, L_{x^{\iota}}]$. On the other hand, by Corollary 3.5, there are some positive roots α, β such that $\alpha = \dot{\alpha} + \sigma, \beta = -\dot{\alpha} + \delta$ where $\sigma \in S_1, \delta \in S_2$ or $\sigma \in S_2, \delta \in S_1$, and $\sigma + \delta = \varepsilon$.

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Then by (4.8), $[\mathcal{L}_\alpha, \mathcal{L}_\beta] = [L_{x^\sigma}, L_{x^\delta}]$. Since $\mathcal{L}_\varepsilon = \sum_{\zeta+\iota=\varepsilon}[L_{x^\zeta}, L_{x^\iota}]$, it follows that the only choices for ζ, ι are $\zeta \in S_1, \iota \in S_2$ or vice versa. Therefore $[\mathcal{L}_\alpha, \mathcal{L}_\beta]$ generates $\mathcal{L}_\varepsilon$.

The proof of the second assertion in the statement is analogous to the first assertion. □

We now define the following subalgebras of $\mathcal{L}$:

$$\mathcal{L}^+ := \langle\{\mathcal{L}_\beta \mid \beta \in R^+\}\rangle, \quad \mathcal{L}^- := \langle\{\mathcal{L}_\beta \mid \beta \in R^-\}\rangle. \tag{4.15}$$

One knows that the core $\mathcal{L}_c$ of an extended affine Lie algebra $\mathcal{L}$, the subalgebra generated by root spaces associated to non-isotropic roots, plays the most fundamental role in the structure theory of $\mathcal{L}$. The following theorem gives a new description of $\mathcal{L}_c$ in terms of positive roots.

Theorem 4.2. *$\mathcal{L}_c = \langle \mathcal{L}_\alpha \mid \alpha \in R^+\rangle$. In particular, $\mathcal{L} = \mathcal{H} + \langle \mathcal{L}_\alpha \mid \alpha \in R^+\rangle$.*

Proof. It is enough to show that $\mathcal{L}_c \subseteq \langle \mathcal{L}_\alpha \mid \alpha \in R^+\rangle$. Let $\beta = \epsilon\dot\alpha + \tau$ be an arbitrary element of R^-, where $\epsilon \in \{-1, +1\}$ and $\tau \in R^0$. By Theorem 3.4, there exists a negative root $\gamma = -\epsilon\dot\alpha + \eta$ where $\eta \in R^0$ and $\eta \neq -\tau$. Then by Corollary 3.5, $\sigma := \beta + \gamma \in R^0 \setminus \{2k\sigma_1 + 2k'\sigma_2 \mid k, k' \geq 0\}$. Hence $\beta = -\gamma + \sigma$ where $-\gamma \in R^+$ and $\sigma \in R^0 \setminus \{2k\sigma_1 + 2k'\sigma_2 \mid k, k' \geq 0\}$. On the other hand by Lemma 3.6, $-\gamma = \alpha' + k\sigma_i$ where $\alpha' \in R^+$ and $k \in 2\mathbb{Z} + 1$. So $\beta = \alpha' + \delta$, where $\delta \in R^0 \setminus \{-2k\sigma_1 - 2k'\sigma_2 \mid k, k' \geq 0\}$ and by Lemma 4.1, $\mathcal{L}_\delta \in \mathcal{L}^+$. Therefore $\mathcal{L}_\beta \in \mathcal{L}^+$ and $\mathcal{L}^- \subseteq \mathcal{L}^+$. Then $\mathcal{L}_c \subseteq \langle \mathcal{L}_\alpha \mid \alpha \in R^+\rangle$ and the result holds.

The second assertion in the statement is immediate as by [1, III, §1], $\mathcal{L} = \mathcal{H} + \mathcal{L}_c$. □

Remark 4.3. *We note that the concepts of positive and negative roots depend on the choice of the considered reflectable base. In this work we fixed the reflectable base $\Pi = \{\dot\alpha, \sigma_1 - \dot\alpha, \sigma_2 - \dot\alpha\}$. Now a natural question which arises here is that if Theorem 4.2 holds for all reflectable bases of R. The answer is not known in general, but using similar calculations given in this work, we have checked that the answer is positive for the reflectable base $\Pi' = \{\dot\alpha, \dot\alpha + \sigma_1, \dot\alpha + \sigma_2\}$ of R.*

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