NILPOTENT GRAPHS OF SKEW POLYNOMIAL RINGS OVER NON-COMMUTATIVE RINGS

MOHAMAD JAVAD NIKMEHR* AND ABDOLREZA AZADI

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Abstract. Let $R$ be a ring and $\alpha$ be a ring endomorphism of $R$. The undirected nilpotent graph of $R$, denoted by $\Gamma_N(R)$, is a graph with vertex set $Z_N(R)^*$, and two distinct vertices $x$ and $y$ are connected by an edge if and only if $xy$ is nilpotent, where $Z_N(R) = \{x \in R \mid xy$ is nilpotent, for some $y \in R^*\}$. In this article, we investigate the interplay between the ring theoretical properties of a skew polynomial ring $R[x; \alpha]$ and the graph-theoretical properties of its nilpotent graph $\Gamma_N(R[x; \alpha])$. It is shown that if $R$ is a symmetric and $\alpha$-compatible with exactly two minimal primes, then $\text{diam}(\Gamma_N(R[x, \alpha])) = 2$. Also we prove that $\Gamma_N(R)$ is a complete graph if and only if $R$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

1. Introduction

Throughout this paper $R$ denotes an associative ring with unity and $\alpha$ is a ring endomorphism of $R$. We denote by $R[x; \alpha]$ the skew polynomial ring whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xr = \alpha(r)x$ for any $r \in R$. We write $Z(R)$ and $Z(R)^* = Z(R) \setminus \{0\}$ for the set of zero-divisors of $R$ including zero and the set of nonzero zero-divisors of $R$, respectively. An element $x$ is called nilpotent if $x^m = 0$ for some positive integer $m$ and the set of all nilpotent elements in $R$ is denoted by $\text{Nil}(R)$. A ring $R$ is called an NI


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*Corresponding author.

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ring if Nil(R) forms an ideal. For X ⊆ R, the ideal generated by X is denoted by <X>. Note that R* = R \ {0}.

Recall that a graph is said to be connected if for each pair of distinct vertices x and y, there is a finite sequence of distinct vertices x = x1, x2, . . . , xn = y such that each pair xi, xj is an edge. Such a sequence is said to be a path and for two distinct vertices a and b in the simple graph Γ, the distance between a and b denoted by d(a, b), is the length of a shortest path connecting a and b, if such a path exists, otherwise we put d(a, b) = ∞. Recall that the diameter of a graph Γ is defined as follows:

\[ \text{diam}(\Gamma) = \text{sup}\{|d(a, b)| \mid a \text{ and } b \text{ are distinct vertices of } \Gamma\} \]

The girth of a simple connected graph Γ, denoted by gr(Γ), is the length of the shortest cycle in Γ, provided Γ contains a cycle, otherwise gr(Γ) = ∞.

In 1988, Beck began to investigate the possibility of coloring a commutative ring R by associating to the ring a zero-divisor graph, defined as a simple graph, the vertices of which are the elements of the ring R, with two distinct elements x and y being adjacent if and only if xy = 0 [4]. However, in 1999, Anderson and Livingston [2] modified and studied the zero-divisor graph Γ(R), whose vertices are the nonzero zero-divisors Z(R)* of commutative ring R.

If R is a noncommutative ring, Redmond [16], defined an undirected zero-divisor graph of a non-commutative ring, the graph Γ(R), with vertices in the set Z(R)* = Z(R) \ {0} such that for distinct vertices a and b there is an edge connecting them if and only if ab = 0 or ba = 0. Several papers are devoted to studying the relationship between the zero-divisor graphs and algebraic properties of rings [2, 1, 3, 4].

In [8], Chen defined a kind of graph structure of rings. He let all the elements of ring R be the vertices of the graph and two vertices x and y are adjacent if and only if xy is nilpotent. However, in 2010, Li and Li [13] modified and studied the nilpotent graph ΓN(R) of R as a graph with vertex set ZN(R)*, and two distinct vertices x and y were adjacent if and only if xy is nilpotent, where ZN(R) = \{x ∈ R \mid xy \text{ is nilpotent, for some } y ∈ R^*\}. Note that the usual zero-divisor graph Γ(R) is a subgraph of the graph ΓN(R). There is a considerable interest in studying if and how certain graph-theoretic properties of rings are preserved under various ring-theoretic extensions. The first such extensions that come to mind are those of matrix, polynomial and power series extensions. Nikmehr [15] studied the diameter and the girth of nilpotent graph of matrix rings over noncommutative rings. Also, Axtell et al. [3] examined the preservation of diameter and girth of zero-divisor graphs of commutative rings under extensions to polynomial and power series rings.

According to Colm [8], a ring R is called reversible if ab = 0 implies that ba = 0 for each a, b ∈ R. Clearly, reduced rings (i.e. rings with no nonzero nilpotent elements) and commutative rings are reversible. Moreover, a ring R is called symmetric if abc = 0 implies that acb = 0 for each a, b, c ∈ R. Simple computations show that reversible rings are symmetric. Following [10], a ring R is called α-compatible if for each a, b ∈ R, ab = 0 ⇔ aα(b) = 0. Recall that an ideal P of R is completely prime if ab ∈ P implies a ∈ P or b ∈ P for a, b ∈ R. In this paper, our first main result states that,
\(N(R)\) is a complete graph if and only if \(R \cong \mathbb{Z}_2 \times \mathbb{Z}_2\). We also prove that if \(R\) is a symmetric and \(\alpha\)-compatible with exactly two minimal primes, then \(\text{diam}(\Gamma_N(R[x, \alpha])) = 2\). Moreover, we examine the preservation and lack thereof of the girth of the nilpotent graph of a noncommutative ring under extension to skew polynomial ring.

2. On Nilpotent Properties of Skew Polynomial Rings

In this section, we show that if \(R\) is symmetric, then \(N(R)\) is always connected and we determine which complete graphs may be realized as \(N(R)\). We also examine the preservation and lack thereof of the diameter and girth of the nilpotent graph of a noncommutative ring under extension to skew polynomial ring. The following lemma, which has been proved in [10, Lemma 2.1], will be helpful in our main results.

**Lemma 2.1.** Let \(R\) be an \(\alpha\)-compatible ring. Then we have the following:

1. If \(ab = 0\), then \(a\alpha^n(b) = \alpha^n(a)b = 0\) for any positive integer \(n\).
2. If \(\alpha^k(a)b = 0\) for some positive integer \(k\), then \(ab = 0\).

Note that the set of nilpotent elements of a ring is not ideal in general, (see [17]). According to [5], a ring \(R\) is called semi-commutative if \(ab = 0\) implies \(aRb = 0\). If \(R\) is a semi-commutative ring, then \(\text{Nil}(R)\) is an ideal of \(R\), by [18, Lemma 3.1]. It is well known that every symmetric ring is semicommutative. Then we have the following result:

**Lemma 2.2.** Let \(R\) be a symmetric ring. Then \(\text{Nil}(R)\) is an ideal of \(R\).

Now we bring the following lemma, which has a key role in our main results.

**Lemma 2.3.** Let \(R\) be a symmetric \(\alpha\)-compatible ring. Then \(\text{Nil}(R[x, \alpha])\) is an ideal.

**Proof.** The proof is a consequence of the following claims:

**Claim 1.** If \(ab \in \text{Nil}(R)\), then \(a\alpha^j(b) \in \text{Nil}(R)\), for all positive integers \(j\).

**Proof of Claim 1.** If \(ab \in \text{Nil}(R)\), then \((ab)^k = 0\), for some \(k \in \mathbb{N}\). So \(a\alpha^j(b)\alpha^j(ab\cdots ab) = 0\), and thus \(a\alpha^j(b)ab\cdots ab = 0\). Continuing this procedure yields that \((a\alpha^j(b))^k = 0\), this means \(a\alpha^j(b) \in \text{Nil}(R)\).

**Claim 2.** If \(a\alpha^m(b) \in \text{Nil}(R)\), for some positive integer \(m\), then \(ab \in \text{Nil}(R)\).

**Proof of Claim 2.** The proof is similar to that of Claim 1.

**Claim 3.** \(\text{Nil}(R)[x; \alpha]\) is an ideal of \(R[x; \alpha]\).

**Proof of Claim 3.** Recall that an ideal \(I\) of a ring \(R\) is called \(\alpha\)-ideal if \(\alpha(I) \subseteq I\). If an ideal \(I\) of \(R\) is \(\alpha\)-ideal, then \(I[x; \alpha]\) is an ideal of \(R[x; \alpha]\). On the other hand, \(\text{Nil}(R)\) is an \(\alpha\)-ideal, by Claim 2. Thus \(\text{Nil}(R)[x; \alpha]\) is an ideal of \(R[x; \alpha]\), and the result follows.

**Claim 4.** \(\text{Nil}(R[x; \alpha]) = \text{Nil}(R)[x; \alpha]\).

**Proof of Claim 4.** First we show that \(\text{Nil}(R[x, \alpha]) \subseteq \text{Nil}(R)[x; \alpha]\). Suppose \(f(x) \in \text{Nil}(R[x; \alpha])\). DOI: http://dx.doi.org/10.22108/toc.2019.117529.1651
There exists some positive integer $k$ such that $f(x)^k = (a_0 + a_1 x + \cdots + a_m x^m)^k = 0$. Then we have

$$h(x) + a_m \alpha^m (a_m) \alpha^{2m} (a_m) \cdots \alpha^{(k-1)m} (a_m) x^{kn} = 0,$$

where $h(x)$ is a polynomial in $R[x; \alpha]$ with degree at most $(km - 1)$. Hence $a_m \alpha^m (a_m) \alpha^{2m} (a_m) \cdots \alpha^{(k-1)m} (a_m) = 0$. Since $R$ is $\alpha$-compatible, $a_m \in \text{Nil}(R)$. Moreover, since $\text{Nil}(R)[x; \alpha]$ is an ideal of $R[x; \alpha]$, by Claim 3, and for each $g_1(x)$ and $g_2(x) \in R[x; \alpha]$, $a_m g_1(x), g_1(x)a_m$, $f(x)^k \in \text{Nil}(R)[x; \alpha]$, we have

$$(a_0 + a_1 x + \cdots + a_{m-1} x^{m-1})^k = h'(x) + a_{m-1} \alpha^{m-1} (a_{m-1}) \cdots \alpha^{(k-1)(m-1)} (a_{m-1}) x^{k(m-1)} \in \text{Nil}(R)[x; \alpha],$$

where $h'(x)$ is a polynomial in $R[x; \alpha]$ with degree at most $k(m-1) - 1$. Hence

$$a_{m-1} \alpha^{m-1} (a_{m-1}) \cdots \alpha^{(k-1)(m-1)} (a_{m-1}) \in \text{Nil}(R)$$

and similar discussion yields that $a_{m-1} \in \text{Nil}(R)$ for each $1 \leq i \leq m$. This means that $f(x) \in \text{Nil}(R)[x; \alpha]$. 

Next we show that $\text{Nil}(R)[x; \alpha] \subseteq \text{Nil}(R[x; \alpha])$. Suppose that $a_i^{m_i} = 0$, for $i = 0, \ldots, m$. Let $k = m_0 + \cdots + m_n + 1$. Then we have $(f(x))^k = \sum [(a_0^{i_0} (a_1 x)^{i_1} \cdots (a_m x^m)^{i_m}) (a_0^{i_0_1} (a_1 x)^{i_1_1} \cdots (a_m x^m)^{i_m_1}) \cdots (a_0^{i_0_k} (a_1 x)^{i_1_k} \cdots (a_m x^m)^{i_m_k})]$, where $i_0 + i_1 + \cdots + i_{m_r} = 1$, for $r = 1, \ldots, k$ and $i_{s_i} \in \{0, 1\}$. It is easy to see that each coefficient of $[f(x)]^k$ is a sum of such elements $((f_{a_0}^{a_0_1} (a_0))^{i_0} (f_{a_1}^{a_1_1} (a_1))^{i_1} \cdots (f_{a_m}^{a_m_1} (a_m))^{i_m}) \cdots (f_{a_0}^{a_0_k} (a_0))^{i_0_1} (f_{a_1}^{a_1_k} (a_1))^{i_1_1} \cdots (f_{a_m}^{a_m_k} (a_m))^{i_m_1})$, where $i_0 + i_1 + \cdots + i_{m_r} = 1$, and $f_j^g = \alpha^j$. It can be easily checked that there exists $a_h \in \{a_0, a_1, \ldots, a_m\}$ such that $i_{h_1} + i_{h_2} + \cdots + i_{h_k} \geq m_h$. Since $a_h^{m_h} = 0$, we deduce that $a_h^{i_{h_1} + i_{h_2} + \cdots + i_{h_k}} = 0$, and hence, by Lemma 2.1, we have $((f_{a_0}^{a_0_1} (a_h))^{i_{h_1}} (f_{a_0}^{a_0_2} (a_h))^{i_{h_2}} \cdots (f_{a_0}^{a_0_k} (a_h))^{i_{h_k}}) = 0$. Thus each coefficient of $(f(x))^k$ is zero, since $R$ is a symmetric ring. Hence $[f(x)]^k = 0$. Therefore, $f(x) \in \text{Nil}(R[x; \alpha])$ as desired. \qed

Note that reduced rings are symmetric, so we can deduce the following result. Notice that our proof is simple and short.

**Corollary 2.4.** Let $R$ be a reduced ring with $\alpha$-compatible property. Then the skew polynomial ring $R[x; \alpha]$ is reduced.

**Remark 2.5.** Let $R$ be a non-reduced ring. If $R$ is symmetric, then $Z_N(R) = R$. In fact, if $R$ is a non-reduced symmetric ring, then $\text{Nil}(R)$ is an ideal of $R$ and so $r$ and $x$ are adjacent for any $r \in R^*$ and $x \in \text{Nil}^+(R)$. Therefore, $Z_N(R) = R$. 

**Remark 2.6.** Let $R$ be a NI ring. If there exists a nonzero nilpotent element of $R$, then all nonzero elements of $R$ are adjacent to it. Thus, for any NI ring $R$ whose $Z_N(R)$ contains at least one nonzero nilpotent element, then $\Gamma_N(R)$ is connected, $\text{diam}(\Gamma_N(R)) \leq 2$, and $\text{gr}(\Gamma_N(R)) = 3$ or $\infty$. 

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Theorem 2.7. Let $R$ be a ring. Then $\Gamma_N(R)$ is complete if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. (\(\Rightarrow\)) By definition.

(\(\Rightarrow\)) Suppose that $\Gamma_N(R)$ is complete. We claim that $R$ is a reduced ring. To prove this, suppose that $x \in \text{Nil}(R)^*$, for some $x \in Z(R)^*$. Since $x \in \text{Nil}(R)^*$, we deduced that there exists a positive integer $n \in \mathbb{N}$ such that $x^n \neq 0$ and $x^i \neq 0$, for all $1 \leq i \leq n-1$. It is clear that $(1-x)(1+x+x^2+\cdots+x^{n-1}) = 1$. Thus $1 \neq 1 - x \in U(R)$ and so $|U(R)| \geq 2$. Let $a, b \in U(R)$, and let $x \in \text{Nil}^*(R)$. Since $a, a^{-1}x$ and $b, b^{-1}x$ are contained in $\text{Nil}(R)$, $a^{-1}x$ and $b^{-1}x$ are also nonzero, we conclude that $a, b \in Z_N^*(R)$, with $a^{-1}, b^{-1} \in U(R)$. Also we have $ab \notin \text{Nil}(R)$. But since $\Gamma_N(R)$ is complete, we reach a contradiction. This proves the claim. Thus $\Gamma_N(R) = \Gamma(R)$ and so by [1, Theorem 5], $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. \(\square\)

We have also the following theorem, its proof is essentially same as that of [14, Theorem 2.2].

Theorem 2.8. Let $R$ be a reduced ring. If $Z(R)$ is not an ideal, then $R$ has exactly two minimal primes if and only if diam$(\Gamma(R))$ is less than or equal to 2.

Following [6], an ideal $I$ of $R$ is called completely semiprime ideal if $R/I$ is a reduced ring. Lam et al. [12, Lemma 12.6], proved that, if $R$ is a reduced ring, then each minimal prime ideal of $R$ is completely prime. Also each minimal prime ideal is a union of annihilators. Thus, if $P$ is a minimal prime ideal of a reduced $\alpha$-compatible ring $R$, then $\alpha(P) \subseteq P$, and so $P[x; \alpha]$ is an ideal of $R[x; \alpha]$, by [9, Theorem 3.1]. One can easily prove that $P[x; \alpha]$ is a minimal prime ideal of $R[x; \alpha]$.

Theorem 2.9. Let $R$ be a symmetric and $\alpha$-compatible ring. If $R$ has exactly two minimal primes, then $\text{diam}(\Gamma_N(R[x; \alpha])) = 2$.

Proof. First suppose that $R$ is a non-reduced ring. Lemma 2.3 and Remark 2.6 yield that $\text{diam}(\Gamma_N(R[x; \alpha])) \leq 2$. Also by Theorem 2.7, we conclude that $\text{diam}(\Gamma_N(R[x; \alpha])) \geq 2$ and so $\text{diam}(\Gamma_N(R[x; \alpha])) = 2$.

Next suppose that $R$ is reduced and $\text{Min}(R) = \{P_1, P_2\}$. Clearly, $\Gamma(R) = \Gamma_N(R)$, $Z_N(R) = Z(R)$ and these primes are completely primes. Since $R$ is reduced, $< 0 >$ is a completely semiprime ideal. Thus $Z(R)$ is the union of the completely prime ideals, by [6, Theorem 2.10]. This together with [6, Corollary 2.3] imply that $P \cup Q = Z(R)$. By using Corollary 2.4 and a similar argument as used in the proof of part (3) of [11, Theorem 2.7], we have $R[x, \alpha]$ is a reduced ring, and $P[x, \alpha]$ and $Q[x, \alpha]$ are two minimal primes of $R[x, \alpha]$. Then by [11, Theorem 2.7] $Z_N(R[x, \alpha])$ is not ideal of $R[x, \alpha]$, and $|P[x, \alpha]| \geq 3$ and $|Q[x, \alpha]| \geq 3$. Let $f(x), g(x)$ be distinct nonzero elements of $Q[x, \alpha]$. If $f(x)g(x) = 0$, then by reduced property of $R[x, \alpha]$ we obtain $f(x)R[x, \alpha]g(x) = 0$, and then $f(X) \in P[x, \alpha]$ or $g(X) \in P[x, \alpha]$, which is a contradiction. Thus, $d(f(x), g(x)) \geq 2$ and so $\text{diam}(\Gamma_N(R[x; \alpha])) = \text{diam}(\Gamma(R[x; \alpha])) = 2$, by Theorem 2.8. \(\square\)

Theorem 2.10. Let $R$ be a symmetric $\alpha$-compatible ring. Then $2 \leq \text{diam}(\Gamma_N(R[x; \alpha])) \leq 3$.
Proof. First we show that $2 \leq \text{diam} (\Gamma_N(R[x,a]))$. It is easy to check that
\[
\text{diam}(\Gamma_N(R)) \leq \text{diam} (\Gamma_N(R[x,a])).
\]
So if $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then, by Theorem 2.7, $2 \leq \text{diam}(\Gamma_N(R))$ and so $2 \leq \text{diam}(\Gamma_N(R[x,a]))$.

Now, let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Since $R$ is reduced with exactly two minimal primes, by Theorem 2.9, $2 = \text{diam}(\Gamma_N(R[x,a]))$.

Next we show that $\text{diam}(\Gamma_N(R[x,a])) \leq 3$. To prove this, we consider two following cases:

Case 1: $R$ is a non-reduced ring. By a similar argument as used in the proof of Theorem 2.9, we conclude that $\text{diam}(\Gamma_N(R[x,a])) = 2 \leq 3$.

Case 2: $R$ is a reduced ring. It follows from [16, Theorem 3.2].

\begin{theorem}
Let $R$ be a symmetric ring. Then the following hold:

1. $\Gamma_N(R)$ is connected,
2. $\text{diam}(\Gamma_N(R)) \leq 3$,
3. If $\Gamma_N(R)$ contains a cycle, then $\text{gr}(\Gamma_N(R)) \leq 4$. Moreover, if $R$ is non-reduced, then $\text{gr}(\Gamma_N(R)) = 3$.
\end{theorem}

Proof. If $R$ is reduced, then $\Gamma_N(R)$ is actually the usual zero-divisor graph $\Gamma'(R)$. We have $\Gamma_N(R)$ is connected, $\text{diam}(\Gamma_N(R)) \leq 3$ and $\text{gr}(\Gamma_N(R)) \leq 4$ by [16, Theorem 3.2]. If $R$ is non-reduced, then since $R$ is symmetric, by Remark 2.6 and Lemma 2.3, we conclude that $\Gamma_N(R)$ is connected, $\text{diam}(\Gamma_N(R)) \leq 2$ and $\text{gr}(\Gamma_N(R)) = 3$ or $\infty$.

\begin{lemma}
Let $R$ be a reduced and $\alpha$-compatible ring. If $Z_N^*(R) \neq \emptyset$, then $\text{gr}(\Gamma_N(R[x,a])) \leq 4$.
\end{lemma}

Proof. Since $R$ is reduced and $Z_N^*(R) \neq \emptyset$, there exist nonzero elements $a, b \in Z_N(R)$ such that $ab = 0$ and so by $\alpha$-compatibility of $R$, $ba(a) = aa(b) = 0$. Now consider the cycle $a - bx - ax - b - a$ of length four in $\Gamma_N(R[x,a])$. Therefore $\text{gr}(\Gamma_N(R[x,a])) \leq 4$.

\begin{theorem}
Let $R$ be a symmetric and $\alpha$-compatible ring. Then $\text{gr}(\Gamma_N(R)) \geq \text{gr}(\Gamma_N(R[x,a]))$. In addition, if $\Gamma_N(R)$ contains a cycle, then $\text{gr}(\Gamma_N(R)) = \text{gr}(\Gamma(R[x,a]))$.
\end{theorem}

Proof. If $Z_N^*(R) = \emptyset$, then $\text{gr}(\Gamma_N(R)) = \infty = \text{gr}(\Gamma_N(R[x,a]))$. So we may assume that $Z_N^*(R) \neq \emptyset$. Since the graph $\Gamma_N(R)$ is a subgraph of $\Gamma_N(R[x,a])$, we have $\text{gr}(\Gamma_N(R)) \geq \text{gr}(\Gamma_N(R[x,a]))$. We consider the following two cases:

Case 1: $R$ is a non-reduced ring. By Theorem 2.11 (3), $\text{gr}(\Gamma_N(R)) = 3$. Since $\text{gr}(\Gamma_N(R)) \geq \text{gr}(\Gamma_N(R[x,a]))$, we conclude that $\text{gr}(\Gamma_N(R[x,a])) = \text{gr}(\Gamma_N(R)) = 3$.

Case 2: $R$ is a reduced ring. By Lemma 2.12, we have $\text{gr}(\Gamma_N(R[x,a])) \leq 4$. Now suppose that $\text{gr}(\Gamma_N(R[x,a])) = 4$. Since $\Gamma_N(R)$ contains a cycle, by Theorem 2.11 (3), $\text{gr}(\Gamma_N(R)) \leq 4$. This together with $\text{gr}(\Gamma_N(R)) \geq \text{gr}(\Gamma_N(R[x,a])) = 4$ imply that $\text{gr}(\Gamma_N(R)) = 4$.

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Case (2): \( R \) is a reduced ring. By Lemma 2.2, \( \text{Nil}(R) \) is an ideal of \( R \). We claim that \( |\text{Nil}(R)| = 2 \).

To see this, suppose to the contrary, \( |\text{Nil}(R)| \geq 3 \). This imply that \( |U(R)| \geq 3 \). Assume that \( a, b \in \text{Nil}^*(R) \) and \( 1 \neq c \in U(R) \). By Remark 2.6, \( a \) and \( b \) are adjacent to all vertices contained \( Z_N^*(R) = R^* \). Thus \( a - b - c - a \) forms a triangle and so \( gr(\Gamma_N(R)) = 3 \), a contradiction. Therefore \( |\text{Nil}(R)| = 2 \). Now we show that \( gr(\Gamma_N(R[x, \alpha])) = 3 \). Suppose \( a \in \text{Nil}(R) \). Since \( a^2 = 0 \), \( a - ax - ax^2 - a \) forms a triangle for nonzero element \( a \in \text{Nil}(R) \), and so the claim is proved.

Case (2): \( R \) is a reduced ring. By Lemma 2.12, we have \( gr(\Gamma_N(R[x, \alpha])) \leq 4 \). Now if \( gr(\Gamma_N(R[x, \alpha])) = 3 \), then by a similar way as used in Case (2) of Theorem 2.13, we can show that \( gr(\Gamma_N(R)) = 3 \), a contradiction, and so \( gr(\Gamma_N(R[x, \alpha])) = 4 \). □

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**Mohamad Javad Nikmehr**
Faculty of Mathematics, K. N. Toosi University of Technology, 16765-3381, Tehran, Iran

*Email: nikmehr@kntu.ac.ir*

**Abdolreza Azadi**
Faculty of Mathematics, K. N. Toosi University of Technology, 16765-3381, Tehran, Iran

*Email: abdoreza.azadi@kntu.ac.ir*

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