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CONNECTED ZERO FORCING SETS AND CONNECTED PROPAGATION TIME OF GRAPHS

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Abstract. The zero forcing number $Z(G)$ of a graph G is the minimum cardinality of a set S with colored (black) vertices which forces the set $V(G)$ to be colored (black) after some times. “color change rule”: a white vertex is changed to a black vertex when it is the only white neighbor of a black vertex. In this case, we say that the black vertex forces the white vertex. We investigate here the concept of connected zero forcing set and connected zero forcing number. We discusses this subject for special graphs and some products of graphs. Also we introduce the connected propagation time. Graphs with extreme minimum connected propagation times and maximum propagation times $|G|-1$ and $|G|-2$ are characterized.

1. Introduction

Let $V(G)$ and $E(G)$ be the vertex set and the edge set of a graph $G=(V,E)$, respectively. For a vertex $v\in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u\in V(G) : uv\in E(G)\}$. The *closed neighborhood* $N[v]=N(v)\cup\{v\}$. Simple graph containing no graph loops or multiple edges. In this paper each graph is undirected, finite, simple and the vertex set is nonempty. A graph $G_0=(V_0,E_0)$ is a *subgraph* of graph $G=(V,E)$ if $V'\subseteq V$ and $E'\subseteq E$.

A *complete graph* is a graph in which every two distinct vertices are adjacent. The complete graph on n vertices is denoted by K_n. A graph (V,E) is *bipartite* if the vertex set V can be partitioned

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into two nonempty subsets X and Y such that every edge of graph has one vertex (endpoint) in X and one in Y. A *complete bipartite graph* is bipartite graph $K_{n,m}$ with $|X| = n$, $|Y| = m$ and $E = \{\{u, w\} : u \in U, w \in W\}$. The *path* is the graph P_n with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\}$. The *cycle* is the graph C_n with $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$.

The following graph operations are used to construct families of graphs:

• The *Cartesian product* of two graphs G and H, is denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if $(1)u = u'$ and $vv' \in E(H)$, or $(2)v = v'$ and $uu' \in E(G)$.

• The *strong product* of two graphs G and H, is denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if $(1)u = u'$ and $vv' \in E(H)$, or $(2)v = v'$ and $uu' \in E(G)$, or $(3)uu' \in E(G)$ and $vv' \in E(H)$.

• The *corona product* of two graphs G and H, denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by taking one copy of G and $|G|$ copies of H, and jointing all the vertices in the ith copy of H to the ith vertex of G.

• The *generalized corona* of G with $H_1, H_2, \ldots, H_n$, is defined as the graph obtained by joining all vertices of H_i to the $i-$th vertex of G. We denote this graph by $G\langle H_1, H_2, \ldots, H_n\rangle$.

• The nth *supertraingle* is denoted by T_n is an equilateral triangular grid such that each side of it contains n vertices (see Figure 1.3).

• The *wheel graph* W_n of order n (n-wheel) is a graph that contains a cycle of order $n - 1$, and for which every graph vertex in the cycle is connected to one other graph vertex.

• The *star graph* S_n of order n (n-star) complete bipartite graph $K_{1,n-1}$.

Zero forcing sets and the zero forcing number were introduced in [1]. The zero forcing number is a useful tool for determining the minimum rank of structured families of graphs and small graphs, and is motivated by simple observations about null vectors of matrices.

Definition 1.1. [1] *Color-change rule: Let G be a graph with each vertex is colored either white or black, u be a black vertex of G, and exactly one neighbor v of u be white. Then change the color of v to black. When this rule is applied, we say u forces v, and write $u \to v$.*

Definition 1.2. [1] *A $zero\ forcing\ set$ (or ZFS for brevity) of a graph G is a subset Z of vertices such that if initially the vertices in Z are colored black and remaining vertices are colored white, the entire graph G may be colored black by repeatedly applying the color-change rule. The zero forcing number of $G, Z(G)$, is the minimum size of a zero forcing set. Any zero forcing set of order $Z(G)$ is called a minimum zero forcing set.*

For a coloring of G, the *derived coloring* is the result of applying the color-change rule until no more changes are possible. For the black set of vertices B, the derived coloring is denoted by $der(B)$ and it is unique [1].

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Example 1.3. *In the following figure, a zero forcing set for each of T_4 and C_8 is shown.*

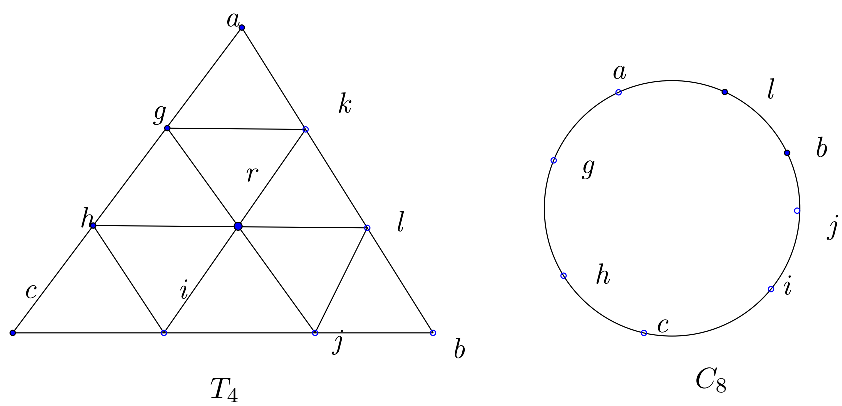

FIGURE 1. Let $ZFS(T_4) = \{a, g, h, c\}$, $ZFS(C_8) = \{a, g\}$

Definition 1.4. [11] *Let B be a zero forcing set of G and $B^0 = B$. For $t \geq 0$, we define $B^{(t+1)}$ to be the set of vertices w for which there exists a vertex $b \in \cup_{s=0}^{t} B^{(s)}$ such that w is the only neighbor of b not in $\cup_{s=0}^{t} B^{(s)}$. The propagation time of B in G, denoted $pt(G, B)$, is the smallest integer t' such that $V = \cup_{s=0}^{t'} B^{(s)}$.*

In other word, the propagation time is the number of steps it take for an initial zero forcing set to force all vertices of a graph to black.

It is possible that two minimum zero forcing sets of one graph have the different propagation time (see Example 1.2 of [11]). Zero forcing parameters were studied and applied to the minimum rank problem and quantum systems [6, 7, 11]. Also, this concept was named as graph infection or graph propagation [1]. For more information about the propagation time, see [2, 3, 4, 6, 7, 9, 10, 12, 14]. This parameter is investigated for graphs with extra condition or some type of the zero forcing set [8, 15].

Definition 1.5. [11] *The minimum propagation time of G is $pt(G) = min\{pt(G, B)|B$ is a minimum zero forcing set of $G\}$.*

Definition 1.6. [11] *The maximum propagation time of G is defined as $PT(G) = max\{pt(G, B)|B$ is a minimum zero forcing set of $G\}$.*

Note that $Z(G)$ and $pt(G)$ are not subgraph monotone. That is, a graph may have a subgraph with greater zero forcing number or minimum propagation time, as we see in the following example. Also, see the Example 1.5 of [11].

Example 1.7. *In Figure 1.3, Let $ZFS(T_4) = \{a, g, h, c\}$, $ZFS(C_8) = \{a, g\}$ and $ZFS(P_6) = \{a\}$ Let $B_0 = ZFS(T_4) = \{a, g, h, c\}$, $B_1 = ZFC(C_8) = \{a, g\}$ and $B_2 = ZFC(P_6) = \{a\}$. Then we have:*

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i	$B_0^{(i)}$	$B_1^{(i)}$	$B_2^{(i)}$
0	$B_0 = \{a, g, h, c\}$	$B_1 = \{a, g\}$	$B_2 = \{a\}$
1	$\{k, i\}$	$\{h, l\}$	$\{g\}$
2	$\{r\}$	$\{c, b\}$	$\{h\}$
3	$\{l, j\}$	$\{i, j\}$	$\{c\}$
4	$\{b\}$	$--$	$\{i\}$
5	$--$	$--$	$\{j\}$

So, $pt(T_4, B_0) = 4$, $pt(C_8, B_1) = 3$ and $pt(P_6, B_2) = 5$. It is easy to see that The minimum propagation time of T_4 is 4, the minimum propagation time of its subgraph C_8 is 3 and the minimum propagation time of its subgraph P_6 is 5. So, this concept is not subgraph monotone (The path P_6 is subgraph of the graph T_4.).

2. Connected zero forcing set

In this section, we investigate the concept of connected zero forcing. We do a comparison between the zero forcing set and connected zero forcing set for special graphs. In [5], it is mentioned that $R \subseteq V(G)$ is a connected forcing set of G, If the subset R is forcing set and it induces a connected subgraph. Now we have the following definition.

Definition 2.1. *A $connected\ zero\ forcing\ set$ (or $CZFS$ for brevity) of a graph G, is a zero forcing set such that be connected in components of G. The $connected\ zero\ forcing\ number$ of $G, Z_c(G)$, is the minimum size of connected zero forcing sets. Any connected zero forcing set of order $Z_c(G)$ is called a minimum connected zero forcing set.*

Applying Definition , we obtain the following proposition.

Proposition 2.2. *Every CZFS is a ZFS and if a ZFS is connected in components, then it is a CZFS.*

Corollary 2.3. *For any graph G, $Z(G) \leq Z_c(G)$.*

It may be interesting to compare the zero forcing number and connected zero forcing number of some well known graphs:

Example 2.4. *In Figure 2, we see the ZFS and CZFS for the star graph S_6.*

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FIGURE 2. ZFS and CZFS of star graph S_6

From definition, we can easily observe that these zero forcing sets and connected zero forcing sets are minimum. Thus, we have the following proposition for the star graphs and other well known graphs.

Proposition 2.5. $Z(K_n) = Z_c(K_n) = n-1$; $Z(P_n) = Z_c(P_n) = 1$; $Z(C_n) = Z_c(C_n) = 2$; $Z(W_n) = Z_c(W_n) = 3$; $Z(T_n) = Z_c(T_n) = n$; $Z(S_n) = n-2$, $Z_c(S_n) = n-1$.

Theorem 2.6. $Z(K_{n_1,n_2,\ldots,n_t}) = Z_c(K_{n_1,n_2,\ldots,n_t}) = n_1 + n_2 + \cdots + n_t - 2$.

Proof. Denote the parts of graph by U_1, $U_2 \ldots, U_t$. Let $v_1 \in U_1$. If this vertex is black and has $n_2 + n_3 + \cdots + n_t - 1$ black neighbors, then it can change the color of another vertex. But the color of vertices $U_1 - \{v_1\}$ can be changed by a vertex of another part for example $v_2 \in U_2$. For this aim, all neighbors of v_2 must be black, except one of them. Therefore $U_1 - \{v_1\}$ at most has one white vertex that is, $Z(K_{n_1,n_2,\ldots,n_t}) = Z_c(K_{n_1,n_2,\ldots,n_t}) = n_1 + n_2 + \cdots + n_t - 2$. □

Now, we find a bound for connected zero forcing number of different product of two graphs. First the strong product of a cycle and a path, we give a simple example. The general case can be found by a similar way.

Example 2.7. *In the following figure, we see a CZFS of size* $n + 2m - 2 = 5 + 6 - 2 = 9$, *for* $C_n \boxtimes P_m, n = 5$ *and* $m = 3$.

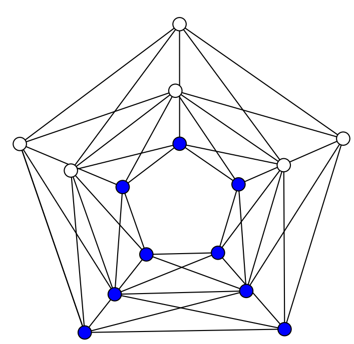

FIGURE 3. $C_n \boxtimes P_m, n = 5$ and $m = 3$

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Theorem 2.8. *$Z(C_n \boxtimes P_m)$ and $Z_c(C_n \boxtimes P_m) \leq n + 2m - 2$.*

Theorem 2.9.

- $Z_c(G\Box P_t) = Z(G\Box P_t) = |G|$.
- $Z_c(G\Box H) \leq min\{Z_c(G)|H|, Z_c(H)|G|\}$.

Proof. The proof is resulted from definitions. □

Theorem 2.10. *$Z(P_n \boxtimes P_m) = Z_c(P_n \boxtimes P_m) \leq n + m - 1$.*

Proof. The graph $P_n \boxtimes P_m$ is a $n \times m$ grid. We know that $n + m - 1 \geq Z(P_n \boxtimes P_m)$ [1]. Now consider the CZFS contains all vertices of the first column and the first row of this grid. This set is connected and ZFS. The size of this set is $n + m - 1$. □

Theorem 2.11. *Let G be a connected graph with $|G| = n$. Then*

$$Z_c(G\langle H_1, H_2, \ldots, H_n\rangle) \leq |G| + \sum_{i=1}^{n} Z(H_i).$$

In particular, if H_i's are connected and $|H_i| > 1$ for all i, then

$$Z_c(G\langle H_1, H_2, \ldots, H_n\rangle) = |G| + \sum_{i=1}^{n} Z(H_i).$$

Proof. Let Z_i be a minimum zero forcing set of H_i for all $1 \leq i \leq n$. It is easy, to see that $V(G) \cup Z_1 \cup \cdots \cup Z_n$ is a ZFS for $G\langle H_1, H_2, \ldots, H_n\rangle$. Also this set is connected, because the graph G is connected. Thus

$$Z_c(G\langle H_1, H_2, \ldots, H_n\rangle) \leq |G| + \sum_{i=1}^{n} Z(H_i).$$

Now, we claim that if $|H_i| > 1$ for all i, then this set is minimum. First notice that every CZFS contains all vertices of G. Because if a vertex of G such as v_i is initially white, then by connectedness of zero forcing set, all of the vertices of H_i must be white or the CZFS is a subset of $V(G)$. In the first case, forcing of vertices of H_i should be begin with a force by v_i, which is not possible where $|H_i| > 1$. In the second case, there exists an vertex v_i such that it is the white neighbor of each vertex in H_i. All vertices in H_i should be black and the rest of graph is a path. Thus, it easily follows that $|G| = 1$ and H_i is complete graph. So the result follows.

Now, if $v_i \to u$ is a force on this graph, it means that u is the only white vertex in H_i. The graph H_i is connected, so there exists another vertex $u_0 \in H_i$ which is a neighbor of u. Thus we can replace $v_i \to u$ by $u_0 \to u$. Therefore, all the forces are in H_i. So, for every minimum CZFS A of $G\langle H_1, H_2, \ldots, H_n\rangle$, $A \cap V(H_i)$ is a ZFS for H_i. Now, it follows that $V(G) \cup Z_1 \cup \cdots \cup Z_n$ is minimum. □

Corollary 2.12. *For any graph G and H, we have*

- $Z(G \circ H) \leq Z(G) + |G|Z(H)$.
- $Z_c(G \circ H) \leq Z_c(G) + |G|Z(H)$.

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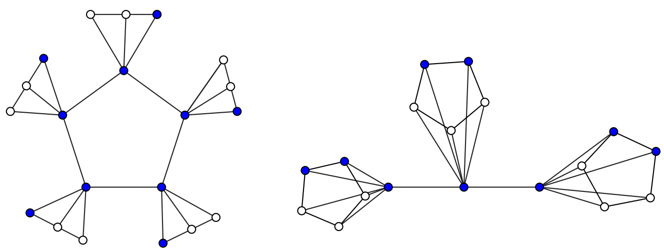


$C_5 \circ P_3$ $P_3 \circ C_6$

Remark 2.13. *The integer of $Z_c(G) - Z(G)$ can be very large, for example if $G = C_n$ and $H = P_m$ where $m > 1$, then $Z(G \circ H) = n+2$ and $Z_c(G \circ H) = 2n$ while if we set $G = P_n$ and $H = C_m$, $m > 2$ then $Z(G \circ H) = 2n+1$ and $Z_c(G \circ H) = 3n$.*

3. Connected propagation time

In this section, we introduce the concept of connected propagation time for graph G. We characterize graphs G having extreme minimum and maximum connected propagation time $|G|-1$, $|G|-2$ and 0.

Definition 3.1. *Let B be a connected zero forcing set of G and $B^0 = B$. for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \cup_{s=0}^{t} B^{(s)}$ such that w is the only neighbor of b not in $\cup_{s=0}^{t} B^{(s)}$. The connected propagation time of B in G, denoted by $pt_c(G,B)$, is the smallest integer t' such that $V = \cup_{s=0}^{t'} B^{(s)}$.*

Definition 3.2. *The minimum connected propagation time of G is $pt_c(G) = min\{pt_c(G,B) | B$ is a minimum connected zero forcing set of $G\}$.*

Definition 3.3. *The maximum connected propagation time of G is defined as $PT_c(G) = max\{pt_c(G,B) | B$ is a minimum connected zero forcing set of $G\}$.*

Here, the case of connected propagation time $|G|-1$ and $|G|-2$ are investigated.

Theorem 3.4. *Let G be a connected graph. Then the following conditions are equivalent:*

- $Z_c(G) = 1$;
- $pt_c(G) = |G| - 1$;
- $Pt_c(G) = |G| - 1$;
- *G is a path.*

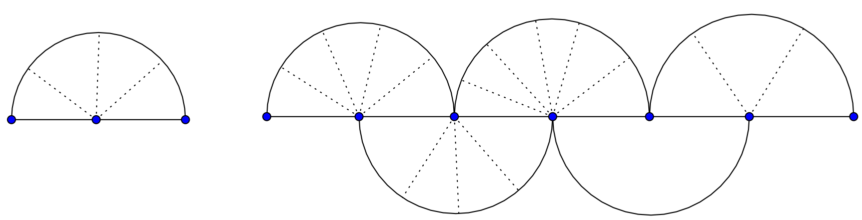
Proof. We prove the sequence $(ii) \to (iii) \to (i) \to (iv) \to (ii)$.

First, note that for each graph G, we have $pt_c(G) \leq Pt_c(G) \leq |G| - Z_c(G) \leq |G| - 1$. Thus, from (ii), we can deduce (iii), and if (iii) holds, then $Z_c(G) = 1$. In addition, if $Z_c(G) = 1$, then $Z(G) = 1$ and therefore G is a path. The last part is trivial. □

Definition 3.5. *For $k \geq 1$ and nonnegative integer numbers $n_1, \ldots, n_k$ (n_is can be zero), by $PC(n_1, n_2, \ldots, n_k)$ we mean a collection of graphs which contains a path $P_{k+2} = (v_1, \ldots, v_{k+2})$ and k cycles, with the following conditions:*

- *the ith cycle ($1 \leq i \leq k$) is $(v_i, v_{i+1}, v_{i+2}, u_1^i, \ldots, u_{n_i}^i)$ with n_i new vertices, which are probably neighbors of v_{i+1}.*
- *For each i, j, s, t we have $u_j^i \neq u_s^t$ and $u_j^i \neq v_t$.*

For instance, the following figures are PC (3) and PC (4,3,5,0,2).



Definition 3.6. *Let G and H be disjoint graphs, each with a vertex labeled v. Then $G \oplus_v H$ is the graph obtained by identifying the vertex v in G with the vertex v in H.*

Theorem 3.7. *Let G be a graph with $Pt_c(G) = |G| - 2$. Then*

(1) *If G is not connected, then $G = K_1 \dot{\cup} P_{|G|-1}$.*

(2) *If G is connected, then G is of the form $PC(n_1, n_2, \ldots, n_{k-1}, 0)$ or $PC(n_1, n_2, \ldots, n_k) \oplus_{v_{k+1}} P_m$ where $(v_1, \ldots, v_{k+2})$ is the path of PC and P_m is a path with $m \geq 2$ and pendent vertex v_{k+1}.*

Proof. From $Pt_c(G) \leq |G| - Z_c(G)$, it follows that $Z_c(G) \leq 2$. If $Z_c(G) = 1$ then by previous theorem $Pt_c(G) = |G| - 1$, which is a contradiction. So, $Z_c(G) = 2$.

If G is not connected then it contain two component, each of them has the CZFS of size one thus each component is one path and in every stage, only the color of one vertex will be changed. So, one path is $P_1 = K_1$ and the graph G is $G = K_1 \dot{\cup} P_{|G|-1}$.

If G is connected, then the connected zero forcing set is consist of two vertices v_1 and v_2 which are adjacent by an edge e. Observe that in each stage, only the color of one vertex will be changed. The edge e is not a cutting- edge, otherwise $G \setminus \{e\}$ has two components G_1 and G_2 with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, thus $\{v_1\}$ is a zero forcing set for G_1 and $\{v_2\}$ is a zero forcing set for G_2. Therefore, G_1 and G_2 are path and so G is a path and $\{v_1, v_2\}$ is not an minimum connected zero forcing set. Thus, there is another path between v_1 and v_2.

Consider a path between v_1 and v_2 with largest possible length, namely, $P : v_1, u_1, u_2, \ldots, u_n, v_2$. Just one vertex of v_1 or v_2 can have neighborers in $V(G) - V(P)$. Otherwise v_1 and v_2 can not change the color of vertices $V(G) - \{v_1, v_2\}$. Because both of them have at least two white neighbors.

Without loss of generality, let v_1 has no neighbors except u_1 and v_2. There exist two cases for v_2.

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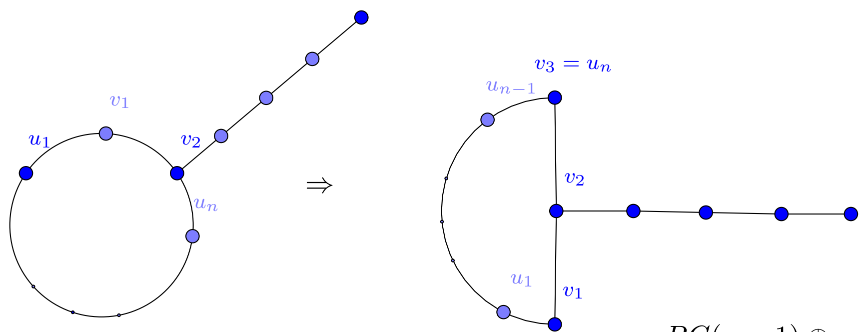
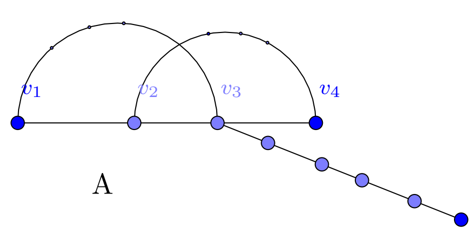
Case 1. If v_2 has another neighbors which does not belong to this path, then in the first step v_1 forces u_1. Again, u_1 can not have more than one white neighbor, which is u_2. So, the only neighbors of u_1 are v_1, u_2 and probably v_2. Continuing this way, in the ith step, u_{i-1} forces u_i and all of the u_is with $i < n$ are adjacent to u_{i-1}, u_{i+1} and probably v_2. After n steps, the set $\{u_n, v_2\}$ is a zero forcing set for $G \setminus \{v_1, u_1, \ldots, u_{n-1}\}$ and since the path P has maximum length, it follows that there is no other path except the edge $\{u_n, v_2\}$ between these two vertices $G \setminus \{v_1, u_1, \ldots, u_{n-1}\}$. This means that this edge, is a cutting edge. Since in each step only one vertices can be forced by black vertices, it can be deduced that u_n has no other neighbors and a path can be jointed to v_2. Putting $v_3 = u_n$, G is of the form $PC(n-1) \oplus_{v_2} P_m$ with $m \geq 2$ (see following figure).



Case 2. All of the neighbors of v_2 belongs to the path. If $n = 1$, put $v_3 = u_1$ and $G = PC(0) = K_3$, else let $1 \leq n_1 < n$, be the largest number for which u_{n_1} is adjacent to v_2. Put $v_3 = u_{n_1}$. As previous part, it can be seen that each u_i $(1 \leq i \leq i_2)$ are at most of degree 3 with two neighbors of the path and probably v_2 and in the ith step, u_{i-1} forces u_i.

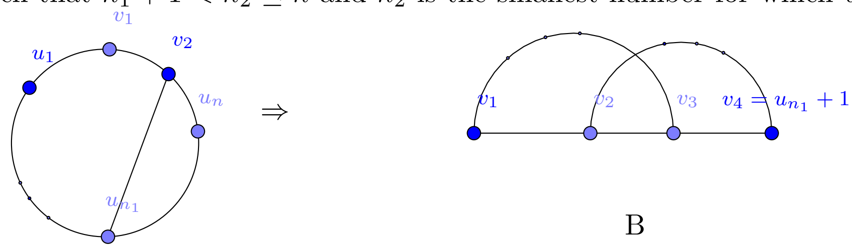
Now $\{v_2, v_3\}$ is a CZFS for $G \setminus \{v_1, u_1, \ldots, u_{n_1-1}\}$ such that in each step only one vertex becomes black. So by repeating the proof, there exist two cases, either G is of the form $PC(n_1 - 1, n - n_1) \oplus_{v_3} P_m$ with $v_4 = u_{n_1+1}$ or all neighbors of v_3 belongs to the path. In the first case the graph has form A (see following figure).



A

When all neighbors of v_3 belong to the path, if $n - n_1 = 1$, then put $v_4 = u_{n_1+1}$ and $G = PC(n_1 - 1, 0)$ and graph is of the form B (see following figure). Else we can choose

$v_4 = u_{n_2}$, such that $n_1 + 1 < n_2 \leq n$ and n_2 is the smallest number for which u_{n_2} is adjacent



to v_3.

Going on, we obtain a path $v_1, v_2, \ldots, v_{k+2}$ with $v_i = u_{n_{i-2}}$ and $1 \leq n_1 < n_3 < \cdots < n_4 < n_2 < n$ and G is of the form $PC(n_1 - 1, n - n_2, n_3 - n_1 - 1, n_2 - n_4 - 1, \ldots, n_{k-1} - n_{k-3} - 1, |n_k - n_{k-2}| - 1) \oplus_{v_{k+1}} P_m$ or $|n_k - n_{k-2}| = 1$. Notice that $n_1 - 1$ is the number of vertices between v_1 and v_3. $n - n_2 = n - (n_1 + 1)$ is the number of vertices between v_2 and v_4, in the fact ith integer is the number between v_i and v_{i+2}. So, G is of the form $PC(n_1 - 1, n - n_2, n_3 - n_1 - 1, n_2 - n_4 - 1, \ldots, n_{k-1} - n_{k-3} - 1, |n_k - n_{k-2}| - 1, 0)$.

□

Theorem 3.8. *For connected graph G, we have $pt_c(G) = |G| - 2$ if and only if G is as Theorem* **??** *?t2?? and in addition*

(1) *If $G = PC(n) \oplus_{v_2} P_m$ for $m \geq 1$, with cycle $v_1, u_1, \ldots, u_n, v_3, v_2$, then u_1 and u_n are adjacent to v_2.*

(2) *If $G = PC(n_1, n_2, \ldots, n_k) \oplus_{v_{k+1}} P_m$ for some $k > 1$ and $m \geq 1$, then in the first cycle $v_1, u_1, \ldots, u_n, v_3, v_2$, we have u_1 is adjacent to v_2.*

Proof. If $pt_c(G) = |G| - 2$, then $Pt_c(G) \geq |G| - 2$. From Theorem 3.4, it follows that $Pt_c(G) = |G| - 2$. Thus, G needs to be of one of the forms described in Theorem 3.7.

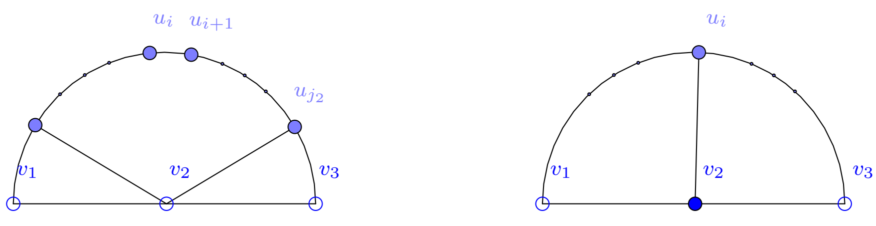
One should note that if u_1 in cycle $v_1, u_1, \ldots, u_{n_1}, v_3, v_2$, (first cycle) not to be a neighbor of v_2 then the set $\{v_1, u_1\}$ is a CZFS of G and in the first step two vertices u_2 and v_2 become black. So, $pt_c(G) < |G| - 2$.

Similarly, if $k = 1$, the set $\{v_3, u_n\}$ is another CZFS of G. So, the vertex u_n must be adjacent to v_2.

Conversely, if G has the above conditions and $k > 1$, we can easily seen that even all the vertices of ith cycle with $1 < i$ can not be a ZFS. Except in the case that $m = 1$, in the kth cycle each of the $\{v_{k+1}, v_{k+2}\}$ or $\{v_k, v_{k+2}\}$ is a CZFS with propagation time $|G| - 2$. In the case that $k = 1$, if $m = 1$, then $G = K_3$.

Let $m > 1$ and u_1 and u_n are adjacent to v_2. Then for each $1 \leq i \leq n - 1$, the set $\{u_i, u_{i+1}\}$ can not be a ZFS. To see this, let $1 \leq j_1 \leq i < i + 1 \leq j_2 \leq n$ such that j_1 is the largest number not greater than i such that u_{j_1} is adjacent to v_2 and j_2 is the smallest number greater than i for which u_{j_2} is adjacent to v_2. Then it is easily seen that $\{u_{j_1}, u_{j_1+1}, \ldots, u_{j_2}\}$ can not be a ZFS (see following figure, case A).

In addition, for each i, if u_i is a neighbor of v_2, then the set $\{u_i, v_2\}$ is not a ZFS (see following figure, case B). Because both of them have degree greater than two. Therefore, the only minimum CZFS's are $\{v_1, v_2\}$, $\{u_1, v_1\}$, $\{v_3, v_2\}$, and $\{u_n, v_3\}$ with propagation times $|G| - 2$.



A B □

Remark 3.9. *In [15], Row investigated the graph with $Z(G) = 2$. He proved that $Z(G) = 2$ if and only if G is a graph of two parallel paths. In his proof, he used a result of [3] for characterization the graphs with maximum nullity 2.*

Applying Row's theorem, in [11], the authors define a special type of graphs with two parallel paths, name zigzag graphs, and show that a connected graph with $pt(G) = |G| - 2$ is a zigzag graph with special conditions.

Of course the graphs are introduced in Theorem **??** *?t2?? and Corollary 3.8, are a special type of the zigzag graph and since every graph with $Z_c(G) = 2$ satisfies $Z(G) = 2$, we can state a proof based on these known results. But here we present an elementary and straightforward proof for these theorems.*

References

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler and S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen and A. Wangsness), Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.*, **428** (2008) 1628–1648.

[2] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, H. van der Holst and B. Shader, Zero forcing parameters and minimum rank problems, *Linear Algebra Appl.*, **433** (2010) 401–411.

[3] F. Barioli, S. Fallat, D. Hershkowitz, H. T. Hall, L. Hogben, H. van der Holst and B. Shader, On the minimum rank of not necessarily symmetric matrices: a preliminary study, *Electron. J. Linear Algebra*, **18** (2009) 126–145.

[4] F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, Parameters related to tree-width, zero forcing, and maximum nullity of a graph, *J. Graph Theory*, **72** (2013) 146–177.

[5] B. Brimkov and R. Davila, *Characterizations of the Connected Forcing Number of a Graph*, https://arxiv.org/pdf/1604.00740.

[6] S. Butler and M. Young, Throttling zero forcing propagation speed on graphs, *Australas. J. Combin.*, **57** (2013) 65–71.

[7] K. B. Chilakamarri, N. Dean, C. X. Kang and E. Yi, Iteration index of a zero forcing set in a graph, *Bull. Inst. Combin. Appl.*, **64** (2012) 57–72.

DOI: http://dx.doi.org/10.22108/toc.2020.115286.1617

[8] J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. Row, N. Warnberg and M. Young, Positive semidefinite zero forcing, *Linear Algebra Appl.*, **439** (2013) 1862–1874.

[9] S. Fallat and L. Hogben, The minimum rank of symmetric matrices described by a graph: a survey, *Linear Algebra Appl.*, **426** (2007) 558–582.

[10] L. Hogben, Minimum rank problems, *Linear Algebra Appl.*, **432** (2010) 1961–1974.

[11] L. Hogben, M. Huynh, N. Kingsley, S. Meyer, Sh. Walker and M. Young, Propagation time for zero forcing on a graph, *Discrete Appl. Math.*, **160** (2012) 1994–2005.

[12] L.-H. Huang, G. J. Chang and H.-G. Yeh, On minimum rank and zero forcing sets of a graph, *Linear Algebra Appl.*, **432** (2010) 2961–2973.

[13] C. R. Johnson, R. Loewy and P. A. Smith, The graphs for which the maximum multiplicity of an eigenvalue is two, *Linear Multilinear Algebra*, **57** (2009) 713–736.

[14] T. Peters, Semidefinite maximum nullity and zero forcing number, *Electron. J. Linear Algebra*, **23** (2012) 815–830.

[15] D. D. Row, A technique for computing the zero forcing number of a graph with a cut-vertex, *Linear Algebra Appl.*, **436** (2012) 4423–4432.

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