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$Kite_{p+2,p}$ IS DETERMINED BY ITS LAPLACIAN SPECTRUM

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Abstract. $Kite_{n,p}$ denotes the kite graph that is obtained by appending complete graph with order $p \geq 4$ to an endpoint of path graph with order $n-p$. It is shown that $Kite_{n,p}$ is determined by its adjacency spectrum for all p and n [H. Topcu and S. Sorgun, The kite graph is determined by its adjacency spectrum, *Applied Math. and Comp.*, **330** (2018) 134–142]. For $n-p=1$, it is proven that $Kite_{n,p}$ is determined by its signless Laplacian spectrum when $n \geq 4$, $n \neq 5$ and is also determined by its distance spectrum when $n \geq 4$ [K. C. Das and M. Liu, Kite graphs are determined by their spectra, *Applied Math. and Comp.*, **297** (2017) 74–78]. In this note, we say that $Kite_{n,p}$ is determined by its Laplacian spectrum for $n-p \leq 2$.

1. Introduction

Throughout this paper, all graphs are simple and undirected. A *graph* G consists of a vertex set $V(G) = \{v_1, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, \ldots, e_m\}$ and it is denoted by $G = (V(G), E(G))$. The *order* of G is number of vertices in G, that is $|V(G)| = n$ and the *size* of G is number of edges in G that is $|E(G)| = m$. For any two vertices v_i, v_j in $V(G)$, if there is and edge between v_i and v_j , then it is denoted by $v_i \sim v_j$. As usual, P_n and K_n denotes path graph and complete graph with n vertices, respectively. $\overline{G} = (V(\overline{G}), E(\overline{G}))$ is the *complement of* G with order n such that $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_n) - E(G)$. *Degree* of a vertex $v_i \in V(G)$ is number of vertices in $V(G)$ which are adjacent to v_i and it is denoted by d_i. In the rest of the paper, for degree sequence of G, we assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. For any vertex $v_i \in V(G)$, the *neighborhood set* of v_i is defined

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as $N(v_i) = \{v_j \in V(G) | v_i \sim v_j\} \subseteq V(G)$. If a vertex in a graph G has degree one, then it is called a *pendant vertex*. An edge in $E(G)$ is said to be a *pendant edge* if and only if it contains a pendant vertex. For a given graph $H = (V(H), E(H))$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, H is a *subgraph of G*. If a subgraph H, could be obtained from G by deleting the some of the vertices in $V(G)$ and edges that are incident to these vertices, then H is an *induced subgraph of G*. If a graph G can not contain H as an induced subgraph, then H is called a *forbidden subgraph* (shortly, *forbidden*) for G. A *clique of G* is an induced complete subgraph of G. The largest possible size of a clique in G is called the *clique number of G* and denoted by $w(G)$. In particular, if a clique in G has size 3, then it is called a *triangle of G*. *Adjacency matrix of G* is $A(G) = [a_{ij}]_{n \times n}$ such that

$$a_{ij} = \begin{cases} 1, & v_i \sim v_j; \\ 0, & v_i \nsim v_j. \end{cases}$$

Degree matrix of G is diagonal matrix $D(G) = diag(d_1, \dots, d_n)$. *Laplacian matrix of G* is $L(G) = D(G) - A(G)$ and *signless Laplacian matrix of G* is $Q(G) = D(G) + A(G)$. For any graph matrix $M(G)$, the *M-spectrum of G* is the set of the eigenvalues (with their multiplicities) of $M(G)$. If two graphs share the same spectrum, then they are called *M-cospectral* graphs. For any graph G, if there is no cospectral but non-isomorphic graph with G, then G is called *determined by its M-spectrum*, shortly *DMS*. For more information about spectral graph theory we refer [14, 15].

There are many papers about the graphs that are determined (or not determined) by their spectrum [4, 5, 6, 7, 8, 9]. We refer well-known surveys about this topic [1, 2]. The *pineapple graph* K_p^q is obtained by appending q pendant edges to a vertex of a complete graph K_p ($q \geq 1, p \geq 3$)[4, 5]. $Kite_{n,p}$ denotes the *kite graph* that is obtained by appending complete graph with order $p \geq 4$ to an endpoint of path graph with order $n - p$. In the literature, it is seen that $Kite_{n,p}$ attracts attention from researchers[10, 11, 12, 13]. When $n = p + 1$, $Kite_{n,p}$ and K_p^1 are actually same graphs and it is shown that this graph is determined by its adjacency spectrum and Laplacian spectrum, respectively, in [9, 10, 4]. Also, it is shown that $Kite_{p+2,p}$ is determined by its adjacency spectrum in [10]. Then, it is proven that, this situation holds for the general form of $Kite_{n,p}$ [12]. According to the signless Laplacian and distance matrices, $Kite_{p+1,p}$ is determined by its spectrum when $n \neq 5, n \geq 4$ and $n \geq 4$,respectively [11]. For the Laplacian matrix, characteristic polynomial of $Kite_{n,p}$ is obtained in [13]. It is shown that any connected graph with the same clique number with $Kite_{n,p}$ is isomorphic to $Kite_{n,p}$ under certain conditions[13]. By the motivation of these results, in this note we say that $Kite_{p+2,p}$ is determined by its Laplacian spectrum.

2. **Preliminaries**

Here we give some well-known results that will be used for the next section.

Theorem 2.1. [15] *Let Q be a real $n \times m$ matrix such that $Q^T Q = I$, and let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If the eigenvalues of $Q^T AQ$ are $\mu_1 \geq \cdots \geq \mu_m$*

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then

$$\lambda_{n-m+i} \leq \mu_i \leq \lambda_i \ (i = 1, \ldots, m)$$

Corollary 2.2. [15] *Let G be a graph with n vertices and eigenvalues $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$, and let H be an induced subgraph of G with m vertices. If the eigenvalues of H are $\lambda_1(H) \geq \cdots \geq \lambda_m(H)$ then*

$$\lambda_{n-m+i}(G) \leq \lambda_i(H) \leq \lambda_i(G) \ (i = 1, \ldots, m)$$

This result is a very useful tool for the adjacency matrix, to say whether a graph G' is an induced subgraph of a given graph G or not. However, $L(G')$ is in general not a principal submatrix of $L(G)$. But $L(G') + D'$ is a principal submatrix of $L(G)$ for some non-negative diagonal matrix D'. Actually, adding D' can not decrease the eigenvalues of $L(G')$. So the right hand inequalities in the above corollary still hold for $L(G')$.

Theorem 2.3. [1] *For the Laplacian matrix, the following can be deduced from the Laplacian spectrum.*

- *a The number of vertices*
- *The number of edges*
- *The number of components*
- *The number of spanning trees*
- *The sum of squares of vertices' degrees*

Theorem 2.4. [15] *The multiplicity of 0 as an eigenvalue of $L(G)$ is equal to the number of components in G.*

Theorem 2.5. [15] *If e is an edge of the graph G and $G_1 = G - e$ then for the Laplacian eigenvalues*

$$0 = \mu_n(G_1) = \mu_n(G) \leq \mu_{n-1}(G_1) \leq \mu_{n-1}(G) \leq \cdots \leq \mu_2(G) \leq \mu_1(G_1) \leq \mu_1(G)$$

Proposition 2.6. [15] *For the Laplacian eigenvalues, we have $\mu_n(G) = \mu_n(\overline{G}) = 0$ and $\mu_i(\overline{G}) = n - (\mu_{n-i}(G))$ where $i = 1, 2, \cdots, n-1$.*

Theorem 2.7. [13] *Laplacian characteristic polynomial of $Kite_{n,n-p}$ is as follows.*

$$\begin{aligned} char(L(Kite_{n,n-p}))(x) &= (x-p)^{p-2}\Big[(x-p)\prod_{i=1}^{n-p+1}(x - 4sin^2\frac{\pi(i-1)}{2n-2p+2}) \\ &+ (1-p)\prod_{i=1}^{n-p}(x - 4sin^2\frac{\pi(i-1)}{2n-2p})\Big] \end{aligned}$$

3. $Kite_{p+2,p}$ is DLS

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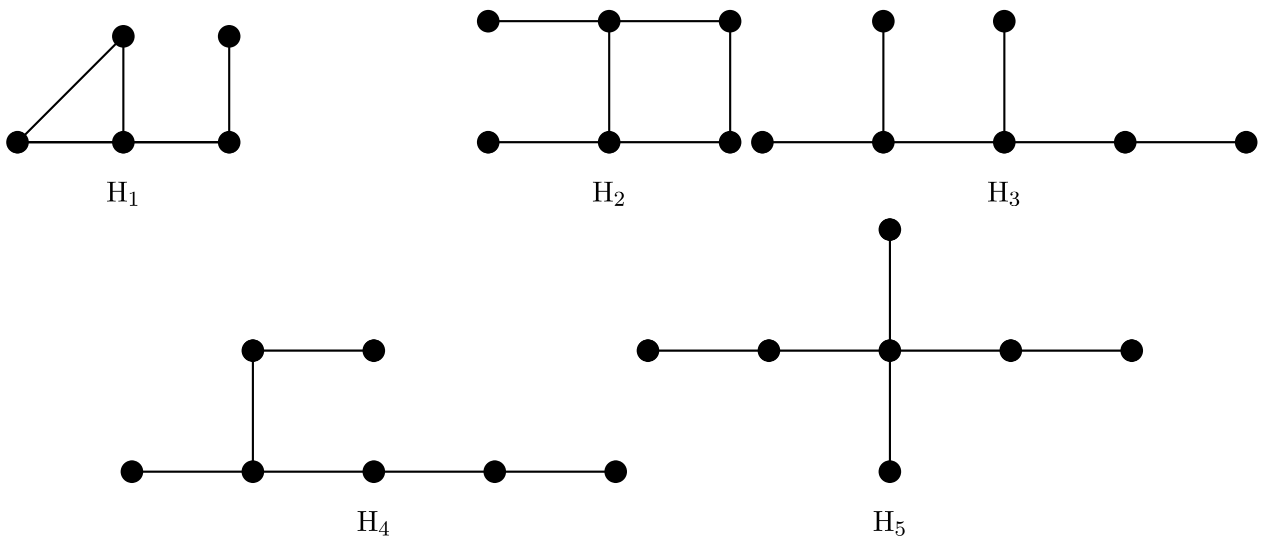

FIGURE 1. Forbidden subgraphs for $\overline{Kite_{p+2,p}}$.

Theorem 3.1. *$\overline{Kite_{p+2,p}}$ is determined by its Laplacian spectrum.*

Proof. From Theorem 2.7, if we put $n = p+2$, then we get the Laplacian characteristic polynomial of $Kite_{p+2,p}$ as follows.

$$char(L(Kite_{p+2,p}))(x) = x(x-p)^{p-2}(x^3 - (p+4)x^2 + (3p+4)x - p - 2)$$

Let us denote the eigenvalues of $L(Kite_{p+2,p})$ with $\mu_1' \geq \mu_2' \geq \cdots \geq \mu_{n-1}' \geq \mu_n' = 0$. Then we get, $p+1 < \mu_1' < p+2$, $\mu_2' = \cdots = \mu_{n-3}' = p$, $2 < \mu_{n-2}' < 3$, $0 < \mu_{n-1}' < 1$. If we denote the eigenvalues of $L(\overline{Kite_{p+2,p}})$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$, then by Proposition 2.6, we obtain that $p+1 < \mu_1 < p+2$, $p-1 < \mu_2 < p$, $\mu_3 = \cdots = \mu_{n-2} = 2$ and $0 < \mu_{n-1} < 1$.

Let a given graph G is Laplacian cospectral with $\overline{Kite_{p+2,p}}$. Clearly, G has $p+2$ vertices and $2p-1$ edges. Since $\mu_{n-1} > \mu_n = 0$, we say that G is connected. Also K_4, $K_3 \cdot \cup K_3$ and graphs given in Figure 1 are forbidden subgraphs for G, from Corollary 2.2 This means that, $w(G) \leq 3$ for the clique number of G.

First we will show that $w(G) = 2$. Hence we assume otherwise, that is $w(G) = 3$. In this case, G must contain at least one triangle. We denote the vertices of this triangle with $u, v, w \in V(G)$ and denote by T_u the set of all vertices of G which are exactly adjacent to u and non-adjacent to v and w. Similarly, we have $T_v, T_w, T_{uv}, T_{uw}, T_{vw}, T_{uvw}$. Now, we examine all of these sets. Clearly, $T_{uvw} = \emptyset$. Suppose that $a \in T_{uv}$, then we get $T_{uw} = T_{vw} = T_w = \emptyset$ and $N(a) = \{u, v\}$ since H_1 is forbidden. Also, for any $u_1 \in T_u$ or $v_1 \in T_v$, we get $N(u_1) = \{u\}$ and $N(v_1) = \{v\}$. Thus, we obtain that $V(G) = \{u, v, w\} \cup T_u \cup T_v$ and T_u, T_v contains just pendant vertices. Since G must contain $2p-1$ edges, then we say that $|T_{uv}| = p-3$ and $|T_u| + |T_v| = 2$. But, for a graph in this form, sum of the squares of degree vertices can not be same with $\overline{Kite_{p+2,p}}$. Hence, T_{uv} must be empty and $V(G) = \{u, v, w\} \cup T_u \cup T_v \cup T_w$. Since H_1 is forbidden, T_u, T_v and T_w may contain just pendant

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vertices. So we get $p = 3$. This gives a contradiction again. Thus we obtain that $w(G) \neq 3$, which means that $w(G) = 2$ and so $t(G) = 0$.

Now on, we continue with considering the sets $T_x, T_y, T_z, T_{xy}, T_{xz}, T_{yz}, T_{xyz}$ for an induced path P_3 of G with vertex set $\{x, y, z\}$ such that $x \sim y$ and $x \sim z$. It is clear that $T_{xy} = T_{xz} = T_{xyz} = \emptyset$. Let $T_{yz} = \emptyset$. Since H_3, H_4, H_5 are forbidden, if T_x, T_y and T_z contains at least two vertices, then these vertices must be pendant. In this case, edge number of G can not reach $2p - 1$. Thus, we say $T_{yz} \neq \emptyset$. Suppose that $u^{'} \in T_{yz}$. Let $a^{'} \in T_y$, so we say $T_{u^{'}} = T_x = \emptyset$ and $a^{'} \nsim u^{'}$ because H_1 and H_2 are forbidden. If $|T_y| \geq 2$, then for all $b \in T_y$, we get $N(b) = \{y\}$ because H_4 is forbidden. Similarly, we say if $|T_z| \geq 2$, then for all $c \in T_z$, $N(c) = \{z\}$. At the same time, if a complete bipartite graph $K_{m,n}$ is an induced subgraph of G, then we get $\min\{m, n\} \leq 2$ because H_2 is forbidden. Since $|T_y \cup T_z \cup T_{yz}| = p - 1$ and G must contain $2p - 1$ edges, we conclude that $G \cong \overline{Kite_{p+2,p}}$. □

By using Proposition 2.6 and Theorem 3.1, the following corollary can be easily obtained.

Corollary 3.2. *$Kite_{p+2,p}$ is determined by its Laplacian spectrum.*

4. Conclusion

From [9, 10, 11, 12, 13] and the result (Theorem 3.1) given in this note, we may conclude that $Kite_{n,p}$ is mostly determined by its spectrum, according to the different graph matrices. Hence, we give below some open cases.

Problem 1: $Kite_{n,p}$ is determined by its Laplacian spectrum for all values of $p \geq 4$ and n.

Problem 2: For which values of n and p, $Kite_{n,p}$ is determined by its signless Laplacian spectrum or distance spectrum, respectively?

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