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ZERO-SUM FLOW NUMBER OF CATEGORICAL AND STRONG PRODUCT OF GRAPHS

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Abstract. A zero-sum flow is an assignment of nonzero integers to the edges such that the sum of the values of all edges incident with each vertex is zero, and we call it a zero-sum k-flow if the absolute values of edges are less than k. We define the zero-sum flow number of G as the least integer k for which G admitting a zero sum k-flow.?

?In this paper we gave complete zero-sum flow and zero sum numbers for categorical and strong product of two graphs namely cycle and paths.

1. Introduction

Nowhere-zero flows were firstly defined by W. T. Tutte in [11] where he discussed some contribution to the theory of chromatic polynomials. The definition of nowhere-zero flows on signed graphs comes naturally from the deep study of embeddings of graphs in non-orientable surfaces, where nowhere-zero flows emerge as the dual notion to local tensions. There is a close relationship between nowhere-zero flows of graphs and circuit covers of graphs since every nowhere-zero flow on a graph G determines a covering of G by circuits. This type of relation is also maintained by signed graphs, although a signed version of the definition of circuit is required.

Let G be a directed graph. A nowhere-zero flow on G is an assignment of non-zero integers to each edge of G such that for every vertex the the sum of the values of incoming edges is equal to the sum of the values of outgoing edges. A nowhere-zero k-flow is a nowhere-zero flow using edge labels

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with maximum absolute value $k-1$. Since for a directed graph that admitting nowhere-zero flows is independent of the choice of the orientation of the graph, therefore an undirected graph can be considered for the analogue concept. A conjecture by Tutte in 1954 says that

Conjecture 1.1. (Nowhere-Zero Sum 5-Flow Conjecture, [11])
Every bridgeless graph has a nowhere-zero 5*-flow.*

There are some less stronger versions of this conjecture for example, F. Jaeger in [6] showed that every bridgeless graph has a nowhere-zero 8-flow, and P. Seymour proved that every bridgeless graph has a nowhere-zero-6-flow [8] in 1981. Nevertheless, the original Tuttes conjecture is still remains open to be proved(or disprove).

As an analogous concept of a nowhere-zero flow for directed graphs, we consider zero-sum flows for undirected graphs in this paper.

Definition 1.1. *For an undirected graph G, a zero-sum flow is an assignment of non-zero integers to the edges such that the sum of the values of all edges incident with each vertex is zero. A zero-sum k-flow is a zero-sum flow whose values are integers with absolute value less than k.*

Note that from algebraic point of view finding such zero-sum flows is the same as finding nowhere zero vectors in the null space of the incidence matrix of the graph. For an undirected graph G, the incidence matrix $W(G)$ of G, is defined as follows:

$$W(G) = \begin{cases} 1, & \text{if } e_j \text{ and } v_i \text{ are incident;} \\ 0, & \text{otherwise.} \end{cases}$$

An element of the null space of $W(G)$ is a function $f : E(G) \to R$ such that for all vertices $v \in V(G)$ we have $\sum_{u \in N(v)} f(uv) = 0$, where $N(v)$ denotes the set of adjacent vertices to vertex v. If f never takes the value zero, then it is called a zero-sum flow on G. A zero-sum k-flow is a zero-sum flow whose values are integers with absolute value less than or equal to $k-1$.

In literature there is a conjecture for zero-sum flows similar to the Tuttes 5-flow Conjecture for nowhere-zero flows. Let G be an undirected graph with incidence matrix W. S. Akbari et al in [2] raised a conjecture for zero-sum flows similar to the Tuttes 5-flow Conjecture for nowhere-zero flows as follows:

Conjecture 1.2. (Zero-Sum 6-Flow Conjecture)
If G is a graph with a zero sum flow, then G admits a zero-sum 6*-flow.*

So in the linear algebra sense if the null space of W contains a vector whose entries are non-zero real numbers, then there exists a vector in that null space of W whose entries are non-zero integers with absolute value less than 6 also.

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In 2010 it was proved, by Akbari et al. [1], that the above stated conjecture is equivalent to the Bouchets 6-Flow Conjecture for bidirected graphs. In literatures a more general concept flow number, which is defined as the least integer k for which a graph may admit a k-flow, has been studied for both directed graphs and bidirected graphs. T.M Wang and S.W Hu extend the concept in 2011 to the undirected graphs and call it zero-sum flow numbers, and also considered general constant-sum flows for regular graphs [13]. A more general concept is considered in the study of nowhere-zero sum, namely, the least number of k for which a graph may admit a k-flow. In [13] T.M Wang and S.W Hu consider similar concepts for zero-sum k-flows.

Definition 1.2. *Let G be a undirected graph. The zero-sum flow number $F(G)$ is defined as the least number of k for which G may admit a zero-sum k-flow. $F(G) = \infty$ if no such k exists.*

The grid graphs are very useful in all areas of applied sciences like of computer science and electronic science. One of the main usage, for example, is as the discrete approximation to a continuous domain or surface. Numerous algorithms in computer graphics, numerical analysis, computational geometry, robotics and other fields are based on grid computations. In [14] and [15] the authors calculated the zero-sum flow number of triangular and Hexagonal grids.

In this paper, we calculate zero-sum flow number of grids which are obtained by taking the categorical product and strong product of two graphs as well as Octagonal Grid and Generalized Prism.

Definition 1.3. *The categorical product $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u, u') and (v, v') are adjacent if and only if u, v are adjacent in G and u', v' are adjacent in H see e.g. [5], [7] and [10]. Moreover Categorical product is sometimes referred to as the tensor product or direct product.*

Definition 1.4. *The strong product $G_1 \boxtimes G_2$ of graphs G_1 and G_2 has as vertices the pairs (x, y) where $x \in V(G_1)$ and $y \in V(G_2)$. Vertices (x_1, y_1) and (x_2, y_2) are adjacent if either x_1x_2 is an edge of G_1 and $y_1 = y_2$ or if $x_1 = x_2$ and y_1y_2 is an edge of G_2 or if x_1x_2 is an edge of G_1 and y_1y_2 is an edge of G_2. Note that the edge set of the strong product $G_1 \boxtimes G_2$ is the union of the edge sets of the Cartesian product $G_1 \Box G_2$ and categorical product $G_1 \times G_2$, see e.g. [5].*

2. Zero-Sum Flow Number of categorical product of two paths

For integers a and b let $[a, b]$ be an interval of integers x, $a \leq x \leq b$. We deal with a categorical product $G \times H$. If we consider graph G as the path P_n with

$$V(P_n) = \{x_i : i \in [1, n]\}, \quad E(P_n) = \{x_ix_{i+1} : i \in [1, n-1]\}$$

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and graph H as the path P_m with

$$V(P_m) = \{y_j : j \in [1, m]\}, \quad E(P_m) = \{y_j y_{j+1} : j \in [1, m-1]\}$$

then

$$V(P_n \times P_m) = \{(x_i, y_j) : i \in [1, n],\ j \in [1, m]\}$$

is the vertex set of categorical product $P_n \times P_m$ and

$$E(P_n \times P_m) = \{(x_i, y_j)(x_k, y_l) : i, k \in [1, n],\ j, l \in [1, m],\ |i - k| = 1,\ |j - l| = 1\}$$

is the edge set of categorical product $P_n \times P_m$.

On Figure 1 it is depicted the categorical product $P_6 \times P_4$.

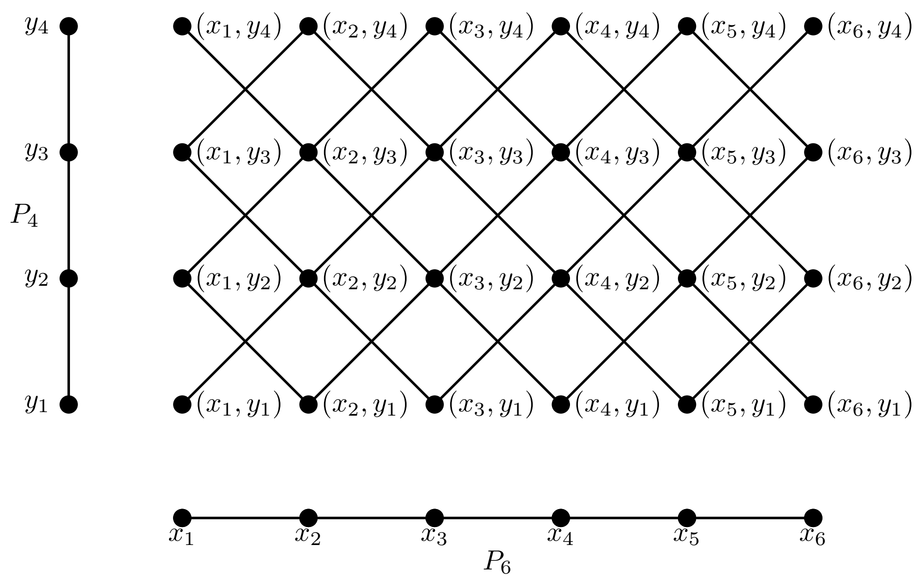

FIGURE 1. Categorical product of two paths $P_6 \times P_4$.

So, $P_n \times P_m$ is the graph of order nm and size $2(n-1)(m-1)$. Since there are four vertices in $P_n \times P_m$ of degree 1 therefore $F(P_n \times P_m) = \infty$

3. Zero-Sum Flow Number of categorical product of cycle and path

If we consider graph G as a cycle C_n with

$$V(C_n) = \{u_i : i \in [1, n]\}, \quad E(C_n) = \{u_i u_{i+1} : i \in [1, n-1]\} \cup \{u_n u_1\}$$

and graph H as a path P_m with

$$V(P_m) = \{v_j : j \in [1, m]\}, \quad E(P_m) = \{v_j v_{j+1} : j \in [1, m-1]\}$$

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then

$$V(C_n \times P_m) = \{(u_i, v_j) : i \in [1, n],\ j \in [1, m]\}$$

is the vertex set of the categorical product of cycle and path $C_n \times P_m$ and

$$\begin{aligned} E(C_n \times P_m) &= \{(u_i, v_j)(u_{i+1}, v_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \\ &\cup \{(u_{i+1}, v_j)(u_i, v_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \\ &\cup \{(u_1, v_j)(u_n, v_{j+1}) : j \in [1, m-1]\} \cup \{(u_n, v_j)(u_1, v_{j+1}) : j \in [1, m-1]\} \end{aligned}$$

is the edge set of the categorical product of cycle and path $C_n \times P_m$. So $C_n \times P_m$ is the graph of order nm and size $2n(m-1)$.
On Figure 2, it is depicted the Categorical product of cycle and path $C_7 \times P_5$.

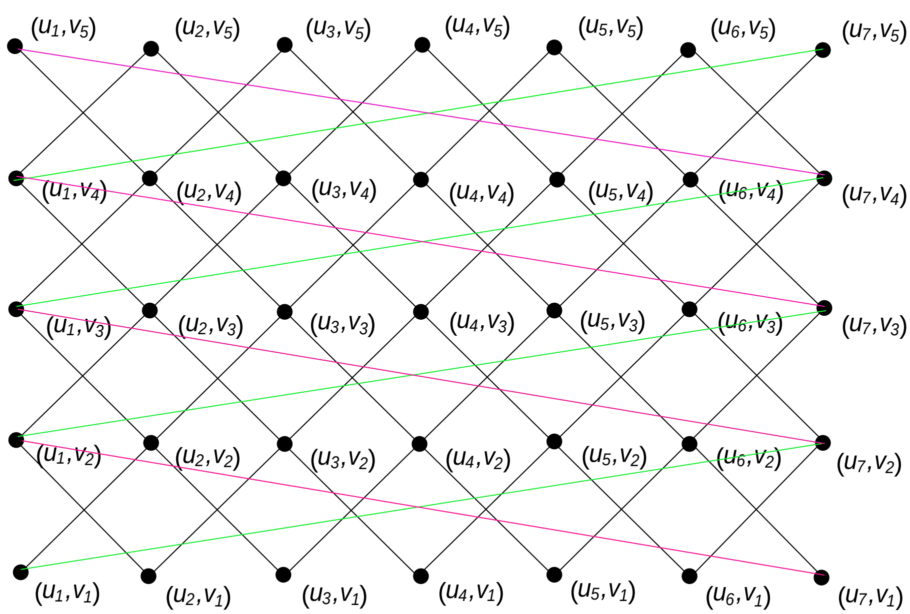

FIGURE 2. Categorical product of cycle and path path $C_7 \times P_5$.

Theorem 3.1. *The zero-sum flow number $F(C_n \times P_m)$ of $C_n \times P_m$ is 2 for all $n \geq 3$, $m \geq 2$.*

Proof. To prove the statement we shall consider the following edge labeling $\varphi : E(C_n \times P_m) \to \{-1, 1\}$.

$$\begin{aligned} \varphi((u_i, v_j)(u_{i+1}, v_{j+1})) &= 1, \quad \text{for } 1 \leq i \leq n-1,\ 1 \leq j \leq m-1 \\ \varphi((u_{i+1}, v_j)(u_i, v_{j+1})) &= -1, \quad \text{for } 1 \leq i \leq n-1,\ 1 \leq j \leq m-1 \\ \varphi((u_1, v_j)(u_n, v_{j+1})) &= -1, \quad \text{for } 1 \leq j \leq m-1 \\ \varphi((u_n, v_j)(u_1, v_{j+1})) &= 1, \quad \text{for } 1 \leq j \leq m-1. \end{aligned}$$

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We can see that φ is an edge labeling from $E(C_n \times P_m)$ to $\{-1,1\}$. Now we show that all the weights(sum of labels of edge incident to a vertex are zero);

$$\begin{aligned}
wt(u_1,v_1) &= \varphi((u_1,v_1)(u_2,v_2)) + \varphi((u_1,v_1)(u_n,v_2)) \\
wt(u_1,v_m) &= \varphi((u_1,v_m)(u_2,v_{m-1})) + \varphi((u_1,v_m)(u_n,v_{m-1})) \\
wt(u_n,v_1) &= \varphi((u_n,v_1)(u_{n-1},v_2)) + \varphi((u_n,v_1)(u_1,v_2)) \\
wt(u_n,v_m) &= \varphi((u_n,v_m)(u_{n-1},v_{m-1})) + \varphi((u_n,v_m)(u_1,v_{m-1})) \\
wt(u_i,v_1) &= \varphi((u_i,v_1)(u_{i+1},v_2)) + \varphi((u_i,v_1)(u_{i-1},v_2)), \quad \text{for } 2 \le i \le n-1 \\
wt(u_i,v_m) &= \varphi((u_i,v_m)(u_{i+1},v_{m-1})) + \varphi((u_i,v_m)(u_{i-1},v_{m-1})), \quad \text{for } 2 \le i \le n-1 \\
wt(u_1,v_j) &= \varphi((u_1,v_j)(u_2,v_{j-1})) + \varphi((u_1,v_j)(u_2,v_{j+1})) + \varphi((u_1,v_j)(u_n,v_{j-1})) \\
&\quad + \varphi((u_1,v_j)(u_n,v_{j+1})), \quad \text{for } 2 \le j \le m-1 \\
wt(u_n,v_j) &= \varphi((u_n,v_j)(u_{n-1},v_{j-1})) + \varphi((u_n,v_j)(u_{n-1},v_{j+1})) + \varphi((u_n,v_j)(u_1,v_{j-1})) \\
&\quad + \varphi((u_n,v_j)(u_1,v_{j+1})), \quad \text{for } 2 \le j \le m-1 \\
wt(u_i,v_j) &= \varphi((u_i,v_j)(u_{i-1},v_{j-1})) + \varphi((u_i,v_j)(u_{i+1},v_{j-1})) + \varphi((u_i,v_j)(u_{i-1},v_{j+1})) \\
&\quad + \varphi((u_i,v_j)(u_{i+1},v_{j+1})), \quad \text{for } 2 \le i \le n-1.\ 2 \le j \le m-1.
\end{aligned}$$

These computations shows that that φ is indeed a zero-sum 2-flow and we get $F(C_n \times P_m) = 2$. This concludes the result. □

4. Zero-Sum Flow Number of categorical product of two cycles

If we consider graph G as the cycle C_n with

$$V(C_n) = \{a_i : 1 \le i \le n\}, \quad E(C_n) = \{a_i a_{i+1} : 1 \le i \le n-1\} \cup \{a_n a_1\}$$

, and graph H as the cycle C_m with

$$V(C_m) = \{b_j : 1 \le j \le m\}, \quad E(C_m) = \{b_j b_{j+1} : 1 \le j \le m-1\} \cup \{b_m b_1\}$$

then

$$V(C_n \times C_m) = \{(a_i, b_j) : 1 \le i \le n,\ 1 \le j \le m\}$$

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is the vertex set of categorical product of two cycles $C_n \times C_m$ and

$$\begin{aligned} E(C_n \times C_m) &= \{(a_i, b_j)(a_{i+1}, b_{j+1}) : 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\} \\ &\cup \{(a_{i+1}, b_j)(a_i, b_{j+1}) : 1 \leq i \leq n-1,\ 1 \leq j \leq m-1\} \\ &\cup \{(a_i, b_1)(a_{i+1}, b_m) : 1 \leq i \leq n-1\} \cup \{(a_{i+1}, b_1)(a_i, b_m) : 1 \leq i \leq n-1\} \\ &\cup \{(a_1, b_j)(a_n, b_{j+1}) : 1 \leq j \leq m-1\} \cup \{(a_n, b_j)(a_1, b_{j+1}) : 1 \leq j \leq m-1\} \\ &\cup \{(a_1, by_1)(a_n, b_m)\} \cup \{(a_n, b_1)(a_1, b_m)\} \end{aligned}$$

is the edge set of categorical product of two cycles $C_n \times C_m$. So, $C_n \times C_m$ is the graph of order nm and size $2nm$.
On Figure 3, it is depicted the Categorical product of two cycles $C_7 \times C_5$..

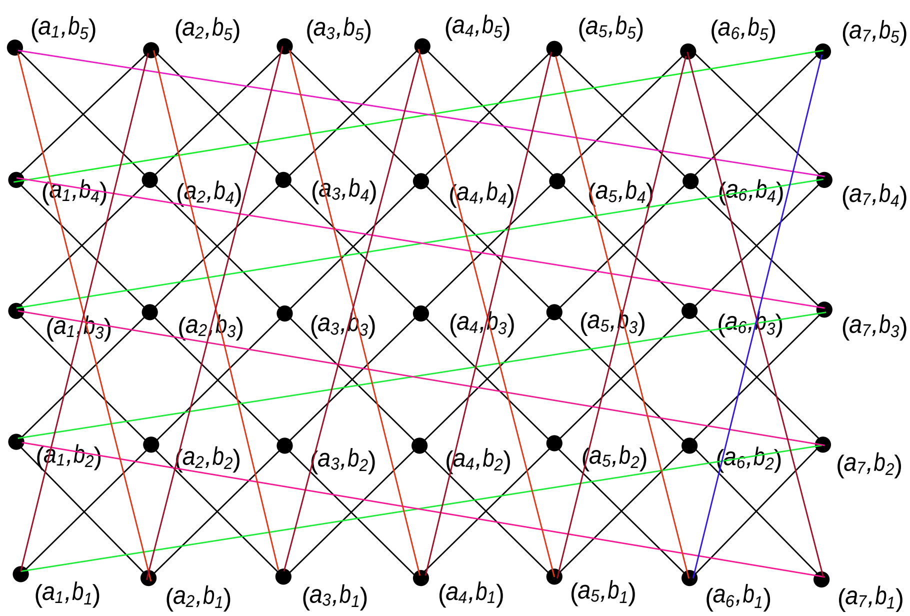

Figure 3. Categorical product of two cycles $C_7 \times C_5$.

Theorem 4.1. *The zero-sum flow number $F(C_n \times C_m)$ of $C_n \times C_m$ is 2 for all $n, m \geq 3$.*

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Proof. To prove the statement we shall consider the following edge labeling $\varphi : E(C_n \times C_m) \to \{-1, 1\}$.

$$\begin{aligned}
\varphi((a_i,b_j)(a_{i+1},b_{j+1})) &= 2, \quad \text{for } 1 \le i \le n-1,\ 1 \le j \le m-1\\
\varphi((a_{i+1},b_j)(a_i,b_{j+1})) &= -2, \quad \text{for } 1 \le i \le n-1,\ 1 \le j \le m-1\\
\varphi((a_1,b_j)(a_n,b_{j+1})) &= -1, \quad \text{for } 1 \le j \le m-1\\
\varphi((a_n,b_j)(a_1,b_{j+1})) &= 1, \quad \text{for } 1 \le j \le m-1.\\
\varphi((a_i,b_1)(a_{i+1},b_m)) &= -1, \quad \text{for } 1 \le i \le n-1\\
\varphi((a_{i+1},b_1)(a_i,b_m)) &= 1, \quad \text{for } 1 \le i \le n-1
\end{aligned}$$

We can see that φ is an edge labeling from $E(C_n \times C_m)$ to $\{-1, 1\}$. Now we show that all the weights(sum of labels of edge incident to a vertex are zero);

$$\begin{aligned}
wt(a_1,b_1) &= \varphi((a_1,b_1)(a_2,b_2)) + \varphi((a_1,b_1)(a_n,b_2)) + \varphi((a_1,b_1)(a_2,b_m))\\
wt(a_1,b_m) &= \varphi((a_1,b_m)(a_2,b_{m-1})) + \varphi((a_1,b_m)(a_n,b_{m-1})) + \varphi((a_1,b_m)(a_2,b_1))\\
wt(a_n,b_1) &= \varphi((a_n,b_1)(a_{n-1},b_2)) + \varphi((a_n,b_1)(a_1,b_2)) + \varphi((a_n,b_1)(a_{n-1},b_m))\\
wt(a_n,b_m) &= \varphi((a_n,b_m)(a_{n-1},b_{m-1})) + \varphi((a_n,b_m)(a_1,b_{m-1})) + \varphi((a_n,b_m)(a_{n-1},b_1))\\
wt(a_i,b_1) &= \varphi((a_i,b_1)(a_{i+1},b_2)) + \varphi((a_i,b_1)(a_{i-1},b_2)) + \varphi((a_i,b_1)(a_{i-1},b_m))\\
&\quad +\varphi((a_i,b_1)(a_{i+1},b_m)), \quad \text{for } 2 \le i \le n-1\\
wt(a_i,b_m) &= \varphi((a_i,b_m)(a_{i+1},b_{m-1})) + \varphi((a_i,b_m)(a_{i-1},b_{m-1})) + \varphi((a_i,b_m)(a_{i-1},b_1))\\
&\quad +\varphi((a_i,b_m)(a_{i+1},b_1)), \quad \text{for } 2 \le i \le n-1\\
wt(a_1,b_j) &= \varphi((a_1,b_j)(a_2,b_{j-1})) + \varphi((a_1,b_j)(a_2,b_{j+1})) + \varphi((a_1,b_j)(a_n,b_{j-1}))\\
&\quad + \varphi((a_1,b_j)(a_n,b_{j+1})), \quad \text{for } 2 \le j \le m-1\\
wt(a_n,b_j) &= \varphi((a_n,b_j)(a_{n-1},b_{j-1})) + \varphi((a_n,b_j)(a_{n-1},b_{j+1})) + \varphi((a_n,b_j)(a_1,b_{j-1}))\\
&\quad + \varphi((a_n,b_j)(a_1,b_{j+1})), \quad \text{for } 2 \le j \le m-1\\
wt(a_i,b_j) &= \varphi((a_i,b_j)(a_{i-1},b_{j-1})) + \varphi((a_i,b_j)(a_{i+1},b_{j-1})) + \varphi((a_i,b_j)(a_{i-1},b_{j+1}))\\
&\quad + \varphi((a_i,b_j)(a_{i+1},b_{j+1})), \quad \text{for } 2 \le i \le n-1,\ 2 \le j \le m-1
\end{aligned}$$

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These computations shows that that φ is indeed a zero-sum 2-flow and we get $F(C_n \times C_m) = 2$. This concludes the result. $\square$

In the next sections, we calculate zero-sum flow number of grids which are obtained by taking the strong product of two graphs.

5. Zero-Sum Flow Number of strong product of two paths

For integers a and b let $[a, b]$ be an interval of integers c, $a \leq c \leq b$. If we consider graph G_1 as the path P_n with

$$V(P_n) = \{x_1, x_2, \ldots, x_n\}, \quad E(P_n) = \{x_i x_{i+1} : i \in [1, n-1]\}$$

and graph G_2 as the path P_m with

$$V(P_m) = \{y_1, y_2, \ldots, y_m\}, \quad E(P_m) = \{y_j y_{j+1} : j \in [1, m-1]\}$$

then

$$V(P_n \boxtimes P_m) = \{(x_i, y_j) : i \in [1, n],\ j \in [1, m]\}$$

is the vertex set and

$$\begin{aligned} E(P_n \boxtimes P_m) &= \{(x_i, y_j)(x_{i+1}, y_j) : i \in [1, n-1],\ j \in [1, m]\} \\ &\cup \{(x_{i+1}, y_j)(x_i, y_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \\ &\cup \{(x_i, y_j)(x_{i+1}, y_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \\ &\cup \{(x_i, y_j)(x_i, y_{j+1}) : i \in [1, n],\ j \in [1, m-1]\} \end{aligned}$$

is the edge set of $P_n \boxtimes P_m$.

Also it is easy to check that $|V(P_n \boxtimes P_m)| = mn$ and $|E(P_n \boxtimes P_m)| = 4mn - 3m - 3n + 2$. On Figure 4, it is depicted the Strong product of two paths $P_7 \boxtimes P_5$.

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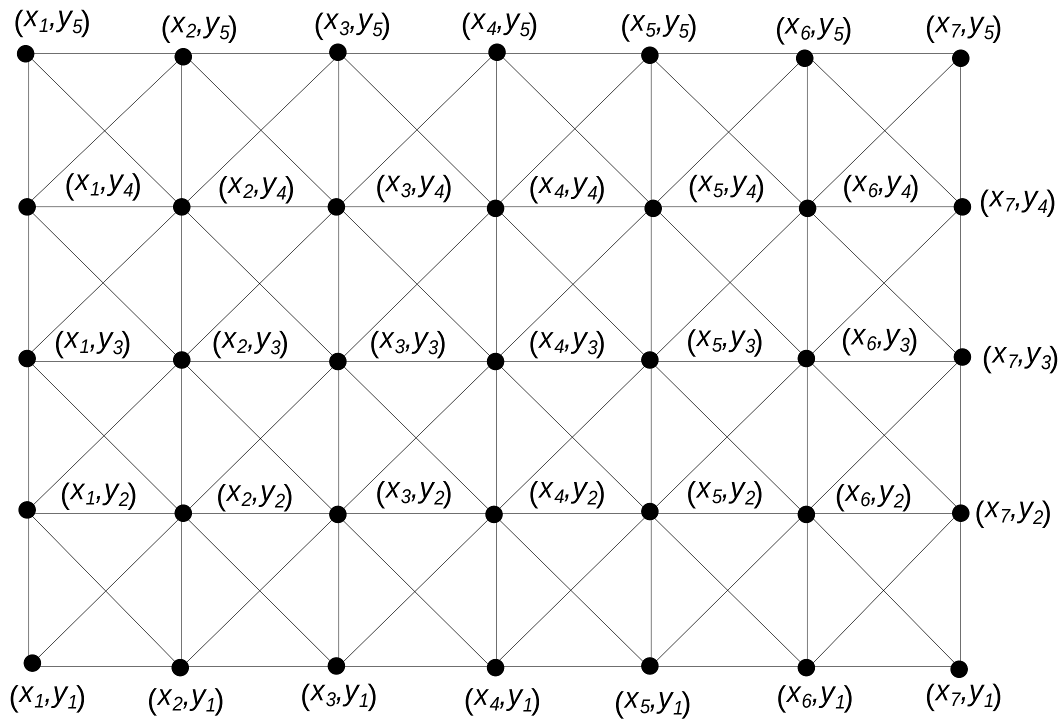

FIGURE 4. Strong product of two paths $P_7 \boxtimes P_5$.

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Theorem 5.1. *The zero-sum flow number $F(P_n \boxtimes P_m)$ of $P_n \boxtimes P_m$ is 3 for all $n, m \geq 2$.*

Proof. Note that there are $2n + 2m - 8$ vertices of degree 5, $mn - 2n - 2m + 4$ vertices of degree 8 and 4 vertices of degree 3 in $P_n \boxtimes P_m$, so a zero-sum flow edge assignment from $\{-1, 1\}$ is not possible. Therefore $F(P_n \boxtimes P_m) \geq 3$. To prove the converse inequality we will consider the following edge labeling $\varphi : E(P_n \boxtimes P_m) \to \{-2, 1, 2\}$.

$$\varphi((x_i, y_j)(x_{i+1}, y_j)) = \begin{cases} 1, & j = 1,\ m,\ 1 \leq i \leq n-1 \\ 2, & 2 \leq j \leq m-1,\ 1 \leq i \leq n-1 \end{cases}$$

$$\varphi((x_i, y_j)(x_i, y_{j+1})) = \begin{cases} 1, & i = 1,\ n,\ 1 \leq j \leq m-1 \\ 2, & 2 \leq i \leq m-1,\ 1 \leq j \leq m-1 \end{cases}$$

$$\varphi((x_{i+1}, y_j)(x_i, y_{j+1})) = \varphi((x_i, y_j)(x_{i+1}, y_{j+1})) = -2, \quad \text{for } 1 \leq j \leq m-1,\ 1 \leq i \leq n-1.$$

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We can see that φ is an edge labeling from $E(P_n \boxtimes P_m)$ to $\{-2, 1, 2\}$. Lets compute all the weights;

$$\begin{aligned} wt(x_1, y_1) &= \varphi((x_1, y_1)(x_2, y_1)) + \varphi((x_1, y_1)(x_1, y_2)) + \varphi((x_1, y_1)(x_2, y_2)) \\ wt(x_1, y_m) &= \varphi((x_1, y_m)(x_1, y_{m-1})) + \varphi((x_1, y_m)(x_2, y_m)) + \varphi((x_1, y_m)(x_2, y_{m-1})) \\ wt(x_n, y_1) &= \varphi((x_n, y_1)(x_{n-1}, y_1)) + \varphi((x_n, y_1)(x_n, y_2)) + \varphi((x_n, y_1)(x_{n-1}, y_2)) \\ wt(x_n, y_m) &= \varphi((x_n, y_m)(x_n, y_{m-1})) + \varphi((x_n, y_m)(x_{n-1}, y_m)) + \varphi((x_n, y_m)(x_{n-1}, y_{m-1})) \end{aligned}$$

$$\begin{aligned} wt(x_i, y_1) &= \varphi((x_i, y_1)(x_{i-1}, y_1)) + \varphi((x_i, y_1)(x_{i+1}, y_1)) + \varphi((x_i, y_1)(x_i, y_2)) \\ &\quad +\varphi((x_i, y_1)(x_{i-1}, y_2)) + \varphi((x_i, y_1)(x_{i+1}, y_2)), \quad \text{for } 2 \leq i \leq n-1. \\ wt(x_i, y_m) &= \varphi((x_i, y_m)(x_{i-1}, y_m)) + \varphi((x_i, y_m)(x_{i+1}, y_m)) + \varphi((x_i, y_m)(x_i, y_{m-1})) \\ &\quad +\varphi((x_i, y_m)(x_{i-1}, y_{m-1})) + \varphi((x_i, y_m)(x_{i+1}, y_{m-1})), \quad \text{for } 2 \leq i \leq n-1. \end{aligned}$$

$$\begin{aligned} wt(x_1, y_j) &= \varphi((x_1, y_j)(x_1, y_{j-1})) + \varphi((x_1, y_j)(x_1, y_{j+1})) + \varphi((x_1, y_j)(x_2, y_j)) \\ &\quad +\varphi((x_1, y_j)(x_2, y_{j-1})) + \varphi((x_1, y_j)(x_2, y_{j+1})), \quad \text{for } 2 \leq j \leq m-1. \\ wt(x_n, y_j) &= \varphi((x_n, y_j)(x_n, y_{j-1})) + \varphi((x_n, y_j)(x_n, y_{j+1})) + \varphi((x_n, y_j)(x_{n-1}, y_j)) \\ &\quad +\varphi((x_n, y_j)(x_{n-1}, y_{j-1})) + \varphi((x_n, y_j)(x_{n-1}, y_{j+1})), \quad \text{for } 2 \leq j \leq m-1. \end{aligned}$$

For $2 \leq i \leq n-1$ and $2 \leq j \leq m-1$,

$$\begin{aligned} wt(x_i, y_j) &= \varphi((x_i, y_j)(x_{i+1}, y_j)) + \varphi((x_i, y_j)(x_{i-1}, y_j)) + \varphi((x_i, y_j)(x_i, y_{j+1})) \\ &\quad +\varphi((x_i, y_j)(x_i, y_{j-1})) + \varphi((x_i, y_j)(x_{i-1}, y_{j-1})) + \varphi((x_i, y_j)(x_{i+1}, y_{j+1})) \\ &\quad +\varphi((x_i, y_j)(x_{i-1}, y_{j+1})) + \varphi((x_i, y_j)(x_{i+1}, y_{j-1})) \end{aligned}$$

These computations shows that that φ is indeed a zero-sum 3-flow and we get $F(P_n \boxtimes P_m) \leq 3$. This concludes the result. □

6. Zero-Sum Flow Number of strong product of cycle and path

If we consider graph G_1 as a cycle C_n with

$$V(C_n) = \{u_i : i \in [1, n]\}, \;\; E(C_n) = \{u_i u_{i+1} : i \in [1, n-1]\} \cup \{u_n u_1\}$$

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and graph G_2 as a path P_m with

$$V(P_m) = \{v_j : j \in [1, m]\}, \quad E(P_m) = \{v_j v_{j+1} : j \in [1, m-1]\}$$

then

$$V(C_n \boxtimes P_m) = \{(u_i, v_j) : i \in [1, n],\ j \in [1, m]\}$$

is the vertex set and

$$\begin{aligned} E(C_n \boxtimes P_m) &= \{(u_i, v_j)(u_{i+1}, v_j) : i \in [1, n-1],\ j \in [1, m]\} \\ &\cup \{(u_{i+1}, v_j)(u_i, v_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \\ &\cup \{(u_i, v_j)(u_{i+1}, v_{j+1}) : i \in [1, n-1],\ j \in [1, m-1]\} \\ &\cup \{(u_i, v_j)(u_i, v_{j+1}) : i \in [1, n],\ j \in [1, m-1]\} \\ &\cup \{(u_1, v_j)(u_n, v_{j+1}) : j \in [1, m-1]\} \cup \{(u_n, v_j)(u_1, v_{j+1}) : j \in [1, m-1]\} \\ &\cup \{(u_1, v_j)(u_n, v_j) : j \in [1, m]\} \end{aligned}$$

is the edge set of $C_n \boxtimes P_m$. Thus $|V(C_n \boxtimes P_m)| = nm$ and $|E(C_n \boxtimes P_m)| = n(4m-3)$.
On Figure 5, it is depicted the Strong product of cycle and path $C_7 \boxtimes P_5$.

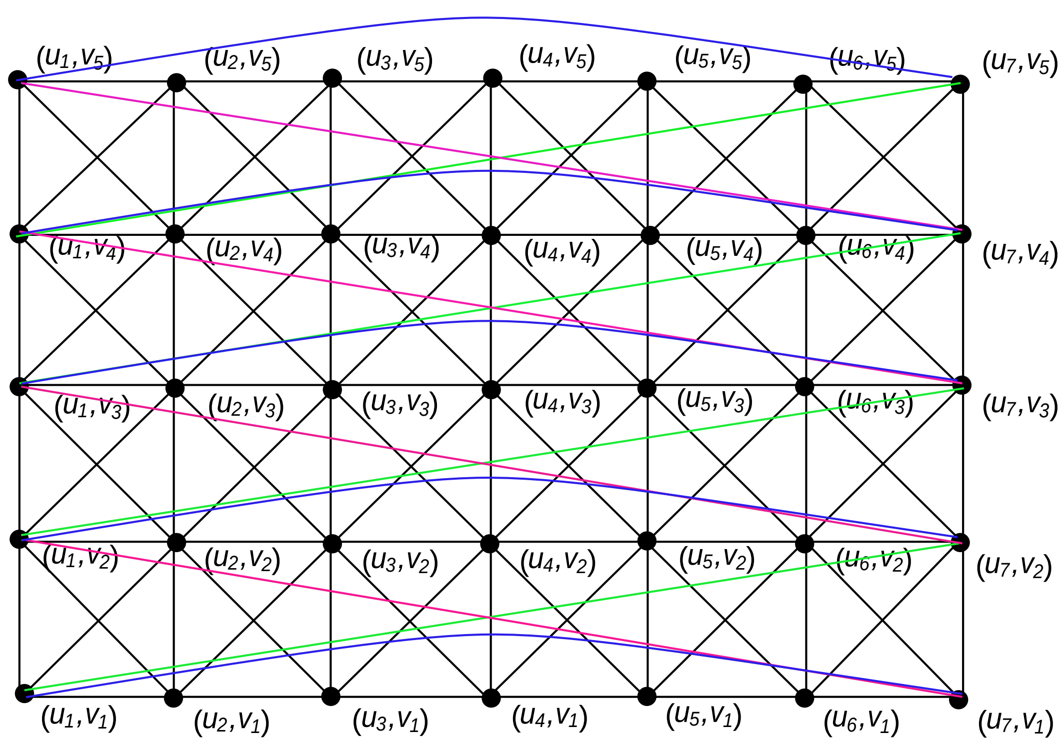

Figure 5. Strong product of cycle and path $C_7 \boxtimes P_5$.

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Theorem 6.1. *The zero-sum flow number $F(C_n \boxtimes P_m)$ of $C_n \boxtimes P_m$ is 3 for all $n \geq 3,\ \geq 2$.*

Proof. Since there are $2n - 2n$ vertices of degree 8 and $2n$ vertices of degree 5 in $C_n \boxtimes P_m$, so a zero-sum flow edge assignment from $\{-1, 1\}$ is not possible. Therefore $F(C_n \boxtimes P_m) \geq 3$. To prove that $F(C_n \boxtimes P_m) \leq 3$ we will consider the following edge labeling $\varphi : E(C_n \boxtimes P_m) \rightarrow \{-2, -1, 1, 2\}$.

$$\varphi((u_i, v_j)(u_{i+1}, v_j)) = \begin{cases} 1, & j = 1,\ m,\ 1 \leq i \leq n-1 \\ 2, & 2 \leq j \leq m-1,\ 1 \leq i \leq n-1 \end{cases}$$

$$\varphi((u_i, v_j)(u_i, v_{j+1})) = \begin{cases} 1, & i = 1,\ n,\ 1 \leq j \leq m-1 \\ 2, & 2 \leq i \leq m-1,\ 1 \leq j \leq m-1 \end{cases}$$

$$\varphi((u_{i+1}, v_j)(u_i, v_{j+1})) = \varphi((u_i, v_j)(u_{i+1}, v_{j+1})) = -2, \quad \text{for } 1 \leq j \leq m-1,\ 1 \leq i \leq n-1,$$

$$\varphi((u_1, v_j)(u_n, v_{j+1})) = \varphi((u_n, v_j)(u_1, v_{j+1})) = 1, \quad \text{for } 1 \leq j \leq m-1$$

$$\varphi((u_1, v_j)(u_n, v_j)) = \begin{cases} -1, & j = 1,\ m \\ -2, & 2 \leq j \leq m-1 \end{cases}$$

We can see that φ is an edge labeling from $E(C_n \boxtimes P_m)$ to $\{-2, -1, 1, 2\}$. Lets compute all the weights;

$$\begin{aligned}
wt(u_1, v_1) &= \varphi((u_1, v_1)(u_2, v_1)) + \varphi((u_1, v_1)(u_1, v_2)) + \varphi((u_1, v_1)(u_2, v_2)) \\
&\quad + \varphi((u_1, v_1)(u_n, v_1)) + \varphi((u_1, v_1)(u_n, v_2)) \\
wt(u_1, v_m) &= \varphi((u_1, v_m)(u_1, v_{m-1})) + \varphi((u_1, v_m)(u_2, v_m)) + \varphi((u_1, v_m)(u_2, v_{m-1})) \\
&\quad + \varphi((u_1, v_m)(u_n, v_m)) + \varphi((u_1, v_m)(u_n, v_{m-1})) \\
wt(u_n, v_1) &= \varphi((u_n, v_1)(u_{n-1}, v_1)) + \varphi((u_n, v_1)(u_n, v_2)) + \varphi((u_n, v_1)(u_{n-1}, v_2)) \\
&\quad + \varphi((u_n, v_1)(u_1, v_1)) + \varphi((u_n, v_1)(u_1, v_2)) \\
wt(u_n, v_m) &= \varphi((u_n, v_m)(u_n, v_{m-1})) + \varphi((u_n, v_m)(u_{n-1}, v_m)) + \varphi((u_n, v_m)(u_{n-1}, v_{m-1})) \\
&\quad + \varphi((u_n, v_m)(u_1, v_m)) + \varphi((u_n, v_m)(u_1, v_{m-1})).
\end{aligned}$$

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$$\begin{aligned} wt(u_i,v_1) &= \varphi((u_i,v_1)(u_{i-1},v_1))+\varphi((u_i,v_1)(u_{i+1},v_1))+\varphi((u_i,v_1)(u_i,v_2)) \\ &\quad +\varphi((u_i,v_1)(u_{i-1},v_2))+\varphi((u_i,v_1)(u_{i+1},v_2)), \quad \text{for } 2\leq i\leq n-1. \\ wt(u_i,v_m) &= \varphi((u_i,v_m)(u_{i-1},v_m))+\varphi((u_i,v_m)(u_{i+1},v_m))+\varphi((u_i,v_m)(u_i,v_{m-1})) \\ &\quad +\varphi((u_i,v_m)(u_{i-1},v_{m-1}))+\varphi((u_i,v_m)(u_{i+1},v_{m-1})), \quad \text{for } 2\leq i\leq n-1. \end{aligned}$$

$$\begin{aligned} wt(u_1,v_j) &= \varphi((u_1,v_j)(u_1,v_{j-1}))+\varphi((u_1,v_j)(u_1,v_{j+1}))+\varphi((u_1,v_j)(u_2,v_j)) \\ &\quad +\varphi((u_1,v_j)(u_2,v_{j-1}))+\varphi((u_1,v_j)(u_2,v_{j+1}))+\varphi((u_1,v_j)(u_n,v_{j-1})) \\ &\quad +\varphi((u_1,v_j)(u_n,v_{j+1}))+\varphi((u_1,v_j)(u_n,v_j)), \quad \text{for } 2\leq j\leq m-1. \\ wt(u_n,v_j) &= \varphi((u_n,v_j)(u_n,v_{j-1}))+\varphi((u_n,v_j)(u_n,v_{j+1}))+\varphi((u_n,v_j)(u_{n-1},v_j)) \\ &\quad +\varphi((u_n,v_j)(u_{n-1},v_{j-1}))+\varphi((u_n,v_j)(u_{n-1},v_{j+1}))+\varphi((u_n,v_j)(u_1,v_{j-1})) \\ &\quad +\varphi((u_i,v_j)(u_1,v_{j+1}))+\varphi((u_i,v_j)(u_1,v_j)), \quad \text{for } 2\leq j\leq m-1. \end{aligned}$$

For $2\leq i\leq n-1$ and $2\leq j\leq m-1$,

$$\begin{aligned} wt(u_i,v_j) &= \varphi((u_i,v_j)(u_{i+1},v_j))+\varphi((u_i,v_j)(u_{i-1},v_j))+\varphi((u_i,v_j)(u_i,v_{j+1})) \\ &\quad +\varphi((u_i,v_j)(u_i,v_{j-1}))+\varphi((u_i,v_j)(u_{i-1},v_{j-1}))+\varphi((u_i,v_j)(u_{i+1},v_{j+1})). \\ &\quad +\varphi((u_i,v_j)(u_{i-1},v_{j+1}))+\varphi((u_i,v_j)(u_{i+1},v_{j-1})). \end{aligned}$$

These computations shows that that φ is indeed a zero-sum 3-flow and we get $F(C_n\boxtimes P_m)\leq 3$. This concludes the result. □

7. Zero-Sum Flow Number of strong product of two cycles

If we consider graph G_1 as a cycle C_n with

$$V(C_n)=\{a_i : i\in[1,n]\},\ \ E(C_n)=\{a_ia_{i+1} : i\in[1,n-1]\}\cup\{a_na_1\}$$

and graph G_2 as a cycle C_m with

$$V(C_m)=\{b_j : j\in[1,m]\},\ \ E(C_m)=\{b_jb_{j+1} : j\in[1,m-1]\}\cup\{b_mb_1\}$$

then

$$V(C_n\boxtimes C_m)=\{(a_i,b_j) : i\in[1,n],\ j\in[1,m]\}$$

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is the vertex set and

$$\begin{aligned}
E(C_n \boxtimes C_m) &= \{(a_i,b_j)(a_{i+1},b_j) : i \in [1,n-1],\ j \in [1,m]\} \\
&\cup \{(a_n,b_j)(a_1,b_j) : j \in [1,m]\} \cup \{(a_i,b_j)(a_i,b_{j+1}) : i \in [1,n],\ j \in [1,m-1]\} \\
&\cup \{(a_i,b_m)(a_i,b_1) : i \in [1,n]\} \cup \{(a_i,b_1)(a_{i+1},b_m) : i \in [1,n-1]\} \\
&\cup \{(a_n,b_1)(a_1,b_m)\} \cup \{(a_i,b_m)(a_{i+1},b_1) : i \in [1,n-1]\} \cup \{(a_n,b_m)(a_1,b_1)\} \\
&\cup \{(a_i,b_j)(a_{i+1},b_{j+1}) : i \in [1,n-1],\ j \in [1,m-1]\} \\
&\cup \{(a_n,b_j)(a_1,b_{j+1}) : j \in [1,m-1]\} \\
&\cup \{(a_i,b_{j+1})(a_{i+1},b_j) : i \in [1,n-1],\ j \in [1,m-1]\} \\
&\cup \{(a_n,b_{j+1})(a_1,b_j) : j \in [1,m-1]\}
\end{aligned}$$

is the edge set of $C_n \boxtimes C_m$. So, $C_n \boxtimes C_m$ is the graph of order mn and size $4mn$.
On Figure 6, it is depicted the Strong product of two cycles $C_7 \boxtimes C_5$.

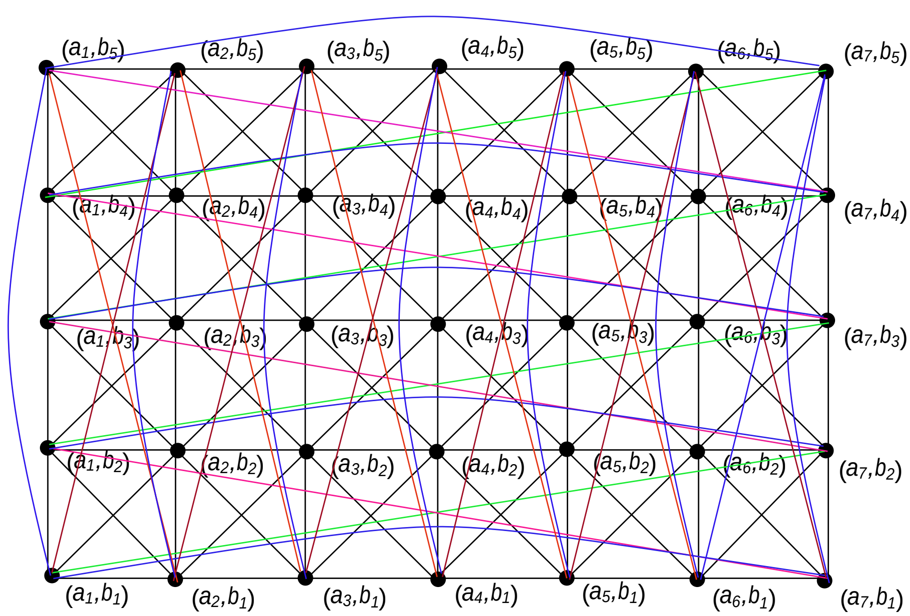

FIGURE 6. Strong product of two cycles $C_7 \boxtimes C_5$.

Theorem 7.1. *The zero-sum flow number $F(C_n \boxtimes C_m)$ of $C_n \boxtimes C_m$ is 3 for all $n, m \geq 3$.*

Proof. Note that there are $2n-4$ vertices of degree 8 and 4 vertices of degree 7 in $C_n \boxtimes C_m$, so a zero-sum flow edge assignment from $\{-1,1\}$ is not possible. Therefore $F(C_n \boxtimes C_m) \geq 3$. To prove that $F(C_n \boxtimes C_m) \leq 3$ we will consider the following edge labeling $\varphi : E(C_n \boxtimes C_m) \to \{-2,-1,1,2\}$.

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$$\varphi((a_i,b_j)(a_{i+1},b_j)) = \begin{cases} 1, & j=1,\ m,\ 1\leq i\leq n-1 \\ 2, & 2\leq j\leq m-1,\ 1\leq i\leq n-1 \end{cases}$$

$$\varphi((a_i,b_j)(a_i,b_{j+1})) = \begin{cases} 1, & i=1,\ n,\ 1\leq j\leq m-1 \\ 2, & 2\leq i\leq m-1,\ 1\leq j\leq m-1 \end{cases}$$

$$\varphi((a_{i+1},b_j)(a_i,b_{j+1})) = \varphi((a_i,b_j)(a_{i+1},b_{j+1})) = -2, \quad \text{for } 1\leq j\leq m-1,\ 1\leq i\leq n-1,$$

$$\varphi((a_1,b_j)(a_n,b_{j+1})) = \varphi((a_n,b_j)(a_1,b_{j+1})) = 1, \quad \text{for } 1\leq j\leq m-1$$

$$\varphi((a_1,b_j)(a_n,b_j)) = \begin{cases} -1, & j=1,\ m \\ -2, & 2\leq j\leq m-1 \end{cases}$$

$$\varphi((a_i,b_1)(a_{i+1},b_m)) = \varphi((a_{i+1},b_1)(a_i,b_m)) = 1, \quad \text{for } 1\leq i\leq m-1$$

$$\varphi((a_i,b_1)(a_i,b_m)) = \begin{cases} -1, & i=1,\ m \\ -2, & 2\leq i\leq n-1 \end{cases}$$

We can see that φ is an edge labeling from $E(C_n \boxtimes C_m)$ to $\{-2,-1,1,2\}$. Lets compute all the weights;

$$\begin{aligned}
wt(a_1,b_1) &= \varphi((a_1,b_1)(a_2,b_1)) + \varphi((a_1,b_1)(a_1,b_2)) + \varphi((a_1,b_1)(a_2,b_2)) \\
&\quad + \varphi((a_1,b_1)(a_n,b_1)) + \varphi((a_1,b_1)(a_n,b_2)) + \varphi((a_1,b_1)(a_1,b_m)) \\
&\quad + \varphi((a_1,b_1)(a_2,b_m)), \\
wt(a_1,b_m) &= \varphi((a_1,b_m)(a_1,b_{m-1})) + \varphi((a_1,b_m)(a_2,b_m)) + \varphi((a_1,b_m)(a_2,b_{m-1})) \\
&\quad + \varphi((a_1,b_m)(a_n,b_m)) + \varphi((a_1,b_m)(a_n,b_{m-1})) + \varphi((a_1,b_1)(a_1,b_m)) \\
&\quad + \varphi((a_2,b_1)(a_1,b_m)), \\
wt(a_n,b_1) &= \varphi((a_n,b_1)(a_{n-1},b_1)) + \varphi((a_n,b_1)(a_n,b_2)) + \varphi((a_n,b_1)(a_{n-1},b_2)) \\
&\quad + \varphi((a_n,b_1)(a_1,b_1)) + \varphi((a_n,b_1)(a_1,b_2)) + \varphi((a_n,b_1)(a_n,b_m)) \\
&\quad + \varphi((a_n,b_1)(a_{n-1},b_m)), \\
wt(a_n,b_m) &= \varphi((a_n,b_m)(a_n,b_{m-1})) + \varphi((a_n,b_m)(a_{n-1},b_m)) + \varphi((a_n,b_m)(a_{n-1},b_{m-1})) \\
&\quad + \varphi((a_n,b_m)(a_1,b_m)) + \varphi((a_n,b_m)(a_1,b_{m-1})) + \varphi((a_n,b_1)(a_n,b_m)) \\
&\quad + \varphi((a_n,b_m)(a_{n-1},b_1))
\end{aligned}$$

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$$
\begin{aligned}
wt(a_i,b_1) &= \varphi((a_i,b_1)(a_{i-1},b_1)) + \varphi((a_i,b_1)(a_{i+1},b_1)) + \varphi((a_i,b_1)(a_i,b_2)) \\
&\quad +\varphi((a_i,b_1)(a_{i-1},b_2)) + \varphi((a_i,b_1)(a_{i+1},b_2)) + \varphi((a_i,b_1)(a_i,b_m)) \\
&\quad +\varphi((a_i,b_1)(a_{i-1},b_m)) + \varphi((a_i,b_1)(a_{i+1},b_m)), \quad \text{for } 2 \le i \le n-1, \\
wt(a_i,b_m) &= \varphi((a_i,b_m)(a_{i-1},b_m)) + \varphi((a_i,b_m)(a_{i+1},b_m)) + \varphi((a_i,b_m)(a_i,b_{m-1})) \\
&\quad +\varphi((a_i,b_m)(a_{i-1},b_{m-1})) + \varphi((a_i,b_m)(a_{i+1},b_{m-1})) + \varphi((a_i,b_1)(a_i,b_m)) \\
&\quad +\varphi((a_i,b_m)(a_{i-1},b_1)) + \varphi((a_i,b_m)(a_{i+1},b_1)), \quad \text{for } 2 \le i \le n-1 \\
wt(a_1,b_j) &= \varphi((a_1,b_j)(a_1,b_{j-1})) + \varphi((a_1,b_j)(a_1,b_{j+1})) + \varphi((a_1,b_j)(a_2,b_j)) \\
&\quad +\varphi((a_1,b_j)(a_2,b_{j-1})) + \varphi((a_1,b_j)(a_2,b_{j+1})) + \varphi((a_1,b_j)(a_n,b_{j-1})) \\
&\quad +\varphi((a_1,b_j)(a_n,b_{j+1})) + \varphi((a_1,b_j)(a_n,b_j)), \quad \text{for } 2 \le j \le m-1, \\
wt(a_n,b_j) &= \varphi((a_n,b_j)(a_n,b_{j-1})) + \varphi((a_n,b_j)(a_n,b_{j+1})) + \varphi((a_n,b_j)(a_{n-1},b_j)) \\
&\quad +\varphi((a_n,b_j)(a_{n-1},b_{j-1})) + \varphi((a_n,b_j)(a_{n-1},b_{j+1})) + \varphi((a_n,b_j)(a_1,b_{j-1})) \\
&\quad +\varphi((a_i,b_j)(a_1,b_{j+1})) + \varphi((a_i,b_j)(a_1,b_j)), \quad \text{for } 2 \le j \le m-1.
\end{aligned}
$$

Finally, for $2 \le i \le n-1$ and $2 \le j \le m-1$

$$
\begin{aligned}
wt(a_i,b_j) &= \varphi((a_i,b_j)(a_{i+1},b_j)) + \varphi((a_i,b_j)(a_{i-1},b_j)) + \varphi((a_i,b_j)(a_i,b_{j+1})) \\
&\quad +\varphi((a_i,b_j)(a_i,b_{j-1})) + \varphi((a_i,b_j)(a_{i-1},b_{j-1})) + \varphi((a_i,b_j)(a_{i+1},b_{j+1})). \\
&\quad +\varphi((a_i,b_j)(a_{i-1},b_{j+1})) + \varphi((a_i,b_j)(a_{i+1},b_{j-1})).
\end{aligned}
$$

These computations shows that that φ is indeed a zero-sum 3-flow and we get $F(C_n \boxtimes C_m) \le 3$. This concludes the result. ☐

8. **Concluding Remark**

In this paper we gave complete zero-sum flow and zero sum numbers for categorical and strong product of two graphs namely cycle and paths. In future we are interested to find the zero-sum flow number for the other graph using categorical and strong product. So we conclude the paper with the following open problems.

Open Problem.1.

Fine the exact value of the zero-sum flow number for the categorical product of path P_n and cycle C_n with complete graph K_n.

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Open Problem.2.

Fine the exact value of the zero-sum flow number for the strong product of path P_n and cycle C_n with complete graph K_n.

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