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GUTMAN INDEX, EDGE-WIENER INDEX AND EDGE-CONNECTIVITY

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Abstract. We study the Gutman index $\mathrm{Gut}(G)$ and the edge-Wiener index $W_e(G)$ of connected graphs G of given order n and edge-connectivity λ. We show that the bound $\mathrm{Gut}(G) \le \frac{2^4 \cdot 3}{5^5(\lambda+1)} n^5 + O(n^4)$ is asymptotically tight for $\lambda \ge 8$. We improve this result considerably for $\lambda \le 7$ by presenting asymptotically tight upper bounds on $\mathrm{Gut}(G)$ and $W_e(G)$ for $2 \le \lambda \le 7$.

1. Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices adjacent to v is the degree $\deg(v)$ of a vertex $v \in V(G)$. The number of edges in a shortest path between two vertices $u, v \in V(G)$ is the distance $d(u,v)$ between u and v. The distance between v and any vertex furthest from v is the eccentricity $ec(v)$ of v in G. The distance between any two furthest vertices in G is the diameter of G. The i-th neig hbourhood $N_i(v)$ of v in G is the set of vertices at distance i from v. $N_1(v) = N(v)$ is the neighbourhood of v and $N[v] = N(v) \cup \{v\}$. The edge-connectivity of G is the smallest number of edges whose removal disconnects G. The number of vertices in a shortest path between two edges $e, f \in E(G)$ is the distance $d(e,f)$ between e and f.

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The Gutman index and the edge-Wiener index have been studied because of their extensive applications. The edge-Wiener index

$$W_e(G) = \sum_{\{e,f\}\subseteq E(G)} d(e,f)$$

of a connected graph G was introduced by Iranmanesh et al. [11] and Khalifeh et al. [12]. Azari and Iranmanesh [3] studied the edge-Wiener index for the sum of graphs, relations between the edge-Wiener index and other indices were investigated in [5], [14] and [18].

The Gutman index

$$\mathrm{Gut}(G) = \sum_{\{u,v\}\subseteq V(G)} \deg(u)\deg(v)d(u,v)$$

of a connected graph G has been introduced in [10]. The Gutman index of acyclic structures was considered by Gutman [10]. unicyclic graphs were investigated by Feng [8], bicyclic graphs by Feng and Liu [9] and graphs of given vertex-connectivity in [16]. Andova et. al. [2] gave bounds on the Gutman index for graphs with maximal and graphs with minimal Gutman index. and lower bounds on this index were studied also by Chen [4]. Knor, Potočnik and Škrekovski [13] studied relations between the Gutman index and the edge-Wiener index. Das, Su and Xiong [7] investigated relations between the Gutman index and the degree distance. Bounds on the number of edges with respect to edge-connectivity and other invariants were given by Ali et al. [1].

Mukwembi [17] proved that for a connected graph G with n vertices,

$$\mathrm{Gut}(G) \le \frac{2^4}{5^5}n^5 + O(n^4). \tag{1.1}$$

The Gutman index of connected graphs G of order n and minimum degree δ was studied in [15]. It was proved that

$$\mathrm{Gut}(G) \le \frac{2^4 \cdot 3}{5^5(\delta+1)}n^5 + O(n^4). \tag{1.2}$$

Since the edge-connectivity is an important graph invariant, in this paper we study the Gutman index of graphs of given order n and edge-connectivity λ. For $\lambda = 1$, from (1.1) we have $\mathrm{Gut}(G) \le \frac{2^4}{5^5}n^5 + O(n^4)$ and the bound is asymptotically tight, since the extremal graph given in [17] has edge-connectivity one.

For every graph G, we have $\lambda \le \delta$, thus from (1.2) we obtain the inequality

$$\mathrm{Gut}(G) \le \frac{2^4 \cdot 3}{5^5(\lambda+1)}n^5 + O(n^4). \tag{1.3}$$

We show that this bound is asymptotically tight for $\lambda \ge 8$. The main challenge of this paper is to obtain asymptotically tight upper bounds on the Gutman index for graphs of given order and edge-connectivity λ, where $2 \le \lambda \le 7$. We prove that the bound (1.3) can be improved considerably for $2 \le \lambda \le 7$. We also obtain asymptotically tight upper bounds on the edge-Wiener index of graphs of given order and edge-connectivity $\lambda \ge 2$.

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2. Results

First, we consider the Gutman index of graphs with edge-connectivity at least 8.

Theorem 2.1. *Let G be any graph with n vertices and edge-connectivity λ, where $\lambda \geq 8$ is a constant. Then*

$$\mathrm{Gut}(G) \leq \frac{2^4 \cdot 3}{5^5(\lambda+1)} n^5 + O(n^4)$$

and the bound is asymptotically tight.

Proof. We obtain the bound from (1.2) by applying the inequality $\lambda \leq \delta$, thus We prove that the bound is asymptotically tight. Let us present the graph G' having diameter $d = 3k+1$ for $k \geq 1$. Let $G_0 = K_{\lfloor \frac{1}{2}[n-k(\lambda+1)] \rfloor}$ and $G_{3k+1} = K_{\lceil \frac{1}{2}[n-k(\lambda+1)] \rceil}$.

If $\lambda \equiv 2 \pmod 3$, then $G_1 = G_2 = \cdots = G_{3k} = K_{\frac{\lambda+1}{3}}$. If $\lambda \equiv 0 \pmod 3$, then $G_{3i-2} = G_{3i-1} = K_{\frac{\lambda}{3}}$ and $G_{3i} = K_{\frac{\lambda}{3}+1}$ for $i = 1, 2, \ldots, k$. If $\lambda \equiv 1 \pmod 3$, then $G_{3i-2} = G_{3i-1} = K_{\frac{\lambda+2}{3}}$ and $G_{3i} = K_{\frac{\lambda-1}{3}}$ for $i = 1, 2, \ldots, k$.

The graph G' consists of the graphs $G_0, G_1, G_2, \ldots, G_{3k+1}$, where each vertex of G_i is adjacent to each vertex of G_{i+1} for $i = 0, 1, 2, \ldots, 3k$. We have $|V(G_i)| + |V(G_{i+1})| + |V(G_{i+2})| = \lambda + 1$ for $i = 1, 2, \ldots, 3k-2$, so the degree of every vertex in $V(G_2) \cup V(G_3) \cup \cdots \cup V(G_{3k-1})$ is λ. Since n is large, the degree of the other vertices is greater than λ. Note that $|V(G_i)||V(G_{i+1})| \geq \lambda$ for every $i = 0, 1, \ldots, 3k$, thus the edge-connectivity of G' is λ.

We have $|V(G_1)| + |V(G_2)| + \cdots + |V(G_{3k})| = k(\lambda+1)$. Since $|V(G_0)| + |V(G_{3k+1})| = n - k(\lambda+1)$, we get $|V(G')| = n$. Let $\mathrm{Gut}(u,v) = \deg(u)\deg(v)d(u,v)$, where $u, v \in V(G')$. Clearly,

$$\mathrm{Gut}(G) = \sum_{\{u,v\} \subseteq V(G')} \mathrm{Gut}(u,v).$$

For any $u \in V(G_0)$ and $v \in V(G_{3k+1})$, we obtain

$$\begin{aligned} \mathrm{Gut}(u,v) &= \left(\left\lfloor \frac{n-k(\lambda+1)}{2} \right\rfloor + O(1) \right) \left(\left\lceil \frac{n-k(\lambda+1)}{2} \right\rceil + O(1) \right) (3k+1) \\ &= \frac{[n-k(\lambda+1)]^2(3k+1)}{4} + O(nk). \end{aligned}$$

Then for $k = \frac{n}{5(\lambda+1)}$, we obtain

$$\begin{aligned} \sum_{u \in V(G_0), v \in V(G_{3k+1})} \mathrm{Gut}(u,v) &= \frac{[n-k(\lambda+1)]^4(3k+1)}{16} + O(n^3k) \\ &= \frac{(\frac{4n}{5})^4(\frac{3n}{5\lambda+1}+1)}{16} + O(n^3k) \\ &= \frac{2^4 \cdot 3}{5^5(\lambda+1)} n^5 + O(n^4). \end{aligned}$$

For any $u \in V(G_1) \cup V(G_{3k})$ and $v \in H = V(G_0) \cup V(G_1) \cup V(G_{3k}) \cup V(G_{3k+1})$, we obtain $\mathrm{Gut}(u,v) = O(n^2k)$ since $\deg(u) = O(n)$. Consequently, $\sum_{u\in V(G_1)\cup V(G_{3k}), v\in H} \mathrm{Gut}(u,v) = O(n^3k)$ since $|V(G_1) \cup V(G_{3k})| = O(1)$ and $|H| = O(n)$.

If $\{u,v\} \subseteq V(G_0)$ or $\{u,v\} \subseteq V(G_{3k+1})$, then $\mathrm{Gut}(u,v) = O(n^2)$ since $d(u,v) = 1$. Thus

$$\sum_{\{u,v\}\subseteq V(G_0)} \mathrm{Gut}(u,v) = O(n^4) \ \text{ and } \sum_{\{\mathrm{u},\mathrm{v}\}\subseteq \mathrm{V}(\mathrm{G}_{3\mathrm{k}+1})} \mathrm{Gut(u,v) = O(n^4)}.$$

Finally, for $u \in H' = V(G_2) \cup V(G_3) \cup \cdots \cup V(G_{3k-1})$ and $v \in V(G')$, we have $\mathrm{Gut}(u,v) = O(nk)$ since $\deg(u) = O(1)$. Consequently, $\sum_{u\in H', v\in V(G')} \mathrm{Gut}(u,v) = O(n^2k^2)$ since $|H'| = O(k)$ and $V(G') = O(n)$. Hence $\mathrm{Gut}(G') = \frac{2^4\cdot 3}{5^5(\lambda+1)}n^5 + O(n^4)$. □

In Lemma 2.2 we study the degrees of vertices in graphs having small edge-connectivity.

Lemma 2.2. *Let G be any graph with n vertices, edge-connectivity λ and diameter d. Let $v, v' \in V(G)$ such that $d(v,v') \geq 3$.*
If $\lambda = 2$, then

$$\deg(v) \leq n - \frac{3}{2}d + O(1) \quad \textit{and} \quad \deg(v) + \deg(v') \leq n - \frac{3}{2}d + O(1).$$

If $\lambda = 3$ or 4, then

$$\deg(v) \leq n - 2d + O(1) \quad \textit{and} \quad \deg(v) + \deg(v') \leq n - 2d + O(1).$$

If $\lambda = 5$ or 6, then

$$\deg(v) \leq n - \frac{5}{2}d + O(1) \quad \textit{and} \quad \deg(v) + \deg(v') \leq n - \frac{5}{2}d + O(1).$$

If $\lambda = 7$, then

$$\deg(v) \leq n - 3d + O(1) \quad \textit{and} \quad \deg(v) + \deg(v') \leq n - 3d + O(1).$$

Proof. Let G be any graph having order n, edge-connectivity λ and diameter d. Let v_0 be a vertex of G having eccentricity d. The i-th neighbourhood of v_0 is denoted by N_i, where $i = 0,1,2,\ldots,d$. Let $v, v' \in V(G)$. Then $v \in N_i$ and $v' \in N_l$, where $i, l \in \{0,1,2,\ldots,d\}$. Note that $N(v) \subseteq N_{i-1} \cup N_i \cup N_{i+1}$ and $N(v) \subseteq N_{l-1} \cup N_l \cup N_{l+1}$. Thus $\deg(v) \leq |N_{i-1}| + |N_i| + |N_{i+1}| - 1$ and $\deg(v') \leq |N_{l-1}| + |N_l| + |N_{l+1}| - 1$. Snce $d(v,v') \geq 3$, we have $N(v) \cap N(v') = \emptyset$. The edge-connectivity of G is λ, thus $|N_j||N_{j+1}| \geq \lambda$ for every $j = 0,1,2,\ldots,d-1$.

If $\lambda = 2$, then $|N_j||N_{j+1}| \geq 2$. Thus $|N_j| + |N_{j+1}| \geq 3$. It follows that $\sum_{j=0}^{i-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \geq \frac{3}{2}(d-2) - 1$, and consequently

$$n = \sum_{j=0}^{d} |N_j| \geq \deg(v) + \frac{3}{2}(d-2) = \deg(v) + \frac{3}{2}d - O(1).$$

Hence $\deg(v) \le n - \frac{3}{2}d + O(1)$. Note that the inequalities hold also if $v \in N_i$ where $i \in \{0, 1, d-1, d\}$. Similarly,

$$\begin{aligned} n &\ge (\deg(v)+1) + (\deg(v')+1) + \frac{3}{2}(d-5) - O(1) \\ &= \deg(v) + \deg(v') + \frac{3}{2}d - O(1). \end{aligned}$$

Thus $\deg(v)+\deg(v') \le n - \frac{3}{2}d + O(1)$.

If $\lambda = 3$, then $|N_j||N_{j+1}| \ge 3$, and if $\lambda = 4$, then $|N_j||N_{j+1}| \ge 4$ for every $j = 0, 1, 2, \dots, d-1$. In both cases, it follows that $|N_j| + |N_{j+1}| \ge 4$. Then $\sum_{j=0}^{i-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \ge 2d - O(1)$. We get

$$n = \sum_{j=0}^{d} |N_j| \ge \deg(v) + 2d - O(1), \quad \text{thus} \quad \mathrm{deg(v) \le n - 2d + O(1).}$$

Similarly, $n \ge \deg(v) + \deg(v') + 2d - O(1)$, thus $\deg(v) + \deg(v') \le n - \frac{3}{2}d + O(1)$.

If $\lambda = 5$, then $|N_j||N_{j+1}| \ge 5$, and if $\lambda = 6$, then $|N_j||N_{j+1}| \ge 6$ for every $j = 0, 1, 2, \dots, d-1$. In both cases, it follows that $|N_j| + |N_{j+1}| \ge 5$. Thus $\sum_{j=0}^{i-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \ge \frac{5}{2}d - O(1)$ and $n \ge \deg(v) + \frac{5}{2}d - O(1)$. Similarly, $n \ge \deg(v) + \deg(v') + \frac{5}{2}d - O(1)$.

If $\lambda = 7$, then $|N_j||N_{j+1}| \ge 7$. Therefore $|N_j| + |N_{j+1}| \ge 6$ for any $j = 0, 1, 2, \dots, d-1$. Thus $\sum_{j=0}^{i-2} |N_j| + \sum_{j=i+2}^{d} |N_j| \ge 3d - O(1)$ and $n \ge \deg(v) + 3d - O(1)$. Similarly, $n \ge \deg(v) + \deg(v') + 3d - O(1)$. □

In the following theorem, we obtain an upper bound on the Gutman index for graphs G of given order, edge-connectivity 2 and diameter.

Theorem 2.3. *Let G be any graph with n vertices, edge-connectivity 2 and diameter d. Then*

$$\mathrm{Gut}(G) \le \frac{d}{16}\Big(n - \frac{3d}{2}\Big)^4 + O(n^4),$$

and the bound is asymptotically tight.

Proof. We denote by $v_0 \in V(G)$ any vertex having eccentricity d. The i-th neighbourhood of v_0 is denoted by N_i, $i = 0, 1, 2, \dots, d$. The edge-connectivity is 2, thus $|N_i||N_{i+1}| \ge 2$ for every $i = 0, 1, 2, \dots, d-1$. Therefore $|N_i| + |N_{i+1}| \ge 3$. For $i = 1, 2, \dots, \lceil \frac{d}{2} \rceil$, each set $N_{2i-2} \cup N_{2i-1}$ contains at least three vertices v_{i1}, v_{i2}, v_{i3}. We define $P_i = \{v_{i1}, v_{i2}, v_{i3}\}$ and $P = \cup_{i=1}^{\lceil \frac{d}{2} \rceil} P_i$. So

$$|P| = 3\Big\lceil \frac{d}{2} \Big\rceil. \tag{2.1}$$

The set P has no vertices from N_d if d is even.

Let us partition the set $Z = \{\{u, v\} : u, v \in V(G)\}$ into three sets A, B and C, where

$$C = \{\{u, v\} : u \in P \text{ and } v \in V(G)\},$$

$$A = \{\{u, v\} \in Z - C : d(u, v) \ge 3\},$$

$$B = \{\{u, v\} \in Z - C : d(u, v) \le 2\}.$$

So $Z = C \cup A \cup B$. Let $|A| = a$ and $|B| = b$. Then $\binom{n}{2} = |C| + a + b$ and from (2.1) we obtain

$$a + b = \binom{n - |P|}{2} = \frac{1}{2}\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil\Big)\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil - 1\Big). \tag{2.2}$$

We have

$$\mathrm{Gut}(G) = \sum_{\{u,v\}\in A} \deg(u)\deg(v)d(u,v) + \sum_{\{u,v\}\in B} \deg(u)\deg(v)d(u,v) + \sum_{\{u,v\}\in C} \deg(u)\deg(v)d(u,v).$$

Let us bound these three terms in the following claims.

Claim 1. $\sum_{\{u,v\}\in C} \deg(u)\deg(v)d(u,v) \le O(n^4)$.

Let $P = V_1 \cup V_2 \cup \cdots \cup V_6$ where

$$V_1 = \{v_{11}, v_{31}, v_{51}, \ldots\}, \quad V_2 = \{v_{12}, v_{32}, v_{52}, \ldots\}, \quad V_3 = \{v_{13}, v_{33}, v_{53}, \ldots\},$$
$$V_4 = \{v_{21}, v_{41}, v_{61}, \ldots\}, \quad V_5 = \{v_{22}, v_{42}, v_{62}, \ldots\}, \quad V_6 = \{v_{23}, v_{43}, v_{63}, \ldots\}.$$

Since $d(u,v) \ge 3$, we get $N(u) \cap N(v) = \emptyset$ for each pair of distinct vertices u, v in the same set V_i, where $i = 1, 2, \ldots, 6$. Therefore $\sum_{u\in V_i} \deg(u) < n$.

Let us define the score $s(u)$ for each vertex $u \in P$ as

$$s(u) = \sum_{v\in V(G)} \deg(u)\deg(v)d(u,v) = \deg(u) \sum_{v\in V(G)} \deg(v)d(u,v). \tag{2.3}$$

Then by Lemma 2.2,

$$\begin{aligned} s(u) &\le \deg(u) \sum_{v\in V(G)} \Big(n - \frac{3}{2}d + O(1)\Big)d(u,v) \\ &= \deg(u)\Big(n - \frac{3}{2}d + O(1)\Big) \sum_{v\in V(G)} d(u,v) \\ &< \deg(u)\Big(n - \frac{3}{2}d + O(1)\Big)nd \end{aligned}$$

and

$$\begin{aligned} \sum_{u\in P} s(u) &= \sum_{u\in V_1} s(u) + \sum_{u\in V_2} s(u) + \cdots + \sum_{u\in V_6} s(u) \\ &< \sum_{u\in V_1} \deg(u)\Big(n - \frac{3}{2}d + O(1)\Big)nd + \cdots + \sum_{u\in V_3} \deg(u)\Big(n - \frac{3}{2}d + O(1)\Big)nd \\ &= \Big(\sum_{u\in V_1} \deg(u) + \sum_{u\in V_2} \deg(u) + \cdots + \sum_{u\in V_6} \deg(u)\Big)\Big(n - \frac{3}{2}d + O(1)\Big)nd \\ &< 6n\Big(n - \frac{3}{2}d + O(1)\Big)nd \le O(n^4). \end{aligned}$$

Since $\sum_{\{u,v\}\in C} \deg(u)\deg(v)d(u,v) \le \sum_{u\in P} s(u)$, the proof of Claim 1 is complete.

Claim 2. $\displaystyle\sum_{\{u,v\}\in B} \deg(u)\deg(v)d(u,v) \le O(n^4)$.

If $\{u,v\} \in B$, then $d(u,v) \le 2$ and $b = O(n^2)$. Using these facts and Lemma 2.2, we get

$$\sum_{\{u,v\}\in B} \deg(u)\deg(v)d(u,v) \;\le\; \sum_{\{u,v\}\in B} 2\Big(n - \frac{3}{2}d + O(1)\Big)^2 = 2b\Big(n - \frac{3}{2}d + O(1)\Big)^2 \le O(n^4).$$

Claim 3. $\displaystyle\sum_{\{u,v\}\in A} \deg(u)\deg(v)d(u,v) \le \frac{d}{16}\Big(n - \frac{3d}{2}\Big)^4 + O(n^4)$.

Let $\{u', v'\}$ be any pair in A, where $\deg(u') + \deg(v')$ is maximum. Let $\deg(u') + \deg(v') = s$. We have $\deg(u')\deg(v') \le \frac{1}{4}(\deg(u') + \deg(v'))^2$, thus

$$\deg(u')\deg(v') \le \frac{1}{4}s^2. \tag{2.4}$$

By (2.2), we get

$$a = \frac{1}{2}\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil\Big)\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil - 1\Big) - b. \tag{2.5}$$

Clearly, all pairs $\{u,v\}, u,v \in N[u'] - P$ and all $\{u,v\}, u,v \in N[v'] - P$ are in B. Since $u' \in N_i$ for some $i = 0,1,\dots,d$, we have $N[u'] \subseteq N_{i-1} \cup N_i \cup N_{i+1}$. Since $|N[u'] \cap P| \le 6$ and $|N[v'] \cap P| \le 6$, we obtain

$$\begin{aligned} b &\ge \binom{\deg(u') + 1 - 6}{2} + \binom{\deg(v') + 1 - 6}{2} \\ &= \frac{1}{2}[(\deg(u'))^2 + (\deg(v'))^2] - 11(\deg(u') + \deg(v')) + 30 \\ &\ge \frac{1}{4}s^2 - 11s + 30. \end{aligned}$$

Then by (2.5),

$$a \le \frac{1}{2}\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil\Big)\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil - 1\Big) - \frac{1}{4}s^2 + 11s - 30,$$

and by (2.4), we get

$$\begin{aligned} \sum_{\{u,v\}\in A} \deg(u)\deg(v)d(u,v) &\le \sum_{\{u,v\}\in A} \frac{s^2 d}{4} = \frac{s^2 da}{4} \\ &\le \frac{s^2 d}{4}\Big[\frac{1}{2}\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil\Big)\Big(n - 3\Big\lceil\frac{d}{2}\Big\rceil - 1\Big) - \frac{1}{4}s^2 + 11s - 30\Big] \\ &= \frac{s^2 d}{4}\Big[\frac{1}{2}\Big(\Big(n - \frac{3d}{2}\Big)^2 + O(n)\Big) - \frac{1}{4}s^2 + O(n)\Big] \\ &= \frac{s^2 d}{4}\Big[\frac{1}{2}\Big(n - \frac{3d}{2}\Big)^2 - \frac{1}{4}s^2\Big] + O(n^4) \end{aligned}$$

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From Lemma 2.2, we have $s \le n - \frac{3d}{2} + O(1)$. Subject to this condition, $\frac{s^2 d}{4}[\frac{1}{2}(n - \frac{3d}{2})^2 - \frac{1}{4}s^2]$ is maximized for $s = n - \frac{3d}{2} + O(1)$ and we obtain

$$\begin{aligned}\sum_{\{u,v\}\in A} \deg(u)\deg(v)d(u,v) &\le \frac{d}{4}\Big[\Big(n-\frac{3d}{2}\Big)^2 + O(n)\Big]\Big[\frac{1}{2}\Big(n-\frac{3d}{2}\Big)^2 - \frac{1}{4}\Big(n-\frac{3d}{2}\Big)^2 + O(n)\Big] + O(n^4)\\ &= \frac{d}{16}\Big(n-\frac{3d}{2}\Big)^4 + O(n^4),\end{aligned}$$

as claimed. By Claims 1, 2 and 3, we have

$$\begin{aligned}\mathrm{Gut}(G) &= \sum_{\{u,v\}\in A} \deg(u)\deg(v)d(u,v) + \sum_{\{u,v\}\in B} \deg(u)\deg(v)d(u,v) + \sum_{\{u,v\}\in C} \deg(u)\deg(v)d(u,v)\\ &\le \frac{d}{16}\Big(n-\frac{3d}{2}\Big)^4 + O(n^4).\end{aligned}$$

We construct the graph $G_{n,d,2}$ to prove that our bound is asymptotically tight. For $1 \le i \le d-1$, let $G_i = K_1$ if i is odd, and $G_i = K_2$ if i is even. $G_0 = K_{\lceil \frac{1}{2}(n - \lfloor \frac{3}{2}(d-1)\rfloor)\rceil}$ and $G_d = K_{\lfloor \frac{1}{2}(n - \lfloor \frac{3}{2}(d-1)\rfloor)\rfloor}$. The graph $G_{n,d,2}$ consists of the graphs $G_0, G_1, G_2, \dots, G_d$, where every vertex of G_{i-1} is adjacent to every vertex of G_i for $i = 1, 2, \dots, d$.

Since $|V(G_1) \cup V(G_2) \cup \dots \cup V(G_{d-1})| = \lfloor \frac{3}{2}(d-1)\rfloor$ and $|V(K_{\lceil \frac{1}{2}(n - \lfloor \frac{3}{2}(d-1)\rfloor)\rceil})| + |V(K_{\lfloor \frac{1}{2}(n - \lfloor \frac{3}{2}(d-1)\rfloor)\rfloor})| = n - \lfloor \frac{3}{2}(d-1)\rfloor$, the graph $G_{n,d,2}$ has n vertices. The diameter of $G_{n,d,2}$ is d, the edge-connectivity is 2 and $\mathrm{Gut}(G_{n,d,2}) = \frac{d}{16}(n - \frac{3d}{2})^4 + O(n^4)$. □

Now we give upper bounds on the Gutman index for graphs with n vertices, diameter d and edge-connectivity λ for $3 \le \lambda \le 7$.

Theorem 2.4. *Let G be any graph with n vertices, edge-connectivity λ and diameter d.*
(a) If $\lambda = 3$ or 4, then $\mathrm{Gut}(G) \le \frac{d}{16}(n - 2d)^4 + O(n^4)$.
(b) If $\lambda = 5$ or 6, then $\mathrm{Gut}(G) \le \frac{d}{16}(n - \frac{5}{2}d)^4 + O(n^4)$.
(c) If $\lambda = 7$, then $\mathrm{Gut}(G) \le \frac{d}{16}(n - 3d)^4 + O(n^4)$.
The bounds are asymptotically tight.

Theorems 2.3 and 2.4 have similar proofs. We include only the main differences between the proofs of Theorem 2.3 and Theorem 2.4, part (b).

If $\lambda = 5$, then $|N_{i-1}||N_i| \ge 5$, and if $\lambda = 6$, then $|N_{i-1}||N_i| \ge 6$ for each $i = 1, 2, \dots, d$. In both cases, $|N_{i-1}| + |N_i| \ge 6$. Let $v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}, v_{i6}$ be any six vertices in $N_{2i-2} \cup N_{2i-1}$ and let $P_i = \{v_{i1}, v_{i2}, v_{i3}, v_{i4}, v_{i5}, v_{i6}\}$, where $i = 1, 2, \dots, \lceil \frac{d}{2} \rceil$. We define $P = \cup_{i=1}^{\lceil \frac{d}{2} \rceil} P_i$. We have $|P| = 6\lceil \frac{d}{2} \rceil$.

Let us partition P into 12 sets

$$\begin{aligned}
V_1 &= \{v_{11}, v_{31}, v_{51}, \ldots\}, \quad V_2 = \{v_{12}, v_{32}, v_{52}, \ldots\},\\
V_3 &= \{v_{13}, v_{33}, v_{53}, \ldots\}, \quad V_4 = \{v_{14}, v_{34}, v_{54}, \ldots\},\\
V_5 &= \{v_{15}, v_{35}, v_{55}, \ldots\}, \quad V_6 = \{v_{16}, v_{36}, v_{56}, \ldots\},\\
V_7 &= \{v_{21}, v_{41}, v_{61}, \ldots\}, \quad V_8 = \{v_{22}, v_{42}, v_{62}, \ldots\},\\
V_9 &= \{v_{23}, v_{43}, v_{63}, \ldots\}, \quad V_{10} = \{v_{24}, v_{44}, v_{64}, \ldots\},\\
V_{11} &= \{v_{25}, v_{45}, v_{65}, \ldots\}, \quad V_{12} = \{v_{26}, v_{46}, v_{66}, \ldots\}.
\end{aligned}$$

So $P = V_1 \cup V_2 \cup \cdots \cup V_12$. The rest of the proof of part (b) can be easily obtained by following the proof of Theorem 2.3 and using Lemma 2.2.

Let us present the graphs $G_{n,d,\lambda}$ for $\lambda = 3, 4, 5, 6, 7$, to prove that the bounds presented in Theorem 2.4 are asymptotically tight. Let $G_{n,d,\lambda}$ be the graph which consists of the graphs $G_0, G_1, G_2, \ldots, G_d$ defined below, where each vertex of G_{j-1} is adjacent to each vertex of G_j, $j = 1, 2, \ldots, d$.

For $\lambda = 3$, let $G_1 = K_1$, $G_2 = K_3$, $G_j = K_2$ for $j = 3, 4, \ldots, d-1$, $G_0 = K_{\lceil \frac{1}{2}(n-2(d-1)) \rceil}$ and $G_d = K_{\lfloor \frac{1}{2}(n-2(d-1)) \rfloor}$.

For $\lambda = 4$, let $G_j = K_2$ where $j = 1, 2, \ldots, d-1$, $G_0 = K_{\lceil \frac{1}{2}(n-2(d-1)) \rceil}$ and $G_d = K_{\lfloor \frac{1}{2}(n-2(d-1)) \rfloor}$.

For $\lambda = 5$, let $G_j = K_2$ if j is odd, and $G_j = K_3$ if j is even, where $3 \le j \le d-1$. Let $G_1 = K_1$, $G_2 = K_5$, $G_0 = K_{\lceil \frac{1}{2}(n - \lfloor \frac{5}{2}(d-1) \rfloor - 1) \rceil}$ and $G_d = K_{\lfloor \frac{1}{2}(n - \lfloor \frac{5}{2}(d-1) \rfloor - 1) \rfloor}$.

For $\lambda = 6$, let $G_j = K_2$ if j is odd, and $G_j = K_3$ if j is even, where $1 \le j \le d-1$. Let $G_0 = K_{\lceil \frac{1}{2}(n - \lfloor \frac{5}{2}(d-1) \rfloor) \rceil}$ and $G_d = K_{\lfloor \frac{1}{2}(n - \lfloor \frac{5}{2}(d-1) \rfloor) \rfloor}$.

For $\lambda = 7$, let $G_1 = K_1$, $G_2 = K_7$, $G_j = K_3$ where $j = 3, 4, \ldots, d-1$, $G_0 = K_{\lceil \frac{1}{2}(n-3d+1) \rceil}$ and $G_d = K_{\lfloor \frac{1}{2}(n-3d+1) \rfloor}$.

The graphs $G_{n,d,\lambda}$ have n vertices, edge-connectivity λ, diameter d and $\mathrm{Gut}(G_{n,d,\lambda})$ is equal to the bound given in Theorem 2.4.

We use Theorems 2.3 and 2.4 to bound the Gutman index only in tetms of order and edge-connectivity 2.

Theorem 2.5. *Let G be any graph with n vertices and edge-connectivity λ.*
(a) If $\lambda = 2$, then $\mathrm{Gut}(G) \le \frac{2^5}{3 \cdot 5^5} n^5 + O(n^4)$.
(b) If $\lambda = 3$ or 4, then $\mathrm{Gut}(G) \le \frac{2^3}{5^5} n^5 + O(n^4)$.
(c) If $\lambda = 5$ or 6, then $\mathrm{Gut}(G) \le \frac{2^5}{5^6} n^5 + O(n^4)$.
(d) If $\lambda = 7$, then $\mathrm{Gut}(G) \le \frac{2^4}{3 \cdot 5^5} n^5 + O(n^4)$.
The bounds are asymptotically tight.

Proof. (a) By Theorem 2.3, $\text{Gut}(G) \le \frac{d}{16}(n - \frac{3d}{2})^4 + O(n^4)$ for graphs G with n vertices, diameter d and edge-connectivity 2. With respect to d,

$$\frac{d}{16}\left(n - \frac{3d}{2}\right)^4$$

is maximized for $d = \frac{2n}{15}$, therefore $\text{Gut}(G) \le \frac{2^5}{3 \cdot 5^5} n^5 + O(n^4)$ for graphs G of order n and edge-connectivity 2.

The bound is asymptotically tight since the Gutman index of the graph $G_{n,\frac{2n}{15},2}$ is $\frac{2^5}{3 \cdot 5^5} n^5 + O(n^4)$ for an integer $d = \frac{2n}{15}$, where $G_{n,d,2}$ is the graph introduced in the proof of Theorem 2.3.

(b) Let $\lambda = 3$ or 4. Then by Theorem 2.4, we get $\text{Gut}(G) \le \frac{d}{16}(n-2d)^4 + O(n^4)$ for graphs G with n vertices and diameter d. Since $\frac{d}{16}(n-2d)^4$ is maximized for $d = \frac{n}{10}$, we obtain $\text{Gut}(G) \le \frac{2^3}{5^5} n^5 + O(n^4)$ for graphs G of order n.

Let $\frac{n}{10}$ be an integer. Then the graphs $G_{n,\frac{n}{10},\lambda}$ described above for $\lambda = 3$ and 4 have the Gutman index $\frac{2^3}{5^5} n^5 + O(n^4)$.

(c) Let $\lambda = 5$ or 6. Then $\frac{d}{16}(n - \frac{5}{2}d)^4$ is maximized for $d = \frac{2n}{25}$ and we obtain the bound $\text{Gut}(G) \le \frac{2^5}{5^6} n^5 + O(n^4)$.

If $\frac{2n}{25}$ is an integer, then the graphs $G_{n,\frac{2n}{25},\lambda}$ described above for $\lambda = 5$ and 6 have the Gutman index $\frac{2^5}{5^6} n^5 + O(n^4)$.

(d) Let $\lambda = 7$. Then $\frac{d}{16}(n - 3d)^4$ is maximized for $d = \frac{n}{15}$ and we obtain the bound $\text{Gut}(G) \le \frac{2^4}{3 \cdot 5^5} n^5 + O(n^4)$.

If $\frac{n}{15}$ is an integer, then the graph $G_{n,\frac{n}{15},\lambda}$ described above for $\lambda = 7$ has the Gutman index $\frac{2^4}{3 \cdot 5^5} n^5 + O(n^4)$. □

Dankelmann et. al. [6] proved the following relation between the Gutman index and the edge-Wiener index.

Lemma 2.6. *Let G be any connected graph with n vertices. Then*

$$\left| W_e(G) - \frac{1}{4} Gut(G) \right| \le \frac{n^4}{8}.$$

We use Lemma 2.6 to get bounds on the edge-Wiener index of graphs of given order and and edge-connectivity.

Corollary 2.7. *Let G be a graph of order n and edge-connectivity λ.*
(a) If $\lambda = 2$, then $W_e(G) \le \frac{2^3}{3 \cdot 5^5} n^5 + O(n^4)$.
(b) If $\lambda = 3$ or 4, then $W_e(G) \le \frac{2}{5^5} n^5 + O(n^4)$.
(c) If $\lambda = 5$ or 6, then $W_e(G) \le \frac{2^3}{5^6} n^5 + O(n^4)$.
(d) If $\lambda = 7$, then $W_e(G) \le \frac{2^2}{3 \cdot 5^5} n^5 + O(n^4)$.
(e) If $\lambda \ge 8$ is a constant, then $W_e(G) \le \frac{2^2 \cdot 3}{5^5(\lambda+1)} n^5 + O(n^4)$.
The bounds are asymptotically tight.

Proof. From Theorem 2.5 and Lemma 2.6 we obtain the results (a), (b), (c) and (d). From Theorem 2.1 and Lemma 2.6 we get $W_e(G) \leq \frac{2^2 \cdot 3}{5^5(\lambda+1)} n^5 + O(n^4)$. The graphs which have the largest Gutman index in terms of order and edge-connectivity $\lambda \geq 2$ achieve also the bounds given in this corollary, thus the bounds on $W_e(G)$ are asymptotically tight. □

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