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ON THE DOMINATED CHROMATIC NUMBER OF CERTAIN GRAPHS

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ABSTRACT. Let G be a simple graph. The dominated coloring of G is a proper coloring of G such that each color class is dominated by at least one vertex. The minimum number of colors needed for a dominated coloring of G is called the dominated chromatic number of G, denoted by $\chi_{dom}(G)$. Stability (bondage number) of dominated chromatic number of G is the minimum number of vertices (edges) of G whose removal changes the dominated chromatic number of G. In this paper, we study the dominated chromatic number, dominated stability and dominated bondage number of certain graphs.

1. **Introduction**

In this paper, we are concerned with simple graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G = (V, E)$ be such a graph and $\lambda \in \mathbb{N}$. A mapping $f : V \longrightarrow \{1, 2, \ldots, \lambda\}$ is called a λ-proper coloring of G if $f(u) \neq f(v)$, whenever the vertices u and v are adjacent in G. A color class of this coloring is a set consisting of all those vertices assigned the same color. If f is a proper coloring of G with the coloring classes $V_1, V_2, \ldots, V_\lambda$ such that every vertex in V_i has color i, sometimes write simply $f = (V_1, V_2, \ldots, V_\lambda)$. The chromatic number $\chi(G)$ of G is the minimum of colors needed in a proper coloring of a graph. The concept of a graph coloring and chromatic number is very well-studied in graph theory (see [4, 15, 17]).

A dominator coloring of G is a proper coloring of G such that every vertex of G dominates all vertices of at least one color class (possibly its own class), i.e., every vertex of G is adjacent to all vertices of at

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least one color class. The dominator chromatic number $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. The concept of dominator coloring was introduced and studied by Gera, Horton and Rasmussen [9]. The total dominator coloring, abbreviated TD-coloring studied well. Let G be a graph with no isolated vertex, the total dominator coloring is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number, abbreviated TD-chromatic number, $\chi_d^t(G)$ of G is the minimum number of color classes in a TD-coloring of G. For more information see [10, 11].

Dominated coloring of a graph is a proper coloring in which each color class is a dominated by a vertex. The least number of colors needed for a dominated coloring of G is called the dominated chromatic number (abbreviated dom-chromatic number) of G and denoted by $\chi_{dom}(G)$ [5, 13]. We call this coloring a dom-coloring, simplicity. It is easy to see that for a graph G of order n with maximum degree $\Delta = n-1$, these parameters are equal.

Proposition 1.1. *If G is a connected graph of order $n \geq 2$ with $\Delta(G) = n-1$, then $\chi_{dom}(G) = \chi_d^t(G) = \chi(G)$.*

A set S of vertices in G is a dominating set of G, if every vertex of $V(G) \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set of G is the domination number of G, and denoted by $\gamma(G)$. A domination-critical (domination-super critical, respectively) vertex in a graph G, is a vertex whose removal decreases (increases, respectively) the domination number. Bauer et al. [3] introduced the concept of domination stability in graphs. The domination stability, or just γ-stability of a graph G, is the minimum number of vertices whose removal changes the domination number. Motivated by domination stability, the stability of some kind of colorings of graphs has introduced and investigated [1, 2].

In the next section, we study and compute the dom-chromatic number of certain graphs. In Section 3, we introduce and study stability and bondage number of dominated chromatic number of graphs.

2. Dominated chromatic number of certain graph

In this section, first we consider graphs obtained by the point-attaching of some another graphs and study their dom-chromatic numbers. Also we compute the dom-chromatic number of circulant graphs and cactus graphs.

2.1. **Dominated chromatic number of point-attaching graphs.** Let G be a connected graph constructed from pairwise disjoint connected graphs $G_1, \ldots, G_k$ as follows. Select a vertex of G_1, a vertex of G_2, and identify these two vertices. Then continue in this manner inductively. Note that the graph G constructed in this way has a tree-like structure, the G_i's being its building stones (see Figure 1). Usually say that G is obtained by point-attaching from $G_1, \ldots, G_k$ and that G_i's are the primary subgraphs of G. A particular case of this construction is the decomposition of a connected graph into blockssee [6].

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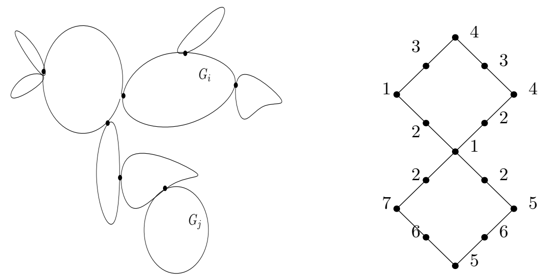

Figure 1. Graph G obtained by point-attaching and dom-coloring of D_8^2, respectively.

In this subsection, we consider some particular cases of these graphs and study their dom-chromatic numbers. The following theorem gives an upper bound for the dom-chromatic number of point-attaching graphs.

Theorem 2.1. *If G is a point-attaching of graphs $G_1, G_2, \ldots, G_k$, then*

$$\chi_{dom}(G) \leq \chi_{dom}(G_1) + \chi_{dom}(G_2) + \cdots + \chi_{dom}(G_k).$$

Proof. We color the graph G_1 with colors $1, 2, \ldots, \chi_{dom}(G_1)$, the graph G_2 with colors $\chi_{dom}(G_1) + 1, \chi_{dom}(G_1) + 2, \cdots, \chi_{dom}(G_1) + \chi_{dom}(G_2)$ and do this action for another G_i's. Therefore we have the result. □

We need the following theorem:

Theorem 2.2. [13] *For $n \geq 3$,*

$$\chi_{dom}(P_n) = \chi_{dom}(C_n) = \begin{cases} \dfrac{n}{2} & \text{if } n \equiv 0 \ (mod\ 4), \\ \lfloor \dfrac{n}{2} \rfloor + 1 & \text{otherwise.} \end{cases}$$

First we consider the ladder graph. Let to recall the definition of Cartesian product of two graphs. Given any two graphs G and H, we define the Cartesian product, denoted $G \square H$, to be the graph with vertex set $V(G) \times V(H)$ and edges between two vertices (u_1, v_1) and (u_2, v_2) if and only if either $u_1 = u_2$ and $v_1 v_2 \in E(H)$ or $u_1 u_2 \in E(H)$ and $v_1 = v_2$. The $n-$ladder graph can be defined as $P_2 \square P_n$ and is denoted by L_n. A graph corresponding to the skeleton of an n-prism is called a prism graph and is denoted by Y_n. A prism graph also called as circular ladder graph, has $2n$ vertices and $3n$ edges. From Figures 2 and 3 which show a dom-coloring of ladder and prism graphs, we have the following result.

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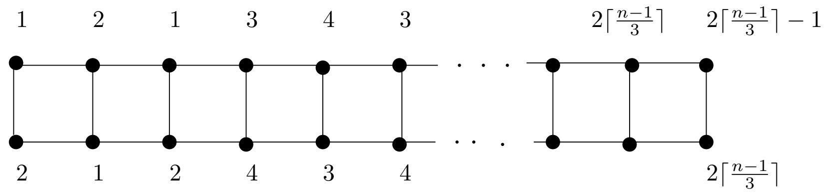

FIGURE 2. The dom-coloring of ladder graph L_n.

Theorem 2.3. (i) *For every $n \geq 2$, $\chi_{dom}(L_n) = 2\lceil\frac{n-1}{3}\rceil$.*
(ii) *For every $n \geq 4$, $\chi_{dom}(Y_n) = \chi_{dom}(L_n)$.*

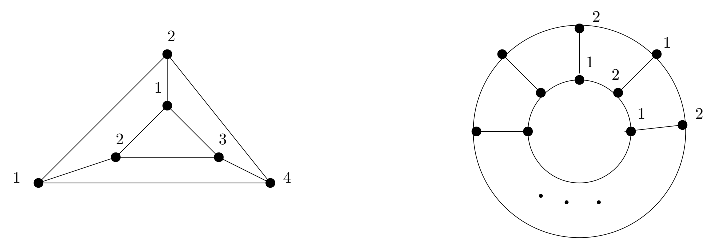

FIGURE 3. The prism graph.

Here, we consider some specific graphs depicted in Figure 4. Note that the graph $Q(m, n)$ is derived from K_m and m copies of K_n by identifying every vertex of K_m with a vertex of one K_n. The following theorem gives the dom-chromatic number of grid graphs $P_m \square P_n$ and graph $Q(m, n)$.

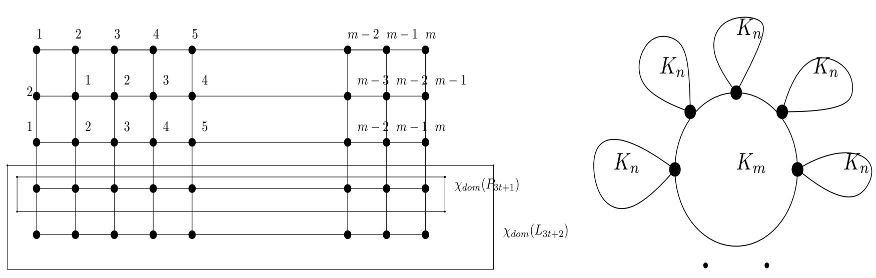

FIGURE 4. The grid graph $P_5 \square P_m$ and $Q(m, n)$ graph, respectively.

Observation 2.4. (i) *Let $m, n \geq 2$. The dom-chromatic number of gird graphs $P_m \square P_n$ is,*

$$\chi_{dom}(P_n \square P_m) = \begin{cases} tm & \text{if } n = 3t, \\ \chi_{dom}(P_m \square P_{3t}) + \chi_{dom}(P_m) & \text{if } n = 3t+1, \\ \chi_{dom}(P_m \square P_{3t}) + \chi_{dom}(L_m) & \text{if } n = 3t+2. \end{cases}$$

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(ii) *For* $m, n \geq 3$, $\chi_{dom}(Q(m,n)) = m(n-1)$.

Proof. (i) It follows from the dom-coloring of $P_n \Box P_m$ depicted in Figure 4.

(ii) Let $\{v_1, v_2, \ldots, v_m\}$ be vertices of K_m and $\{v_j, u_{j_2}, \ldots, u_{j_n}\}$ be vertices of j'th copy of K_n that common in the vertex v_j with K_m. We color the vertices of K_m with $\{1, 2, \ldots, m\}$ such that the color v_i be i. We color the vertices u_{j_2}'s ($j \in \{1, 2, \ldots, m-1\}$) with $j+1$ and the vertex u_{m_2} with color 1 and other vertices with $\{m+1, m+2, \ldots\}$. Therefore $\chi_{dom}(Q(m,n)) = m + m(n-1) - m = m(n-1)$.

□

The friendship graph (or Dutch-Windmill) graph F_n is a graph that can be constructed by joining n copies of the cycle graph C_3 with a common vertex. The generalized friendship graph D_m^n is a collection of n cycles (all af order m), meeting at a common vertex. The generalized friendship graph may also be referred to as flower. We have the following result for the dom-chromatic number of these kind of graphs:

Theorem 2.5. (i) *For* $n \geq 2$, $\chi_{dom}(F_n) = 3$.

(ii)

$$\chi_{dom}(D_m^n) = \begin{cases} \chi_{dom}(D_m^{n-1}) + \lfloor \frac{m}{2} \rfloor & \text{if } n \equiv 1 \ (mod\, 4), \\ \chi_{dom}(D_m^{n-1}) + \lfloor \frac{m}{2} \rfloor - 1 & \text{otherwise.} \end{cases}$$

Proof. (i) Since F_n is the join of K_1 and nK_2, it is suffices to color K_1 with color 1 and every K_2 with colors 2 and 3.

(ii) We color the center vertex with color 1 and the adjacent vertices to center with color 2 and a vertex adjacent with color 2 with color 1. Then for remaining vertices we color as same dominated coloring of P_4 (see Figure 1).

□

As another example of point-attaching graph, consider the graph K_n and n copies of connected graph H. By definition, the graph $Q(n,H)$ is obtained by identifying each vertex of K_n with a fixed vertex of H.

Theorem 2.6. *If the graph* $Q(n,H)$ *is obtained by identifying each vertex of* K_n *with a fixed vertex of* H, *then*

$$n(\chi_{dom}(H) - 1) \leq \chi_{dom}(Q(n,H)) \leq n\chi_{dom}(H).$$

Proof. To obtain a dominated coloring for $Q(n,H)$, we have to color the graph H with $\chi_{dom}(H)$ colors and we shall use new colors for another copies of H. So we need at most $n\chi_{dom}(H)$ colors. Thus $\chi_{dom}(Q(n,H)) \leq n\chi_{dom}(H)$.

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Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. Note that v_i is adjacent with v_{i-1} and v_{i+1}. If we color one of the adjacent vertex of v_i in H, with the color of v_{i-1} and do this for another v_i's, then we have dom-coloring and obviously we need $n(\chi_{dom}(H)-1)$ colors. So $\chi_{dom}(Q(n,H)) \geq n(\chi_{dom}(H)-1)$. □

Here we consider r-gluing of two graphs. Let G_1 and G_2 be two graphs and $r \in \mathbb{N} \cup \{0\}$ with $r \leq min\{\omega(G_1), \omega(G_2)\}$, where $\omega(G)$ shows the clique number of G. Choose a K_r from each G_i, $i = 1, 2$, and form a new graph G from the union of G_1 and G_2 by identifying the two chosen K_r's in an arbitrary manners. The graph G is called r-gluing of G_1 and G_2 and denoted by $G_1 \cup_{K_r} G_2$. If $r = 0$ then $G_1 \cup_{K_0} G_2$ is just disjoint union. The $G_1 \cup_{K_i} G_2$ for $i = 1, 2$, is called vertex and edge gluing, respectively. The following result is about the dominated chromatic number of r-gluing of two graphs.

Theorem 2.7. *For any two connected graphs G_1 and G_2,*

$$max\{\chi_{dom}(G_1), \chi_{dom}(G_2)\} \leq \chi_{dom}(G_1 \cup_{K_r} G_2) \leq \chi_{dom}(G_1) + \chi_{dom}(G_2) - r.$$

Proof. Since we need at least $\chi_{dom}(G_1)$ colors to color G_1 and $\chi_{dom}(G_2)$ colors to color G_2, so we need at least max $\{\chi_{dom}(G_1), \chi_{dom}(G_2)\}$ colors to color $G_1 \cup_{K_r} G_2$, and so we have $max\{\chi_{dom}(G_1), \chi_{dom}(G_2)\} \leq \chi_{dom}(G_1 \cup_{K_r} G_2)$. On the other hand, first we give colors $a_1, a_2, \dots, a_r$ to the vertices of K_r. Then we give $a_{r+1}, \dots, a_{\chi_{dom}(G_1)}$ to the other vertices of G_1 to have a dom-coloring for G_1. Also we give $b_1, b_2, \dots, b_{\chi_{dom}(G_2)-r}$ to the other vertices of G_2 to have a dom-coloring for G_2. So every vertex of $G_1 \cup_{K_r} G_2$ uses the color class which used before and this is a dom-coloring for $G_1 \cup_{K_r} G_2$. So $\chi_{dom}(G_1 \cup_{K_r} G_2) \leq \chi_{dom}(G_1) + \chi_{dom}(G_2) - r$. □

Remark 2. The bounds in the Theorem 2.7 are sharp. For the lower bound it suffices to consider $G_1 = K_4$, $G_2 = K_5$ and $r = 4$. For the upper bound it suffices to consider the complete graph $G_1 = K_5$, $G_2 = K_6$ and $r = 5$.

2.2. **Dom-chromatic number of circulant graphs.** Let $1 \leq a_1 < a_2 < \cdots < a_m \leq \lfloor \frac{n}{2} \rfloor$, where m, n, a_i are integers, $1 \leq i \leq m$, and $n \geq 3$. Set $S = \{a_1, a_2, \dots, a_m\}$. A graph G with the vertex set $\{1, 2, \dots, n\}$ and the edge set $\big\{\{i,j\} \big| |i-j| \equiv a_t (mod\, n)$ for some $1 \leq t \leq m\big\}$ is called circulant graph [14] with respect to set S (or with connection set S), and denoted by $C_n(S)$ or $C_n(a_1, a_2, \dots, a_m)$. Notice that $C_n(S)$ is k-regular, where $k = 2|S| - 1$ if $\frac{n}{2} \in S$ and $k = 2|S|$, otherwise. We need the following results to obtain results about dom-chromatic number of circulant graphs.

Observation 2.8. [5] *For any graph G, $\chi_{dom}(G) \geq \chi(G)$ and $\chi_{dom}(G) = \chi(G)$ if $diam(G) \leq 2$.*

Proposition 2.9. [13]

(i) *Let G be a graph with order at least 2. Then $\chi_{dom}(G) \geq \gamma_t(G)$, where $\gamma_t(G)$ is the total domination number of G.*

(ii) *Let G be a graph without isolated vertices. Then $\chi_{dom}(G) \leq \chi(G).\gamma(G)$.*

From Observation 2.8 and Proposition 2.9 we have the following proposition.

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Proposition 2.10. *If G is a graph without isolated vertices, then*

$$max\{\chi(G),\gamma_t(G)\} \leq \chi_{dom}(G) \leq \chi(G).\gamma(G).$$

We need the following theorem:

Theorem 2.11. [12] *For any $n \geq 4$,*

$$\gamma_t(C_n(1,3)) = \begin{cases} \lceil \frac{n}{4} \rceil + 1 & \text{if } n \equiv 2,4 \ (mod\, 8), \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

Theorem 2.12. $\chi_{dom}(C_6(1,3)) = 3$, $\chi_{dom}(C_7(1,3)) = 4$ *and for $n \geq 8$*

$$\chi_{dom}(C_n(1,3)) = \begin{cases} 2\lfloor \frac{n}{8} \rfloor & \text{if } n \equiv 0 \ (mod\, 8), \\ 2\lfloor \frac{n}{8} \rfloor + 1 & \text{if } n \equiv 1 \ (mod\, 8), \\ 2\lfloor \frac{n}{8} \rfloor + 2 & \text{otherwise.} \end{cases}$$

Proof. We know $\chi_{dom}(G) \geq \gamma_t(G)$ by Proposition 2.10, and so

$$\chi_{dom}(C_n(1,3)) \geq \gamma_t(C_n(1,3)) = \begin{cases} \lceil \frac{n}{4} \rceil + 1 & \text{if } n \equiv 2,4 \ (mod\, 8), \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

Note that $C_6(1,3)$ is isomorphic to complete bipartite graph $K_{3,3}$ and so $\chi_{dom}(C_6(1,3)) = \chi_{dom}(K_{3,3}) = 2$. If $n = 7$, then the color classes are: $\{1,3\},\{2,7\},\{4,6\},\{5\}$. For $n \geq 8$ and $1 \leq i \leq \lfloor \frac{n}{8} \rfloor$ and for $k \in \mathbb{N}$ the color classes are:

$n = 8k$: $A = \{\{1,3,n-1,n-3\} \cup \{2,4,n,n-2\} \cup \{8i-2,8i,8i+2,8i+4\} \cup \{8i-1,8i+1,8i+3,8i+5\}\}$

$n = 8k+1$: $A \cup \{n-4\}$

$n = 8k+2$: $A \cup \{n-4\} \cup \{n-5\}$

$n = 8k+3$: $A \cup \{n-4,n-6\} \cup \{n-5\}$

$n = 8k+4$: $A \cup \{n-4,n-6\} \cup \{n-5,n-7\}$

$n = 8k+5$: $A \cup \{n-4,n-6,n-8\} \cup \{n-5,n-7\}$

$n = 8k+6$: $A \cup \{n-4,n-6,n-8\} \cup \{n-5,n-7,n-9\}$

$n = 8k+7$: $A \cup \{n-4,n-6,n-8,n-10\} \cup \{n-5,n-7,n-9\}$. So we have the result. □

Theorem 2.13. [7] *Let $C_n(a,b)$ be a circulant graph and $gcd(a,n) = 1$. Then the graph $C_n(a,b)$ is isomorphic to graph $C_n(1,c)$ where $c \equiv a^{-1}b \ (mod n)$.*

By Theorems 2.12 and 2.13 we have the following result:

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Theorem 2.14. *Given the circulant graph $C_n(a,b)$, where $n \geq 8$, $gcd(a,n) = 1$ and $a^{-1}b \equiv 3 \ (mod\, n)$, we have*

$$\chi_{dom}(C_n(a,b)) = \begin{cases} 2\lfloor \frac{n}{8} \rfloor & \text{if } n \equiv 0 \ (mod\, 8), \\ 2\lfloor \frac{n}{8} \rfloor + 1 & \text{if } n \equiv 1 \ (mod\, 8), \\ 2\lfloor \frac{n}{8} \rfloor + 2 & \text{otherwise.} \end{cases}$$

2.3. **Dom-chromatic number of cactus graphs.** We consider a class of simple linear polymers called cactus chains [16], in this subsection. A cactus graph is a connected graph in which no edge lies in more than one cycle. Consequently, each block of a cactus graph is either an edge or a cycle. If all blocks of a cactus G are cycles of the same size i, the cactus is i-uniform. A triangular cactus is a graph whose blocks are triangles, i.e., a 3-uniform cactus. A vertex shared by two or more triangles is called a cut-vertex. If each triangle of a triangular cactus G has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles, we say that G is a chain triangular cactus. By replacing triangles in this definitions with cycles of length 4 we obtain cacti whose every block is C_4. We call such cacti square cacti. Note that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square. First we consider a chain triangular. An example of a chain triangular cactus is shown in Figure 5. We call the number of triangles in G, the length of the chain. Obviously, all chain triangular cacti of the same length are isomorphic. Hence, we denote the chain triangular cactus of length n by T_n [16].

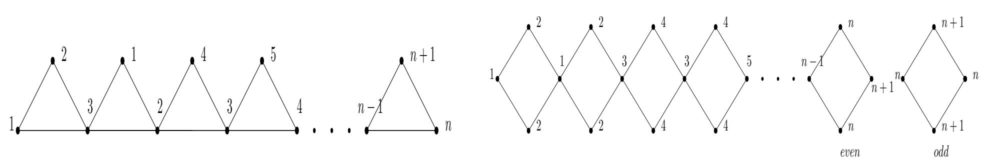

FIGURE 5. The dom-coloring of T_n and Q_n, respectively.

By replacing triangles in the definitions of triangular cactus T_n, with cycles of length 4 we obtain cacti whose every block is C_4. We call such cacti, square cacti. An example of a square cactus chain is shown in Figure 5. We see that the internal squares may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-square; if the cut-vertices are not adjacent, we call the square a para-square. We consider a para-chain of length n, which is denoted by Q_n as shown in Figure 5 and also another kind of square cactus chain and compute its dominated chromatic number (Figure 6).

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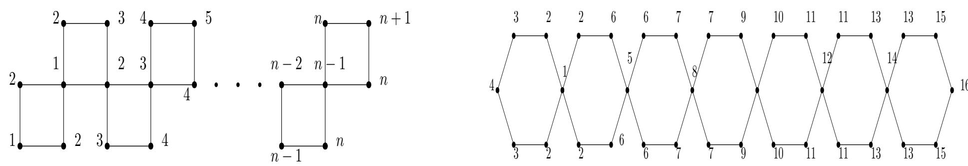

FIGURE 6. The dom-coloring of O_n and H_n, respectively.

Theorem 2.15. (i) *For* $n \geq 3$, $\chi_{dom}(T_n) = n + 1$.

(ii) *For* $n \geq 2$, $\chi_{dom}(O_n) = \chi_{dom}(Q_n) = n + 1$.

Proof. (i) We can see in Figure 5, $\chi_{dom}(T_2) = 3$ and for $n \geq 3$, $\chi_{dom}(T_n) = \chi_{dom}(T_{n-1}) + 1$, and so we have the result.

(ii) We can see in Figure 6, $\chi_{dom}(O_1) = 2$ and for $n \geq 2$, $\chi_{dom}(O_n) = \chi_{dom}(O_{n-1}) + 1$, and so we have the result.

□

Replacing triangles in the definitions of triangular cactus, by cycles of length 6 we obtain cacti whose every block is C_6. We call such cacti, hexagonal cacti. An example of a hexagonal cactus chain is shown in Figure 6. We see that the internal hexagonal may differ in the way they connect to their neighbors. If their cut-vertices are adjacent, we say that such a square is an ortho-hexagonal; if the cut-vertices are not adjacent, we call the square a para-hexagonal. We consider a para-chain of length n, which is denoted by H_n as shown in Figure 6.

Theorem 2.16. *For every* $n \geq 3$, $\chi_{dom}(H_n) = \chi_{dom}(M_n) = n + 4$.

Proof. By dom-coloring which has shown in Figures 6 and 7 we have, $\chi_{dom}(H_2) = \chi_{dom}(M_2) = 6$ and for $n \geq 3$, $\chi_{dom}(H_n) = \chi_{dom}(H_{n-1}) + 2$ and $\chi_{dom}(M_n) = \chi_{dom}(M_{n-1}) + 2$. Therefore we have the result. □

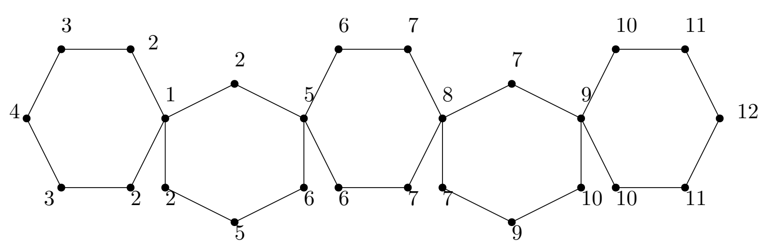

FIGURE 7. Meta-chain square cactus graph M_n.

3. Dom-Stability and dom-bandage number of certain graphs

In this section, we study the stability and bondage number of dominated chromatic number of certain graphs. First we consider stability of certain graphs.

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3.1. **Dom-stability of certain graphs.** Stability of dominated chromatic number of a graph G, $St_{dom}(G)$, is the minimum number of vertices of G whose removal changes the dom-chromatic number of G.

Theorem 3.1. *For $n \geq 4$*

$$St_{dom}(P_n) = \begin{cases} 2 & \text{if } n \equiv 3 \ (mod\, 4), \\ 1 & \text{otherwise} \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. For $n \geq 4$, we consider the following four cases:

(i) If $n = 4k$, for some $k \in \mathbb{N}$, then in this case by removing the vertex v_{4k-1}, we have $P_{4k} - v_{4k-1} = P_{4k-2} \cup K_1$. Therefore $\chi_{dom}(P_{4k} - v_{4k-1}) = \lfloor \frac{4k-2}{2} \rfloor + 1 + 1 \neq \chi_{dom}(P_{4k})$.

(ii) If $n = 4k+1$, for some $k \in \mathbb{N}$, then by removing v_{4k+1}, we have $P_{4k+1} - v_{4k+1} = P_{4k}$. Therefore $\chi_{dom}(P_{4+1}) \neq \chi_{dom}(P_{4k})$.

(iii) If $n = 4k + 2$, for some $k \in \mathbb{N}$, then the proof is similar to proof of Part (i).

(iv) If $n = 4k + 3$, for some $k \in \mathbb{N}$, then by removing the vertex v_{4k+2} and v_{4k+3}, we have $P_{4k+3} - \{v_{4k+2}, v_{4k+3}\} = P_{4k+1}$. Therefore $\chi_{dom}(P_{4k+3}) \neq \chi_{dom}(P_{4k+1})$.

□

Theorem 3.2. *For $n \geq 4$,*

$$St_{dom}(C_n) = \begin{cases} 3 & \text{if } n = 4k, \\ 2 & \text{if } n = 4k+3, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. For $n \geq 4$, we consider the following four cases:

(i) If $n = 4k$, for some $k \in \mathbb{N}$, then by removing the vertices $v_{4k}, v_{4k-1}, v_{4k-2}$, we have $C_{4k} - \{v_{4k}, v_{4k-1}, v_{4k-2}\} = P_{4k-3}$. Therefore $\chi_{dom}(C_{4k}) \neq \chi_{dom}(P_{4k-3})$.

(ii) If $n = 4k + 1$, for some $k \in \mathbb{N}$, then by removing the vertex v_{4k+1}, we have $C_{4k+1} - \{v_{4k+1}\} = P_{4k}$. Therefore $\chi_{dom}(C_{4k+1}) \neq \chi_{dom}(P_{4k})$.

(iii) If $n = 4k + 2$, for some $k \in \mathbb{N}$, then the proof is similar to the proof of Part (ii).

(iv) If $n = 4k + 3$, for some $k \in \mathbb{N}$, then by removing the vertices v_{4k+3}, v_{4k+2}, we have $C_{4k+3} - \{v_{4k+3}, v_{4k+2}\} = P_{4k+1}$. Therefore $\chi_{dom}(C_{4+3}) \neq \chi_{dom}(P_{4k+1})$.

□

The n-book graph ($n \geq 2$) is defined as the Cartesian product $K_{1,n} \Box P_2$. We call every C_4 in the book graph B_n, a page of B_n. All pages in B_n have a common side $v_1 v_2$. The following observation gives the dom-stability of F_n, W_n, D_m^n and B_n.

Observation 3.3. *For $n \geq 2$, $St_{dom}(F_n) = St_{dom}(W_n) = St_{dom}(D_m^n) = St_{dom}(B_n) = 1$.*

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Theorem 3.4. *For every $n \in \mathbb{N}$, there exists a graph G such that $St_{dom}(G) = n$.*

Proof. Consider the graph G of order $2n$ in Figure 8. As observe that, each vertex with color 1 is adjacent with color 2 and each vertex with color 2 is adjacent to every vertex with color 1, and so $\chi_{dom}(G) = 2$. By removing just one vertex of G, the coloring does not change. Suppose that A is the set of vertices whose have color 1. The dom-chromatic number of the induced graph $G - A$ is n. The set A has the minimum number of vertices which changes the dom-chromatic number of these kind of graphs (since K_2 is always a subgraph of these graphs and we do not need to change the color of the graph by removing each vertex). Therefore $\chi_{dom}(G) = n$. □

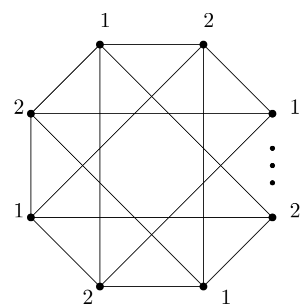

FIGURE 8. n-regular graph of order $2n$.

Theorem 3.5. *There exist graphs G and H with the same dom-chromatic number such that $|St_{dom}(G) - St_{dom}(H)|$ is very large.*

Proof. Suppose that S_{n_1,n_2} is a double-star with degree sequence $(n_1 + 1, n_2 + 1, 1, \ldots, 1)$. It is easy to see that for a bipartite graph $G = (V_1, V_2, E)$, if there exists a vertex of part V_1 which is adjacent to all vertices of V_2 and there exists a vertex in part V_2 such that adjacent to all vertices of V_1, then $\chi_{dom}(G) = 2$. Now let $G = K_{n,n}$ and $H = S_{n_1,n_2}$. So $\chi_{dom}(K_{n,n}) = \chi_{dom}(S_{n_1,n_2}) = 2$ but $St_{dom}(K_{n,n}) = n$ and $St_{dom}(S_{k_1,k_2}) = 1$. Therefore $|St_{dom}(K_{n,n}) - St_{dom}(S_{n_1,n_2})| = n - 1$. □

Proposition 3.6. *If G is a graph and $v \in V(G)$, then:*

$$St_{dom}(G) \leq St_{dom}(G - v) + 1.$$

Proof. If $\chi_{dom}(G) = \chi_{dom}(G - v)$, then $St_{dom}(G) \leq St_{dom}(G - v) + 1$ and if $\chi_{dom}(G) \neq \chi_{dom}(G - v)$, then $St_{dom}(G) = 1$. So we have the result. □

3.2. **Dom-bondage of certain graphs.** Bondage number of dominated coloring of graph G, $B_{dom}(G)$, is the minimum number of edges of G, whose removal changes the dom-chromatic number of G. In this subsection, we study the dom-bondage number of specific graphs.

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Theorem 3.7. *For $n \geq 4$,*

$$B_{dom}(P_n) = \begin{cases} 2 & \text{if } n = 4k+2, \\ \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. For $n \geq 4$, we consider the following four cases:

(i) If $n = 4k$, for some $k \in \mathbb{N}$. In this case, by removing the edge between two vertices v_{4k-1} and v_{4k}, we have the result.

(ii) If $n = 4k+1$, for some $k \in \mathbb{N}$, by removing the edge between two vertices v_{4k-1} and v_{4k}, the new graph is $P_{4k-1} \cup P_2$ and so $B_{dom}(P_{4k+1}) = 1$.

(iii) If $n = 4k+2$, for some $k \in \mathbb{N}$, by removing the edges between two vertices v_{4k-1} and v_{4k+1}, the new graph is $P_{4k-1} \cup P_1 \cup P_2$ and so $\chi_{dom}(P_{4k-1} \cup P_1 \cup P_2) \neq \chi_{dom} P_{4k+2}$.

(iv) If $n = 4k+3$, for some $k \in \mathbb{N}$, the proof is similar to the proof of Part (i).

□

By removing each edge of the graph C_n, we have a path graph of order n and so $\chi_{dom}(C_n) = \chi_{dom}(P_n)$. Therefore we need to remove two edges and so $B_{dom}(C_n) \geq 2$.

Theorem 3.8. *For $n \geq 4$,*

$$B_{dom}(C_n) = \begin{cases} 3 & \text{if } n = 4k+2, \\ \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. For $n \geq 4$, we consider the following four cases:

(i) If $n = 4k$, for some $k \in \mathbb{N}$, then by removing the edges between two vertices v_{4k-1} and v_{v_1}, the new graph is $P_{4k-1} \cup P_1$. Therefore $B_{dom}(C_{4k}) = 2$.

(ii) If $n = 4k+1$, for some $k \in \mathbb{N}$, by removing two edges between vertices v_1 and v_2 and between v_{4k} and v_{4k+1}, we have a dom-coloring for new graph with $\lfloor \frac{4k+1}{2} \rfloor + 2$ colors. So $B_{dom}(C_{4k+1}) = 2$.

(iii) If $n = 4k+2$, for some $k \in \mathbb{N}$, then by removing the edges $\{v_{4k+2}v_1\}$, $\{v_{4k}v_{4k+1}\}$ and $\{v_{4k-3}v_{4k-2}\}$, the new graph is $P_{4k-2} \cup 2P_2$ and so $B_{dom}(C_{4k+2}) = 3$.

(iv) If $n = 4k+3$, for some $k \in \mathbb{N}$, the proof is similar to the proof of Part (i).

□

Observation 3.9. *For $n \geq 2$, $B_{dom}(F_n) = B_{dom}(B_n) = 1$.*

We end the paper with the following proposition:

Proposition 3.10. *We have the following properties for dom-bondage number:*

i) *For any natural number $n \geq 2$, there exists a graph G such that $B_{dom}(G) = n$.*

ii) *There exist graphs G and H with the same dom-chromatic number such that $\mid B_{dom}(G) - B_{dom}(H) \mid$ is very large.*

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Proof. (i) Since for any $m \geq n$, $B_{dom}(K_{m,n}) = n$, so it suffices to consider $G = K_{m,n}$.

(ii) It suffices to consider $G = K_{n,n}$ and $H = S_{n_1,n_2}$.

□

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References

[1] S. Akbari, S. Klavžar, N. Movarraei and M. Nahvi, Nordhaus-Gaddum and other bounds for the chromatic edge-stability number, *European J. Combin.*, **84** (2020) 8 pp.

[2] S. Alikhani and S. Soltani, Stabilizing the distinguishing number of a graph, *Comm. Algebra*, **46** (2018) 5460–5468.

[3] D. Bauer, F. Harary, J. Nieminen and C. L. Suffel, Domination alteration sets in graphs, *Discrete Math.*, **47** (1983) 153–161.

[4] O. V. Borodin, Colorings of plane graphs: A survey, *Discrete Math.*, **313** (2013) 517–539.

[5] F. Choopani, A. Jafarzadeh, A. Erfanian and D. A. Mojdeh, On dominated coloring of graphs and some Nardhaus-Gaddum-type relations, *Turkish J. Math.*, **42** (2018) 2148–2156.

[6] E. Deutsch and S. Klavǩzar, Computing Hosoya polynomials of graphs from primary subgraphs, *MATCH Commun. Math. Comput. Chem.*, **70** (2013) 627–644.

[7] C. Heuberger, On planarity and colorability of circulant graphs, *Discrete Math.* **268** (2003) 153–169.

[8] A. P. Kazemi, Total dominator chromatic number of a graph, *Trans. Comb.*, **4** no. 2 (2015) 57–68.

[9] R. Gera, C. Ramussen and S. Horton, Dominator colorings and safe clique partitions, *Congr. Numer.*, **181** (2006) 19–32.

[10] N. Ghanbari and S. Alikhani, More on the total dominator chromatic number of a graph, *J. Inf. Optim. Sci.*, **40** (2019) 157–169.

[11] N. Ghanbari and S. Alikhani, Total dominator chromatic number of some operations on a graph, *Bull. Comput. Appl. Math.*, **6** (2018) 9–20.

[12] N. Jafari Rad, Domination in circulant graph, *An. tiin. Univ. "Ovidius" Constana Ser. Mat.*, **17** (2009) 169-176.

[13] H. B. Merouane, M. Haddad, M. Chellali and H. Kheddouci, Dominated coloring of graphs, *Graphs Combin.*, **31** (2015) 713–727.

[14] M. Meszka, R. Nedela and A. Rosa, The chromatic number of 5-valent circulants, *Discrete Math.*, **308** (2008) 6269–6284.

[15] F. Ramezani, Coloring problem of signed interval graphs, *Trans. Comb.*, **8** (2019) 1–9.

[16] A. Sadeghieh, N. Ghanbari and S. Alikhani, Computation of Gutman index of some cactus chains, *Electron. J. Graph Theory Appl.*, **6** (2018) 138–151.

[17] P. Zhang, *A Kaleidoscopic View of Graph Colorings*, Springer, 2016.

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