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ON A GENERALIZATION OF LERAY SIMPLICIAL COMPLEXES

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Abstract. We define a refinement of the notion of Leray simplicial complexes and study its properties. Moreover, we translate some of our results to the language of commutative algebra.

1. Introduction

Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. Suppose that M is a graded S-module with minimal free resolution

$$0 \longrightarrow \cdots \longrightarrow \bigoplus_j S(-j)^{\beta_{1j}(M)} \longrightarrow \bigoplus_j S(-j)^{\beta_{0j}(M)} \longrightarrow M \longrightarrow 0.$$

The integers $\beta_{ij}(M)$ are called the *graded Betti numbers* of M. The *Castelnuovo–Mumford regularity* (or simply, *regularity*) of M, denoted by $\operatorname{reg}(M)$, is defined as

$$\operatorname{reg}(M) = \max\{j - i |\ \beta_{ij}(M) \neq 0\}.$$

Kalai and Meshulam [4] proved that for Stanley-Reisner ideals $I, J \subset S$, we have

$$\operatorname{reg}(S/I + J) \leq \operatorname{reg}(S/I) + \operatorname{reg}(S/J),$$

and

$$\operatorname{reg}(S/I \cap J) \leq \operatorname{reg}(S/I) + \operatorname{reg}(S/J) + 1.$$

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Their proof is based on the study of the so-called Leray simplicial complexes. As a generalization of this concept, in Section 3, we define the notion of (k,t)-Leray simplicial complexes and study its properties. More precisely, it is well-known that a simplicial complex Δ is a $(d-1)$-dimensional Cohen-Macaulay complex with n vertices if and only if its Alexander dual $\Delta^\vee$ has complete $(n-d-2)$-skeleton and is $(n-d-1)$-Leray. As a generalization of this fact, in Theorem 3.3 we study the Alexander dual of (k,t)-Leray simplicial complexes and show that it is related to the concept of CM_t simplicial complexes, defined by Haghighi, Zaare-Nahandi and the author [2].

For every integer $t \geq 0$, we define $L^t_{\mathbb{K}}(\Delta)$ to be the minimum integer $k \geq 0$ such that Δ is (k,t)-Leray over $\mathbb{K}$. In Theorems 3.6 and 3.7, we study the behavior of the function $L^t_{\mathbb{K}}(-)$ with respect to the join, intersection and union of simplicial complexes. Our final goal is to translate Theorem 3.7 to the language of commutative algebra. In order to do this, for every graded S-module M we introduce the notion of $\mathrm{reg}^t(M)$ as a refinement of the Castelnuovo–Mumford regularity. As a consequence of Theorem 3.7, we deduce in Theorem 3.9 that for Stanley-Reisner ideals $I, J \subset S$,

$$\mathrm{reg}^t(S/I+J) \leq \mathrm{reg}^t(S/I) + \mathrm{reg}^t(S/J),$$

and

$$\mathrm{reg}^t(S/I\cap J) \leq \mathrm{reg}^t(S/I) + \mathrm{reg}^t(S/J) + 1.$$

2. **Preliminaries**

In this section, we provide the definitions and basic facts which will be used in the next section. A *simplicial complex* Δ on the set of vertices $[n] := \{1,\dots,n\}$ is a collection of subsets of $[n]$ which contains all the singletons and is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ, the *size* of a face F is defined to be $|F|$ and its *dimension* is defined to be $|F|-1$. (As usual, for a given finite set X, the number of elements of X is denoted by $|X|$.) The *dimension* of Δ which is denoted by $\dim\Delta$, is defined to be $d-1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. We say that Δ is *pure* if all facets of Δ have the same cardinality. The d-skeleton of Δ is the subcomplex of Δ consisting of all of the faces of Δ that have dimension at most d. The *link of* Δ *with respect to a face* $F \in \Delta$, denoted by $\mathrm{lk}_\Delta F$, is the simplicial complex $\mathrm{lk}_\Delta F = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}$. For any subset $W \subseteq [n]$, the *induced subcomplex* of Δ on W denoted by $\Delta[W]$ is the simplicial complex with vertex set W and its faces are those faces of Δ which are contained in W. The *Alexander dual* of Δ is defined as $\Delta^\vee = \{[n] \setminus F : F \notin \Delta\}$. For every integer i, the ith *reduced homology of* Δ *with coefficients in* $\mathbb{K}$ is denoted by $\widetilde{H}_i(\Delta;\mathbb{K})$.

Let Δ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i\in F} x_i$. The *Stanley-Reisner ideal of* Δ *over* $\mathbb{K}$ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \notin \Delta$. The *Stanley-Reisner ring of* Δ *over* $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined to be $\mathbb{K}[\Delta] = S/I_\Delta$. The simplicial complex Δ is called *Cohen–Macaulay over* $\mathbb{K}$ (resp. *Buchsbaum over* $\mathbb{K}$), if its Stanley-Reisner ring $\mathbb{K}[\Delta]$ is a Cohen–Macaulay ring (resp. a Buchsbaum ring). Using the celebrated Reisner's theorem, Δ is Cohen–Macaulay over $\mathbb{K}$ if and only if $\widetilde{H}_i(\mathrm{lk}_\Delta F;\mathbb{K}) = 0$, for every face $F \in \Delta$ and every

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integer $i < \dim \mathrm{lk}_\Delta F$. Also, it is well-know that Δ is Buchsbaum over $\mathbb{K}$ if and only if it is pure and $\widetilde{H}_i(\mathrm{lk}_\Delta F;\mathbb{K}) = 0$, for every nonempty face $F \in \Delta$ and every integer $i < \dim \mathrm{lk}_\Delta F$. Based on these facts, Haghighi, Yassemi and Zaare-Nahandi [2] have defined the notion of CM_t simplicial complexes as a generalization of Cohen–Macaulay and Buchsbaum simplicial complexes.

Definition 2.1. *Let t be a nonnegative integer, $\mathbb{K}$ be a field and let Δ be a pure simplicial complex. We say that Δ is a CM_t simplicial complex over $\mathbb{K}$, if $\widetilde{H}_i(\mathrm{lk}_\Delta F;\mathbb{K}) = 0$, for every face $F \in \Delta$ with $|F| \geq t$ and every integer $i < \dim \mathrm{lk}_\Delta F$.*

3. Main results

Let $k \geq 0$ be a nonnegative integer. A simplicial complex Δ is called a *k-Leray* simplicial complex if $\widetilde{H}_i(\Delta[W];\mathbb{K}) = 0$, for every $i \geq k$, and every subset $W \subseteq V(\Delta)$. As a generalization of this concept, we define the notion of (k,t)-Leray simplicial complexes and study its properties.

Definition 3.1. *A simplicial complex Δ is called (k,t)-Leray over $\mathbb{K}$ if $\widetilde{H}_i(\Delta[W];\mathbb{K}) = 0$, for every $i \geq k$ and every subset $W \subseteq V(\Delta)$, with $|W| \leq t$.*

The following proposition is an immediate consequence of the above definition.

Proposition 3.2. *Let Δ be a simplicial complex over $[n]$. Then*

(i) *Δ is (k,n)-Leray if and only if it is k-Leray complex.*
(ii) *If Δ is (k,t)-Leray, for some integer t with $1 \leq t \leq n$, then Δ is $(k,t-1)$-Leray.*
(iii) *Δ is (k,t)-Leray if and only if any induced subcomplex of Δ with t vertices is a k-Leray simplicial complex.*

As the first main result of this paper, we study the Alexander dual of (k,t)-Leray simplicial complexes. As the following theorem shows, it is related to CM_t simplicial complexes.

Theorem 3.3. *Let Δ be a simplicial complex over $[n]$. Assume that d and t are nonnegative integers such that Δ has no facet of dimension at most $t-2$. Then the following conditions are equivalent.*

(1) *Δ is a $(d-1)$-dimensional CM_t simplicial complex.*
(2) *$\Delta^\vee$ is $(n-d-1, n-t)$-Leray and the $(n-d-2)$-skeleton of $\Delta^\vee$ is a simplex.*

Proof. (1) $\Rightarrow$ (2) Let Δ be a $(d-1)$-dimensional CM_t simplicial complex. Then Δ has no face of cardinality $d+1$. Therefore the $(n-d-2)$-skeleton of $\Delta^\vee$ is a simplex. We now prove that $\Delta^\vee$ is a $(n-d-1, n-t)$-Leray simplicial complex. We use Hochster's formula [3, Theorem 8.1.1], which says that for every pair of integers $i, j \geq 0$, we have

$$\beta_{ij}(I_\Delta) = \sum_{W \subseteq V(\Delta), |W| = j} \dim_{\mathbb{K}} \widetilde{H}_{j-i-2}(\Delta[W];\mathbb{K}).$$

We know from [1, Theorem 3.1] that $\beta_{0j}(I_{\Delta^\vee}) = 0$, for every $j \neq n-d$ and $\beta_{ij}(I_{\Delta^\vee}) = 0$, for every i, j with $n-d+i+1 \leq j \leq n-t$. Hence, $\widetilde{H}_k(\Delta[W];\mathbb{K}) = 0$, for every $k > n-d-2$ and every $W \subseteq V(\Delta)$, with $|W| \leq n-t$. This shows that $\Delta^\vee$ is $(n-d-1, n-t)$-Leray.

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(2) $\Rightarrow$ (1) Assume that $\Delta^\vee$ is a $(n-d-1, n-t)$-Leray simplicial complex and that the $(n-d-2)$-skeleton of $\Delta^\vee$ is a simplex. Since the $(n-d-2)$-skeleton of $\Delta^\vee$ is a simplex, we conclude that $\widetilde{H}_k(\Delta^\vee[W];\mathbb{K}) = 0$, for every $k \leq n-d-3$ and every $W \subseteq V(\Delta)$. Then Hochster's formula implies that $\beta_{ij}(I_{\Delta^\vee}) = 0$, whenever $j \leq n-d+1-1$. Since $\Delta^\vee$ is $(n-d-1, n-t)$-Leray, we conclude that $\widetilde{H}_k(\Delta^\vee[W];\mathbb{K}) = 0$, for every $k \geq n-d-1$ and every $W \subseteq V(\Delta)$, with $|W| \leq n-t$. Therefore, again Hochster's formula implies that $\beta_{ij}(I_{\Delta^\vee}) = 0$, whenever $n-d+i+1 \leq j \leq n-t$. Moreover, as Δ has no facet of dimension at most $t-2$, it follows has the minimal generators of $I_{\Delta^\vee}$ have degree at most $n-t$. In other words, $\beta_{0j}(I_{\Delta^\vee}) = 0$, for every $j \neq n-t+1$. Thus, we deduce from [1, Theorem 3.1] that Δ is a$(d-1)$-dimensional CM_t simplicial complex. $\square$

Restricting to the case of $t = 1$, we obtain the following corollary.

Corollary 3.4. *Let Δ be a simplicial complex over $[n]$. Then the following conditions are equivalent.*

(1) *Δ is a $(d-1)$-dimensional Buchsbaum simplicial complex.*
(2) *$\Delta^\vee$ is $(n-d-1, n-1)$-Leray and the $(n-d-2)$-skeleton of $\Delta^\vee$ is a simplex.*

Proof. Note that the above conditions hold if $\Delta = \{\emptyset\}$. Thus, assume that $\Delta \neq \{\emptyset\}$. In this case, the assertion follows immediately from Theorem 3.3. $\square$

Let Δ be a simplicial complex. The quantity $L_{\mathbb{K}}(\Delta)$ is the minimum integer k such that Δ is k-Leray over $\mathbb{K}$ (see e.g., [4]). As a refinement, for every integer $t \geq 0$, we define $L^t_{\mathbb{K}}(\Delta)$ to be the minimum integer $k \geq 0$ such that Δ is (k,t)-Leray over $\mathbb{K}$. It follows from Proposition 3.2 that

$$L^0_{\mathbb{K}}(\Delta) \leq L^1_{\mathbb{K}}(\Delta) \leq \cdots \leq L^n_{\mathbb{K}}(\Delta) = L_{\mathbb{K}}(\Delta),$$

and

$$L^t_{\mathbb{K}}(\Delta) = \max\{L_{\mathbb{K}}(\Gamma) \mid \Gamma \text{ is an induced subcomplex of } \Delta, \text{ with } |V(\Gamma)| = t\}.$$

Let Δ and Δ' be two simplicial complexes whose vertex sets are disjoint. The simplicial join $\Delta * \Delta'$ is defined to be the simplicial complex whose faces are of the form $\sigma \cup \sigma'$ where $\sigma \in \Delta$ and $\sigma' \in \Delta'$.

Lemma 3.5. *Let Δ_1 and Δ_2 be simplicial complexes with disjoint set of vertices. Then*

$$L_{\mathbb{K}}(\Delta_1 * \Delta_2) = L_{\mathbb{K}}(\Delta_1) + L_{\mathbb{K}}(\Delta_2).$$

Proof. Assume that $L_{\mathbb{K}}(\Delta_i) = k_i$, for $i = 1, 2$. Let $W = W_1 \cup W_2$ be a subset of vertices of $\Delta_1 * \Delta_2$, with $W_i \subseteq V(\Delta_i)$ for $i = 1, 2$. Then for every integer k, we have

$$(\dagger) \qquad \widetilde{H}_k((\Delta_1 * \Delta_2)[W];\mathbb{K}) = \textstyle\sum_{p+q=k-1} \widetilde{H}_p(\Delta_1[W_1]) \otimes \widetilde{H}_q(\Delta_2[W_2]).$$

If $k \geq k_1 + k_2$, then for every pair of integers p, q with $p + q = k - 1$, we have either $p \geq k_1$ or $q \geq k_2$. Therefore, using equality $(\dagger)$, we conclude that $\widetilde{H}_k((\Delta_1 * \Delta_2)[W];\mathbb{K}) = 0$. In particular,

$$L_{\mathbb{K}}(\Delta_1 * \Delta_2) \leq k_1 + k_2.$$

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On the other hand, for $i=1,2$, there exists a subset $U_i \subseteq V(\Delta_i)$ with $\widetilde{H}_{k-1}(\Delta_i[U_i];\mathbb{K}) \neq 0$. Hence, we deduce from equality (†) that

$$\widetilde{H}_{k_1+k_2-1}((\Delta_1 * \Delta_2)[U];\mathbb{K}) \neq 0,$$

where $U = U_1 \cup U_2$. Thus,

$$L_{\mathbb{K}}(\Delta_1 * \Delta_2) = k_1 + k_2.$$

□

Using Lemma 3.5, we are able to prove the following theorem.

Theorem 3.6. *Let Δ_1 and Δ_2 be simplicial complexes with disjoint set of vertices. Then for every integer $t \geq 0$,*

$$L^t_{\mathbb{K}}(\Delta_1 * \Delta_2) = \max\{L^{t_1}_{\mathbb{K}}(\Delta_1) + L^{t_2}_{\mathbb{K}}(\Delta_2) : t_1 + t_2 = t\}.$$

Proof. We have

$$\begin{aligned} L^t_{\mathbb{K}}(\Delta_1 * \Delta_2) &= \max\{L_{\mathbb{K}}((\Delta_1 * \Delta_2)[W]) : |W| = t\} \\ &= \max\{L_{\mathbb{K}}(\Delta_1[W_1] * \Delta_2[W_2]) : |W_1| + |W_2| = t\} \\ &= \max\{L_{\mathbb{K}}(\Delta_1[W_1]) + L_{\mathbb{K}}(\Delta_2[W_2]) : |W_1| + |W_2| = t\} \\ &= \max\{L^{t_1}_{\mathbb{K}}(\Delta_1) + L^{t_2}_{\mathbb{K}}(\Delta_2) : t_1 + t_2 = t\}, \end{aligned}$$

where the third equality follows from Lemma 3.5. □

In the following proposition, we study the behavior of $L^t_{\mathbb{K}}(-)$ with intersection and union of simplicial complexes.

Theorem 3.7. *Let $t \geq 0$ be an integer and assume that $\Delta_1, \ldots, \Delta_m$ are simplicial complexes with the same vertex set. Then*

(i) $L^t_{\mathbb{K}}(\bigcap_{i=1}^m \Delta_i) \leq \sum_{i=1}^m L^t_{\mathbb{K}}(\Delta_i)$.
(ii) $L^t_{\mathbb{K}}(\bigcup_{i=1}^m \Delta_i) \leq \sum_{i=1}^m L^t_{\mathbb{K}}(\Delta_i) + m - 1$.

Proof. We only prove (i), as the proof of (ii) is similar. Set $\Delta = \bigcap_{i=1}^m \Delta_i$. Then

$$\begin{aligned} L^t_{\mathbb{K}}(\Delta) &= \max\{L_{\mathbb{K}}(\Delta[W]) : |W| = t\} \leq \max\{\sum_{i=1}^m L^t_{\mathbb{K}}(\Delta_i[W]) : |W| = t\} \\ &\leq \sum_{i=1}^m \max\{L^t_{\mathbb{K}}(\Delta_i[W]) : |W| = t\} = \sum_{i=1}^m L^t_{\mathbb{K}}(\Delta_i), \end{aligned}$$

where the first inequality follows from [4, Theorem 1.2]. □

Let M be a finitely generated graded S-module. For every integer $t \geq 0$, we define

$$\operatorname{reg}^t(M) := \max\{j - i : \beta_{ij}(M) \neq 0, \ for\ some\ j \leq t, i \geq 0\}.$$

Clearly, for every squarefree monomial ideal I, we have $\operatorname{reg}^n(S/I) = \operatorname{reg}(S/I)$.

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Proposition 3.8. *For every simplicial complex* Δ*,*

$$\operatorname{reg}^t(S/I_\Delta) = L^t_{\mathbb{K}}(\Delta).$$

Proof. Let $[n]$ be the vertex set of Δ. It follows from Hochster's formula [3, Theorem 8.1.1] that

$$\begin{aligned}
\operatorname{reg}^t(S/I_\Delta) &= \max\{j - i : \widetilde{H}_{j-i-1}(\Delta[W];\mathbb{K}) \neq 0,\ \mathit{for\ some}\ j \leq t, i \geq 0, W \subseteq [n], |W| = j\} \\
&= \max\{k : \widetilde{H}_{k-1}(\Delta[W];\mathbb{K}) \neq 0,\ \mathit{for\ some}\ W \subseteq [n], |W| \leq t\} \\
&= \min\{k : \widetilde{H}_{k'}(\Delta[W];\mathbb{K}) = 0,\ \mathit{for\ all}\ k' \geq k \text{ and all } W \subseteq [n], |W| \leq t\} \\
&= L^t_{\mathbb{K}}(\Delta) + 1.
\end{aligned}$$

□

As an immediate consequence of Theorem 3.7 and Proposition 3.8, we obtain the following theorem which is a generalization of [4, Theorem 1.4].

Theorem 3.9. *Let* Δ_1, Δ_2 *be simplicial complexes with the same vertex set. Then for every integer* $t \geq 0$*, we have*

(i) $\operatorname{reg}^t(S/(I_{\Delta_1} + I_{\Delta_2})) = \operatorname{reg}^t(S/I_{\Delta_1\cap\Delta_2}) \leq \operatorname{reg}^t(S/I_{\Delta_1}) + \operatorname{reg}^t(S/I_{\Delta_1})$.

(ii) $\operatorname{reg}^t(S/(I_{\Delta_1} \cap I_{\Delta_2})) = \operatorname{reg}^t(S/I_{\Delta_1\cup\Delta_2}) \leq \operatorname{reg}^t(S/I_{\Delta_1}) + \operatorname{reg}^t(S/I_{\Delta_1}) + 1$.

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