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THE VERTEX STEINER NUMBER OF A GRAPH

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Abstract. Let x be a vertex of a connected graph G and $W \subset V(G)$ such that $x \notin W$. Then W is called an x-Steiner set of G if $W \cup \{x\}$ is a Steiner set of G. The minimum cardinality of an *x-Steiner set* of G is defined as *x-Steiner number* of G and denoted by $s_x(G)$. Some general properties satisfied by these concepts are studied. The *x-Steiner numbers* of certain classes of graphs are determined. Connected graphs of order p with x-Steiner number 1 or $p-1$ are characterized. It is shown that for every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $s(G) = a$ and $s_x(G) = b$ for some vertex x in G, where $s(G)$ is the *Steiner number* of a graph.

1. Introduction

By a *graph* $G = (V,E)$, we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1]. The *degree* of the vertex v is the number of edges incident at v and it is denoted by $deg_G(v)$. If $deg_G(v) = 1$, then v is called an *end vertex* of G. For a non empty vertex subset $W \subset V(G)$ of a graph G, an induced subgraph of W in G, denoted by $G[W]$, is the subgraph of G, with vertex set $V(G[W]) = W$ and edge set $E(G[W]) = \{uv \in E(G) : u, v \in W\}$.A vertex v is called a *simplicial vertex* of a graph G if the subgraph induced by its neighbors is complete.

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an *u-v geodesic*. It is known that the distance is a metric on the vertex set of G. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G

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is the *radius*, and denoted by $\text{rad}(G)$ and the maximum eccentricity is its *diameter*, and denoted by $\text{diam}(G)$. If $d(u,v) = \text{diam}(G)$, then u and v are called *antipodal vertices* of G and the path u-v is called a *diametral path* of G. For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree* with respect to W or a *Steiner W-tree*. It is to be noted that $d(W) = d(u,v)$ when $W = \{u,v\}$. If v is an end vertex of a *Steiner W-tree*, then $v \in W$. Also if $G[W]$ is connected, then any *Steiner W-tree* contains the elements of W only. The Steiner distance of a graph is introduced in [3]. The set of all vertices of G that lie on some *Steiner W-tree* is denoted by $S(W)$. If $S(W) = V(G)$, then W is called a *Steiner set* of G. A *Steiner set* of minimum cardinality is a minimum *Steiner set* or simply a *s-set* of G and this cardinality is the *Steiner number* $s(G)$ of G. If W is a Steiner set of G and $x \notin W$, then $W \cup \{x\}$ need not be a Steiner set of G. The *Steiner number* of a graph was introduced in [4] and further studied in [2, 5] and [9, 10, 11, 12, 13] and [15, 16] and [18, 19, 20, 21, 22].

The Steiner tree problem in networks, and particularly in graphs, was formulated in 1971-by Hakimi [7] and Levi [14]. In the case of an unweighted, undirected graph, this problem consists of finding, for a subset of vertices W, a minimal-size connected subgraph that contains the vertices in W. The computational side of this problem has been widely studied, and it is known that it is an NP-hard problem for general graphs [6]. Steiner trees have application to multiprocessor computer networks [8, 23]. For example, it may be desired to connect a certain set of processors with a subnetwork that uses the least number of communication links. A Steiner tree for the vertices, corresponding to the processors that need to be connected, corresponds to such a desired subnetwork. Steiner distance has application to multiprocessor communication. For example, suppose the primary requirement when communicating a message from a processor x to a collection W of other processors is to minimize the number of communication links that are used. Then a Steiner tree for $W \cup \{x\}$ is an optimal way of connecting these vertices. This motivated us to define a new parameter the vertex Steiner number of a graph. Throughout the following G denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

Theorem 1.1. [4] *Each simplicial vertex of a graph G belongs to every Steiner set of G. In particular, each end-vertex of G belongs to every Steiner set of G.*

Theorem 1.2. [4] *Every non-trivial tree with exactly k end-vertices has Steiner number k.*

Theorem 1.3. [4] *For the complete bipartite graph $G = K_{m,n}$ $(2 \le m \le n)$, $s(G) = m$.*

2. The Vertex Steiner Number of Graph

Definition 2.1. *Let x be a vertex of a connected graph G and $W \subset V(G)$ such that $x \notin W$. Then W is called an x-Steiner set of G if $W \cup \{x\}$ is a Steiner set of G. The minimum cardinality of an x-Steiner set of G is defined as the x-Steiner number of G and denoted by $s_x(G)$. Any x -Steiner set of cardinality $s_x(G)$ is called an s_x -set of G.*

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Note 2.2. *Hereafter, we denote any Steiner $W \cup \{x\}$-tree of G as a Steiner W_x-tree of G.*

Example 2.3. *For the graph G in Figure 1, the minimum vertex Steiner sets and the vertex Steiner numbers are given in Table 1.*

TABLE 1.

vertex x	s_x-sets	$s_x(G)$
v_1	$\{v_5, v_7\}$,$\{v_5, v_6\}$	2
v_2	$\{v_1, v_5, v_7\}$,$\{v_1, v_5, v_6\}$	3
v_3	$\{v_1, v_5, v_6, v_7\}$	4
v_4	$\{v_1, v_5, v_7\}$, $\{v_1, v_5, v_6\}$	3
v_5	$\{v_1, v_7\}$, $\{v_1, v_6\}$	2
v_6	$\{v_1, v_5\}$	2
v_7	$\{v_1, v_5\}$	2

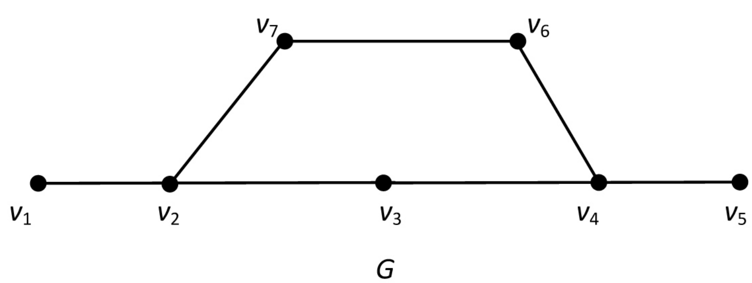

FIGURE 1.

Theorem 2.4. *Every simplicial vertex of G other than the vertex x (whether x is simplicial or not) belongs to every x-Steiner set for any vertex x in G.*

Proof. Let x be a vertex of G. By way of contradiction, suppose G contains a simplicial vertex $v \neq x$ and an x-Steiner set W such that $v \notin W$. Since W is an x-Steiner set of G, the vertex v lies on a Steiner W_x - tree T so that $W \cup \{x\} \subseteq V(T)$ and $v \in V(T)$. Let $deg_T(v) = k$. If $k = 1$, then v is an end vertex of T, it follows that $v \in W$, which is a contradiction. If $k \geq 2$, let $N_T(v) = \{u_1, u_2, \ldots, u_k\}$ be neighborhood of v in T. Since v is a simplicial vertex, it follows that $u_i u_j \in E(G)$ for all i, j with $1 \leq i, j \leq k$ and $i \neq j$. Let $T^{'}$ be the tree in G obtained from T by deleting the vertex v and adding $k-1$ edges $u_i u_{i+1}$ $(1 \leq i \leq k-1)$. Then $W \cup \{x\} \subseteq V(T^{'})$ and $|V(T^{'})| = |V(T)| - 1$, which contradicts the fact that T a Steiner W_x-tree of G. Thus $v \in W$. □

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Corollary 2.5. *For the complete graph K_p $(p \geq 2)$, $s_x(K_p) = p - 1$ for every vertex x in G.*

Corollary 2.6. *For the nontrivial tree T with k end vertices,*

$$s_x(T) = \begin{cases} k & \text{if } x \text{ is a cut vertex of } G \\ k-1 & \text{if } x \text{ is an end vertex of } G. \end{cases}$$

Theorem 2.7. *Let x be a vertex of a connected graph G with v a cut-vertex of G and W an x-Steiner set of G.*
(i) If $x = v$, then every component of $G - v$ contains an element of W.
(ii) If $x \neq v$, then for each component C of $G - v$ with $x \notin C$, $W \cap C \neq \emptyset$.

Proof. Let v be a cut-vertex of G, x a vertex of G and W an x-Steiner set of G.

(i) Let $x = v$. Suppose there exists a component, say G_1 of $G - v$ such that G_1 contains no vertex of W. By Theorem 2.4, W contains all the simplicial vertices of G and hence it follows that G_1 does not contain any simplicial vertex of G. Thus G_1 contains at least one edge, say yz. Since every Steiner W_x-tree T must have its end-vertex in W and v is a cut-vertex of G, it is clear that no Steiner W_x-tree would contain the vertices y and z. This contradicts the fact that W is an x-Steiner set of G.

(ii) Let $x \neq v$. Suppose there exists a component, say C of $G - v$ with $x \notin C$ such that $W \cap C = \emptyset$. Then proceeding as in (i), we get a contradiction. □

Corollary 2.8. *If v is a cut-vertex of a connected graph G and W an x-Steiner set of G, then v lies in every Steiner W_x-tree of G.*

Theorem 2.9. *No cut-vertex of a connected graph G belongs to any minimum x-Steiner set of G.*

Proof. Let v be a cut-vertex of G, x be a vertex of G, and W an *x-Steiner set* of G. If $x = v$, then by the definition of the x-Steiner set, $v \notin W$. So let $x \neq v$.Suppose that $v \in W$. Let $G_1, G_2, \ldots, G_r$ $(r \geqslant 2)$ be the components of G-v. Then by Theorem 2.7(ii), let us assume that $x \in V(G_1)$, each component $G_i (2 \leq i \leq r)$ contains an element of W. We claim that $W^{'}$=$W - \{v\}$ is also an *x-Steiner* set of G instead of a Steiner set of G. Since v is a cut-vertex of G, by Corollary 2.8, each Steiner W_x-tree contains v. Now,since $v \notin W^{'}$, it follows that each *Steiner W_x-tree* is also a *Steiner $W^{'}_x$-tree* of G. Thus $W^{'}$ is an *x-Steiner set* of G such that $|W^{'}| < |W|$ which is a contradiction to W, an *x-Steiner set* of G. Hence the theorem. □

Observation 2.10. *(a). For the cycle $G = C_p$ $(p \geq 3)$, $s_x(C_p) = \begin{cases} 1 & \text{if } p \text{ is even} \\ 2 & \text{if } p \text{ is odd} \end{cases}$ for every x in $V(C_p)$.*
(b). For a complete bipartite graph $G = K_{m,n}$ $(2 \leq m \leq n)$ with bipartite sets $U = \{u_1, u_2, u_3, u_4, \ldots, u_m\}$, and $W = \{w_1, w_2, w_3, w_4, \ldots, w_n\}$, $s_x(K_{m,n}) = \begin{cases} m-1 & \text{if } x \in U \\ n-1 & \text{if } x \in W \end{cases}$

(c). For the wheel $G = W_p = K_1 + C_{p-1}$ $(p \geq 4)$,

$$s_x(G) = \begin{cases} p-1 & \text{if} \quad x \in V(K_1) \\ p-4 & \text{if} \quad x \in V(C_{p-1}) \end{cases}$$

Theorem 2.11. *Let G be a connected graph of order $p \geq 2$. Then for any vertex x in G, $1 \leq s_x(G) \leq p-1$.*

Proof. Any x-Steiner set needs at least one vertex. Therefore $s_x(G) \geq 1$. For a vertex x, $W = V(G) - \{x\}$ is an x-Steiner set of G and so $s_x(G) \leq |W| = p-1$. □

Theorem 2.12. *Let G be a connected graph of order $p \geq 2$. Then for a vertex x in G, $s_x(G) = 1$ if and only if there exists a vertex y such that every vertex of G is on a diametral path joining x and y.*

Proof. Let x and y be vertices of G such that each vertex of G is on a diametral path P joining x and y. Let $W = \{y\}$. Since $s_x(G) \geq 1$ and since P is a geodesic joining x and y such that each vertex of G is on a geodesic joining x and y and also every x-y geodesic is a Steiner W_x-tree of G, it follows that W is a s_x-set of G and hence $s_x(G) = 1$. Conversely, let $s_x(G) = 1$ and let $W = \{y\}$ be a s_x-set of G. Since every Steiner W_x-tree in G is an x-y geodesic, each vertex of G also lies on an x-y geodesic. We claim that $d(x, y) = d(G)$. If $d(x, y) < d(G)$, then let u and v be two vertices of G such that $\text{d}(u, v) = \text{d}(G)$. Now, it follows that u and v lie on distinct geodesics joining x and y. Hence $d(x, y) = d(x, u)+d(u, y) \cdots (1)$ and $d(x, y) = d(x, v)+d(v, y) \cdots (2)$. By the triangle inequality, $d(u, v) \leq d(u, x)+d(x, v) \cdots (3)$. Since $d(x, y) < d(u, v)$, (3) becomes $d(x, y) < d(u, x)+d(x, v) \cdots (4)$. Using (4) in (1), we get $d(u, y) < d(u, x) + d(x, v) - d(x, u) = d(x, v)$. Thus, $d(u, y) < d(x, v) \ldots (5)$. Also, by triangle inequality, we have $d(u, v) \leq d(u, y) + d(y, v) \cdots (6)$. Now,using (5) and (2), (6) becomes $d(u, v) < d(x, v) + d(v, y) = d(x, y)$. Thus, $d(G) < d(x, y)$, which is a contradiction. Hence $d(x, y) = d(G)$ and since $W = \{y\}$ is a s_x-set of G, it follows that each vertex of G is on a diametral path joining x and y. □

Corollary 2.13. *For the n-cube Q_n $(n \geq 2)$, $s_x(Q_n) = 1$ for every vertex in Q_n.*

Proof. Q_n has 2^n vertices, which may be labeled $(a_1a_2a_3 \cdots a_n)$, where each a_i $(1 \leq i \leq n)$ is either 0 or 1. Let a_i' denote the complement of a_i so that a_i' is 0 or 1 according as a_i is 1 or 0. Two vertices in Q_n are adjacent if and only if their binary representations differ exactly in one place. Let $x = (a_1a_2a_3 \cdots a_n)$ be any vertex in Q_n. Let u be any vertex of Q_n. For convenience, let $u = (a_1'a_2a_3 \cdots a_n)$, then u lies on the x-y geodesic $P : x = (a_1a_2a_3 \cdots a_n)$, $(a_1'a_2a_3 \cdots a_n)$, $(a_1'a_2'a_3 \cdots a_n)$, $(a_1'a_2'a_3' \cdots a_n), \ldots, (a_1'a_2'a_3' \cdots a_n') = y$ of length n so that it is on a diametral path joining x and y. Hence by Theorem 2.12, $s_x(G) = 1$ for every vertex x in Q_n. □

Theorem 2.14. *For a connected graph G of order $p \geq 2$, $s_x(G) < p-1$ if and only if there exists an x-Steiner set W such that $G[W \cup \{x\}]$ is disconnected.*

Proof. Let x be any vertex of G. First assume that $s_x(G) < p-1$. Let W be an x-Steiner set of G. Suppose that $G[W \cup \{x\}]$ is connected. Then the Steiner W_x-tree of G contains the elements of $W \cup \{x\}$ only, which is a contradiction to W an x-Steiner set of G. Hence $G[W \cup \{x\}]$ is disconnected. Conversely, let x be any vertex of G and W be an x-Steiner set of G such that $G[W \cup \{x\}]$ is disconnected. We claim that $s_x(G) < p-1$. Suppose $s_x(G) = p-1$. Then it follows that $W = V(G) - \{x\}$ is the unique *x-Steiner set* of G such that $G[W \cup \{x\}]$ is connected, which is a contradiction. □

Theorem 2.15. *For a connected graph G of order $p \geq 2$, $s_x(G) = p-1$ if and only if $deg(x) = p-1$.*

Proof. Assume that x is a vertex of degree $p-1$. Suppose that $s_x(G) < p-1$. Then by Theorem 2.14, there exists an x-Steiner set W such that $G[W \cup \{x\}]$ is disconnected, which is a contradiction to x a vertex of degree $p-1$. Therefore $s_x(G) = p-1$. Conversely let $s_x(G) = p-1$. If $deg(x) < p-1$, then G is non complete. Let y be a vertex of G such that $xy \notin E(G)$. Let $y_1, y_2, \ldots, y_n$ be the non simplicial vertices of G in $G[N(y)]$. It is clear that $|N(y)| \geq 1$. Then $W = V(G) - \{x, y_1, y_2, \ldots, y_n\}$ is an x-Steiner set of G so that $s_x(G) < p-1$, which is a contradiction. □

Corollary 2.16. *A graph G is complete if and only if $s_x(G) = p-1$ for every vertex x in G.*

Proof. This follows from Theorem 2.15. □

For every connected graph G, $\text{rad}(G) \leq \text{diam}(G) \leq 2\,\text{rad}(G)$. Ostrand [17] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the vertex Steiner number can also be prescribed.

Theorem 2.17. *For positive integers r, d and n with $r \leq d \leq 2r$, there exists a connected graph G with $rad(G) = r$, $diam(G) = d$ and $s_x(G) = n$ for some vertex x in G.*

Proof. When $r = 1$, $d = 1$ or 2. If $d = 1$, let $G = K_{n+1}$. Then by Corollary 2.5, $s_x(G) = n$ for any vertex x in G. Let $d = 2$. If $n = 1$, let $G = K_{1,2}$. Let x be an end vertex of G. Then by Corollary 2.6, $s_x(G) = 1$. If $n \geq 2$, let $G = K_{1,n}$. Then by Corollary 2.6, $s_x(G) = n$ for the cut-vertex x in G. Now, let $r \geq 2$. We construct a graph G with the desired properties as follows.

Case I. Suppose that $r = d$. For $n = 1$, let $G = C_{2r}$. Then it is clear that $r = d$. By Observation 2.10 (a), $s_x(G) = 1$ for any vertex x in G. Now, let $n \geq 2$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$, be the cycle of order $2r$. Let G be the graph obtained by adding the new vertices $x_1, x_2, \ldots, x_{n-1}$ and joining each x_i $(1 \leq i \leq n-1)$ with u_1 and u_2 of C_{2r}. The graph G is shown in Figure 2. It is easily verified that the eccentricity of each vertex of G is r so that $\text{rad}(G) = \text{diam}(G) = r$. Let $S = \{x_1, x_2, \ldots, x_{n-1}\}$ be the set of all simplicial vertices of G with $|S| = n-1$ and $x = u_1$. By Theorem 2.4, S is contained in every x-Steiner set of G. It is clear that S is not an x-Steiner set of G and so $s_x(G) \geq n$. Let y be the antipodal vertex of x in C_{2r}. Then it follows from Theorem 2.4 that $S \cup \{y\}$ is an x-Steiner set of G so that $s_x(G) = n$.

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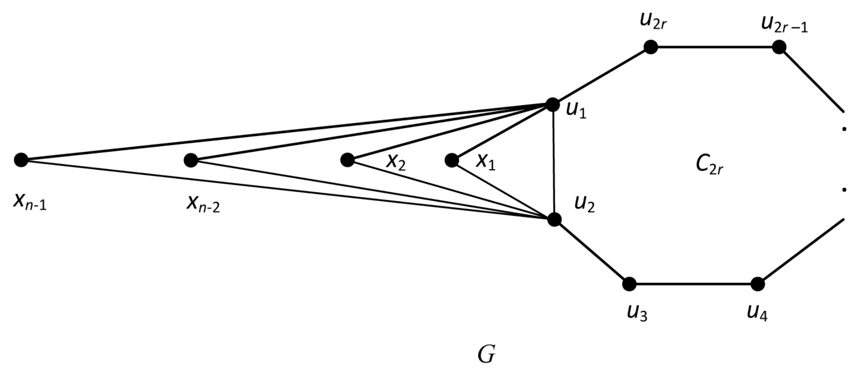

FIGURE 2.

Case II. Suppose $r < d$. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let P_{d-r+1} : $u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} and by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. If $n = 1$ or 2, then let $G = H$. Now $\text{rad}(G) = r$ and $\text{diam}(G) = d$ and G has one end vertex. Clearly, $s_x(G) = 1$ or 2 according as $x = v_{r+1}$ or v_1 respectively. If $n \geq 3$, then add $(n-2)$ new vertices $w_1, w_2, \ldots, w_{n-2}$ to H and join each vertex w_i $(1 \leq i \leq n-2)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 3. Now $\text{rad}(G) = r$ and $\text{diam}(G) = d$ and G has $(n-1)$ end vertices. Let x be any cut vertex of G and $W = \{w_1, w_2, \ldots, w_{n-2}, u_{d-r}\}$ be the set of all simplicial vertices of G with $|W| = n - 1$. By Theorem 2.4, W is contained in every x-Steiner set of G. It is clear that W is not an x-Steiner set of G and so $s_x(G) \geq n$. Let $W' = W \cup \{v_{r+1}\}$. Then W' is an x-Steiner set of G so that $s_x(G) = n$.

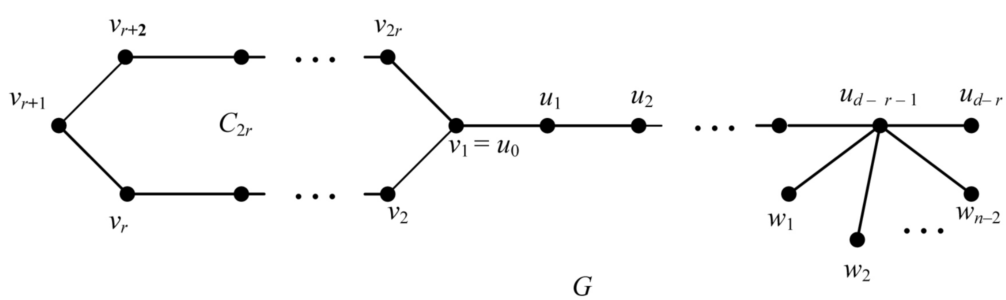

FIGURE 3.

□

3. The Steiner number and the x-Steiner number of a graph

Theorem 3.1. *Let x be a cut vertex of G. Then W is an s_x-set of G if and only if W is a s-set of G.*

Proof. Let x be a cut vertex of G and W is an s_x-set of G. If W is not a s-set of G, then there exists a set W' with $|W'| < |W|$ such that W' is an s-set of G. Since x is a cut vertex of G, x lies in every

Steiner W'- tree of G. Hence it follows that W' is an x-Steiner set of G, which is a contradiction. Therefore, W is a s-set of G. Conversely, let W be a s-set of G. Since x is a cut vertex of G, x lies in every Steiner W tree of G. Hence it follows that W is an x-Steiner set of G. If W is not an s_x-set of G, then there exists a set W' with $|W'| < |W|$ such that W' is an s_x-set of G. Now every vertex of G lies on Steiner W'_x-tree of G. Since x is a cut vertex of G, every Steiner W'_x - tree of G is also a Steiner W'-tree of G, so it follows that W' is a Steiner set of G, which is a contradiction. Therefore, W is an s_x-set of G. □

Corollary 3.2. *Let x be a cut vertex of a connected graph G. Then $s_x(G) = s(G)$.*

Remark 3.3. *The converse of the Corollary 3.2 need not be true. For the graph G given in Figure 4, $W_1 = \{v_1, v_4\}$ and $W_2 = \{v_3, v_6\}$ are the only two s-sets of G so that $s(G) = 2$. Also for the vertex $x = v_5$, $W_3 = \{v_2, v_7\}$ and $W_4 = \{v_1, v_3\}$ are the only two s_x-sets of G so that $s_x(G) = 2 = s(G)$. However, x is not a cut vertex of G.*

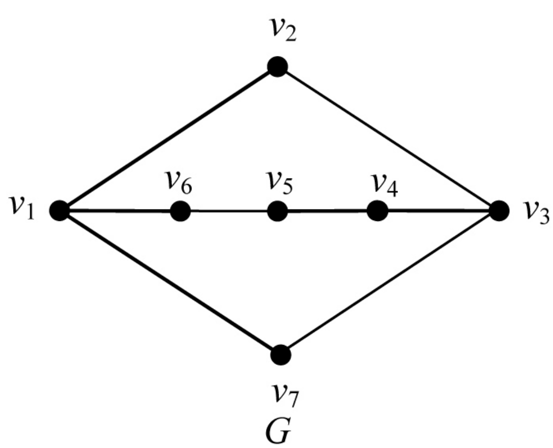

FIGURE 4.

Theorem 3.4. *For any vertex x in G, $s(G) \leq s_x(G) + 1$.*

Proof. Let x be a vertex of G and W be an x-Steiner set of G. Then $W \cup \{x\}$ is a Steiner set of G so that $s(G) \leq |W \cup \{x\}| = s_x(G) + 1$. □

Theorem 3.5. *For any vertex x in G, $s(G) = s_x(G) + 1$ if and only if x belongs to a minimum Steiner set of G.*

Proof. Let W be a minimum Steiner set of G and $x \in W$. Then $W - \{x\}$ is an x-Steiner set of G so that $s_x(G) \leq |W| - \{x\} = s(G) - 1$. This implies that $s_x(G) + 1 \leq s(G)$. Then it follows from Theorem 3.4 that $s(G) = s_x(G) + 1$. Conversely, let $s(G) = s_x(G) + 1$ for any vertex x in G. Let W be an x-Steiner set of G. Then $W' = W \cup \{x\}$ is a Steiner set of G. Since $|W'| = s_x(G) + 1 = s(G)$, it follows that W' is a minimum Steiner set of G with $x \in W'$. Hence the theorem. □

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In the following theorem we give a realization result of Theorem 3.4 for some vertex x in G.

Theorem 3.6. *For any positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $s(G) = a$ and $s_x(G) = b$ for some vertex x in G.*

Proof. If $a = b$, let G be a tree with a end vertices. Then by Theorem 1.2, $s(G) = a$. Let x be a cut vertex of G. Then by Corollary 2.6, $s_x(G) = a$. So, let $2 \leq a < b$. Consider the complete bipartite graph $G = K_{a,b+1}$ with bipartite sets $U = \{u_1, u_2, \ldots, u_a\}$ and $W = \{w_1, w_2, \ldots, w_{b+1}\}$. Then by Theorem 1.3, $s(G) = a$. Also by Observation 2.10 (b), $s_x(G) = b$ for every $x \in W$. □

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