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FORCING EDGE DETOUR MONOPHONIC NUMBER OF A GRAPH

P. TITUS AND K. GANESAMOORTHY*

Abstract. For a connected graph $G = (V, E)$ of order at least two, an *edge detour monophonic set* of G is a set S of vertices such that every edge of G lies on a detour monophonic path joining some pair of vertices in S. The *edge detour monophonic number* of G is the minimum cardinality of its edge detour monophonic sets and is denoted by $edm(G)$. A subset T of S is a *forcing edge detour monophonic subset* for S if S is the unique edge detour monophonic set of size $edm(G)$ containing T. A forcing edge detour monophonic subset for S of minimum cardinality is a *minimum forcing edge detour monophonic subset* of S. The *forcing edge detour monophonic number* $f_{edm}(S)$ in G is the cardinality of a minimum forcing edge detour monophonic subset of S. The *forcing edge detour monophonic number* of G is $f_{edm}(G) = min\{f_{edm}(S)\}$, where the minimum is taken over all edge detour monophonic sets S of size $edm(G)$ in G. We determine bounds for it and find the forcing edge detour monophonic number of certain classes of graphs. It is shown that for every pair a, b of positive integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_{edm}(G) = a$ and $edm(G) = b$.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [1]. The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. For a vertex v in G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The

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∗Corresponding author.
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minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G of G. Two vertices u and v of G are called *antipodal* if $d(u,v) = diam\, G$. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. A vertex v is a *semi-extreme vertex* of G if the subgraph induced by its neighbors has a full degree vertex in $N(v)$. In particular, every extreme vertex is a semi-extreme vertex and a semi-extreme vertex need not be an extreme vertex.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. For every pair x, y of vertices of G, a longest $x - y$ monophonic path is called $x - y$ *detour monophonic path*. A set S of vertices of G is a *detour monophonic set* if each vertex v of G lies on an $x - y$ detour monophonic path for some $x, y \in S$. The minimum cardinality of a detour monophonic set of G is the *detour monophonic number* of G and is denoted by $dm(G)$. The detour monophonic number of a graph was introduced in [3] and further studied in [2].

An *edge detour monophonic set* of G is a set S of vertices such that every edge of G lies on a detour monophonic path joining some pair of vertices in S. The *edge detour monophonic number* of G is the minimum cardinality of its edge detour monophonic sets and is denoted by $edm(G)$. The edge detour monophonic number of a graph was introduced and studied in [4].

For the graph G given in Figure 1, $S_1 = \{x, v_1, v_5\}$, $S_2 = \{x, v_1, v_4\}$, $S_3 = \{x, v_2, v_4\}$, $S_4 = \{x, v_1, v_2\}$, $S_5 = \{x, v_4, v_5\}$ and $S_6 = \{x, v_2, v_5\}$ are the edge detour monophonic sets of minimum size and so $edm(G)$=3.

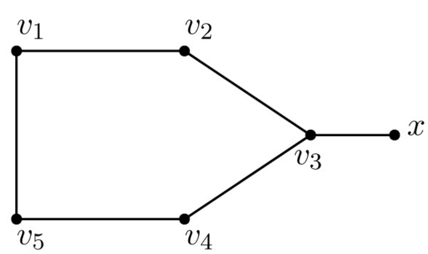

FIGURE 1. G

A connected graph G may contain more than one edge detour monophonic sets of size $edm(G)$. For example, the graph G given in Figure 1 contains six edge detour monophonic sets of size $edm(G)$. For each edge detour monophonic set S of size $edm(G)$ there is always some subset T of S that uniquely determines S as the edge detour monophonic set of size $edm(G)$ containing T. Such sets are called "forcing edge detour monophonic subsets" and we discuss these sets in this paper.

The following theorems will be used in the sequel.

Theorem 1.1. [4] *Each semi-extreme vertex of a graph G belongs to every edge detour monophonic set of G. In particular, if the set S of all semi-extreme vertices of G is an edge detour monophonic set, then S is the unique edge detour monophonic set of size $edm(G)$.*

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Theorem 1.2. [4] *For the complete graph $K_p(p \geq 2)$, $edm(K_p) = p$.*

Theorem 1.3. [4] *For any connected graph G, no cutvertex of G belongs to any edge detour monophonic set of size $edm(G)$.*

Theorem 1.4. [4] *Let G be a connected graph. Then $edm(G) = 2$ if and only if there exist two independent vertices u and v such that every edge of G lies on a $u - v$ detour monophonic path.*

Theorem 1.5. [4] *For the complete bipartite graph $G = K_{m,n}$ $(1 \leq m \leq n)$,*
(i) $edm(G) = 2$ if $m = n = 1$
(iii) $edm(G) = n$ if $m = 1$, $n > 1$
(iii) $edm(G) = m$ if $n \geq 2$.

Theorem 1.6. [4] *For the cycle $C_n(n \geq 3)$,*

$$edm(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$$

Throught this paper G denotes a connected graph with at least two vertices.

2. Forcing Edge Detour Monophonic Number

Definition 2.1. *Let G be a connected graph and let S be an edge detour monophonic set of size $edm(G)$. A subset T of S is a forcing edge detour monophonic subset for S if S is the unique edge detour monophonic set of size $edm(G)$ containing T. A forcing edge detour monophonic subset for S of minimum cardinality is a minimum forcing edge detour monophonic subset of S. The forcing edge detour monophonic number $f_{edm}(S)$ in G is the cardinality of a minimum forcing edge detour monophonic subset of S. The forcing edge detour monophonic number of G is $f_{edm}(G) = min\{f_{edm}(S)\}$, where the minimum is taken over all edge detour monophonic sets S of size $edm(G)$ in G.*

Example 2.2. *For the graph G given in Figure 1, $S_1 = \{x, v_1, v_5\}$, $S_2 = \{x, v_1, v_4\}$, $S_3 = \{x, v_2, v_4\}$, $S_4 = \{x, v_1, v_2\}$, $S_5 = \{x, v_4, v_5\}$ and $S_6 = \{x, v_2, v_5\}$ are the edge detour monophonic sets of size $edm(G)$. If y is an element of $S_i(1 \leq i \leq 6)$, then $\{y\}$ is a subset of more than one edge detour monophonic sets of size $edm(G)$. Since S_1 is the unique edge detour monophonic set of size $edm(G)$ containing $\{v_1, v_5\}$, we have $f_{edm}(S_1) = 2$ and so $f_{edm}(G) = 2$. For the graph G given in Figure 2, $S = \{v_1, v_5\}$ is the unique edge detour monophonic set of size $edm(G)$ and so $f_{edm}(G) = 0$.*

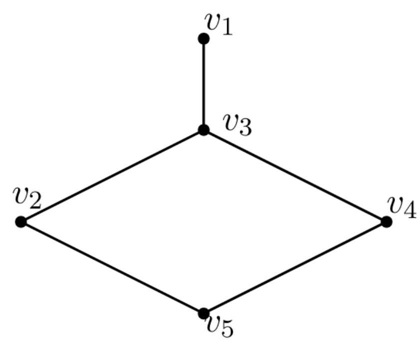

FIGURE 2. G

The next theorem follows immediately from the definitions of the edge detour monophonic number and forcing edge detour monophonic number of a graph G.

Theorem 2.3. *For a connected graph G of order p, $0 \leq f_{edm}(G) \leq edm(G) \leq p$.*

Remark 2.4. *The bounds in Theorem 2.3 are sharp. For the graph G given in Figure 2, $f_{edm}(G) = 0$. By Theorem 1.2, for the complete graph $K_p (p \geq 2)$, $edm(K_p) = p$. The inequalities in Theorem 2.3 are strict. For the graph G given in Figure 1, $edm(G) = 3$ and $f_{edm}(G) = 2$. Thus $0 < f_{edm}(G) < edm(G) < p$.*

The following theorem is an easy consequence of the definitions of the edge detour monophonic number and forcing edge detour monophonic number. In fact, the theorem characterizes graphs G for which the lower bound in Theorem 2.3 is attained and also graphs G for which $f_{edm}(G) = 1$ and $f_{edm}(G) = edm(G)$.

Theorem 2.5. *Let G be a connected graph . Then*
(i) *$f_{edm}(G) = 0$ if and only if G has a unique edge detour monophonic set of size $edm(G)$.*
(ii) *$f_{edm}(G) = 1$ if and only if G has at least two edge detour monophonic sets of size $edm(G)$, one of which is a unique edge detour monophonic set of size $edm(G)$ containing one of its elements, and*
(iii) *$f_{edm}(G) = edm(G)$ if and only if no edge detour monophonic set of size $edm(G)$ is the unique edge detour monophonic set of size $edm(G)$ containing any of its proper subsets.*

Definition 2.6. *A vertex v of a connected graph G is said to be an edge detour monophonic vertex of G if v belongs to every edge detour monophonic set of size $edm(G)$.*

We observe that if G has a unique edge detour monophonic set S of size $edm(G)$, then every vertex in S is an edge detour monophonic vertex of G. Also, if x is a semi-extreme vertex of G, then x is an edge detour monophonic vertex of G.

The following theorem and corollary follows immediately from the definitions of edge detour monophonic vertex and forcing edge detour monophonic subset of G.

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Theorem 2.7. *Let G be a connected graph and let $\Im_{edm}$ be the set of relative complements of the minimum forcing edge detour monophonic subsets in their respective edge detour monophonic sets of size $edm(G)$ in G. Then $\bigcap_{F \in \Im_{edm}} F$ is the set of edge detour monophonic vertices of G.*

Corollary 2.8. *Let G be a connected graph and let S be an edge detour monophonic set of size $edm(G)$. Then no edge detour monophonic vertex of G belongs to any minimum forcing edge detour monophonic subset of S.*

Theorem 2.9. *Let G be a connected graph and let M be the set of all edge detour monophonic vertices of G. Then $f_{edm}(G) \leq edm(G) - |M|$.*

Proof. Let S be any edge detour monophonic set of size $edm(G)$. Then $edm(G) = |S|$, $M \subseteq S$ and S is the unique edge detour monophonic set of size $edm(G)$ containing $S - M$. Thus $f_{edm}(G) \leq |S - M| = |S| - |M| = edm(G) - |M|$. □

Corollary 2.10. *If G is a connected graph with l semi-extreme vertices, then $f_{edm}(G) \leq edm(G) - l$.*

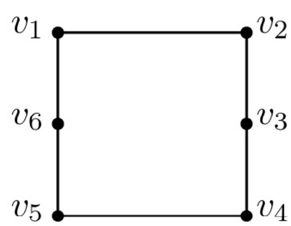

FIGURE 3. G

Remark 2.11. *The bound in Theorem 2.9 is sharp. For the graph G given in Figure 1, $edm(G) = 3$ and $f_{edm}(G) = 2$. Also, x is the unique edge detour monophonic vertex of G and so $f_{edm}(G) = edm(G) - |M|$. Also the inequality in Theorem 2.9 can be strict. For the graph G given in Figure 3, $S_1 = \{v_1, v_4\}$, $S_2 = \{v_2, v_5\}$ and $S_3 = \{v_3, v_6\}$ are the edge detour monophonic sets of minimum size so that $edm(G) = 2$ and $f_{edm}(G) = 1$. Also, no vertex of G is an edge detour monophonic vertex of G, we have $f_{edm}(G) < edm(G) - |M|$.*

Theorem 2.12. *Let G be a connected graph and let S be an edge detour monophonic set of size $edm(G)$. Then no cutvertex of G belongs to any minimum forcing edge detour monophonic subset of S.*

Proof. Let v be a cutvertex of G. By Theorem 1.3, v does not belong to any edge detour monophonic set of size $edm(G)$. Since any minimum forcing edge detour monophonic subset of S is a subset of S, the result follows from Theorem 2.7. □

Theorem 2.13. *If $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$, then $f_{edm}(G) = 0$.*

Proof. It is clear that every vertex of G is a semi-extreme vertex of G except the vertex $V(K_1)$ and the vertex $V(K_1)$ is a cutvertex. Then by Theorems 1.1 and 1.3, we have $S = V(G) - V(K_1)$ is the unique edge detour monophonic set of size $edm(G)$. Hence by Theorem 2.5(i), $f_{edm}(G) = 0$. □

We denote the vertex connectivity of a connected graph G by $\kappa(G)$ or κ.

Theorem 2.14. *If G is a non-complete connected graph such that it has a minimum cutset consisting of κ vertices, then $f_{edm}(G) \leq p - \kappa$.*

Proof. Since G is a non-complete connected graph, it is clear that $1 \leq \kappa \leq p - 2$. Let $U = \{u_1, u_2, u_3, \ldots, u_\kappa\}$ be a minimum cutset of G. Let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of $G - U$ and let $S = V(G) - U$. Then every edge $uu_i (1 \leq i \leq \kappa)$, where $u \in V(G) - U$ is incident to one vertex of G_j for each $j (1 \leq j \leq r)$. It is clear that S is an edge detour monophonic set of G and so $edm(G) \leq |S| = p - \kappa$. Hence by Theorem 2.3, we have $f_{edm}(G) \leq p - \kappa$. □

Theorem 2.15. *If G is a connected graph with $edm(G) = 2$, then $f_{edm}(G) \leq 1$.*

Proof. Let $edm(G) = 2$. Then by Theorem 2.3, $f_{edm}(G) \leq 2$. Suppose that $f_{edm}(G) = 2$. Then by Theorems 2.5(i) and 2.5(iii), G has at least two edge detour monophonic sets of size $edm(G)$ and no edge detour monophonic set of size $edm(G)$ is the unique edge detour monophonic set of size $edm(G)$ containing any of its proper subsets. Since $edm(G) = 2$, there exists a unique element, say x, is common for any two edge detour monophonic sets of size $edm(G)$, say S_1 and S_2. Let $S_1 = \{x, u\}$ and $S_2 = \{x, v\}$. Since S_1 is an edge detour monophonic set of size $edm(G)$, by Theorem 1.4, the edge vv_i, for some $v_i \neq u \in V(G)$ lies on an $x - u$ detour monophonic path. Similarly, since S_2 is an edge detour monophonic set of size $edm(G)$, by Theorem 1.4, uv_j, for some $v_j \neq v \in V(G)$ lies on an $x - v$ detour monophonic path. Then the vertices x, u and v lie on a cycle. Let C be a longest cycle containing the vertices x, u and v. Then the length of C is more than 4. If C is an even cycle, then either S_1 or S_2 is not an edge detour monophonic set of G, which is a contradiction. If C is an odd cycle, then any internal edge of an $x - u$ geodesic does not lie on an $x - u$ detour monophonic path and so S_1 is not an edge detour monophonic set of G, which is a contradiction. Hence $f_{edm}(G) \leq 1$. □

Now, we proceed to determine the forcing edge detour monophonic number of certain classes of graphs.

Theorem 2.16. *For any cycle $C_n (n \geq 4)$, $f_{edm}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$.*

Proof. Let $C_n : v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_n, v_1$ be a cycle of order n.
Case (i) n is even. Let $n = 2m$. Then every edge detour monophonic set of size $edm(C_n)$ consists of a pair of antipodal vertices and C_n has exactly m edge detour monophonic sets of size $edm(C_n)$. Hence C_n has at least two edge detour monophonic sets of size $edm(C_n)$ and one of the edge detour

monophonic set of size $edm(C_n)$ of which containing one of its elements. Then by Theorem 2.5(ii), $f_{edm}(C_n) = 1$.

Case (ii) n is odd. Let $n = 2m + 1$. It is clear that no two point set will form an edge detour monophonic set of C_n. Now, $\{v_1, v_2, v_3\}$ is an edge detour monophonic set of minimum size and so $edm(C_n) = 3$. We observe that any edge detour monophonic set of size $edm(C_n)$ is any one of the following forms.

(i) any three consecutive vertices

(ii) a vertex and its antipodal vertices

(iii) any three non-adjacent vertices

Then clearly no edge detour monophonic set of size $edm(C_n)$ is the unique edge detour monophonic set of size $edm(C_n)$ containing any of its proper subsets. Hence by Theorem 2.5(iii), $f_{edm}(C_n) = edm(C_n) = 3$. □

Theorem 2.17. *For any complete graph $G = K_p (p \geq 2)$ or any non-trivial tree $G = T$, $f_{edm}(G) = 0$.*

Proof. For $G = K_p$, it follows from Theorem 1.2 that the set of all vertices of G is the unique edge detour monophonic set of size $edm(G)$. Now, it follows from Theorem 2.5(i) that $f_{edm}(G) = 0$. If G is a non-trivial tree, then by Theorems 1.1 and 1.3, the set of all endvertices of G is the unique edge detour monophonic set of size $edm(G)$ and so by Theorem 2.5(i), $f_{edm}(G) = 0$. □

Theorem 2.18. *For the complete bipartite graph $G = K_{m,n} (2 \leq m \leq n)$,*

$$f_{edm}(G) = \begin{cases} 0 & \text{if } m < n \\ 1 & \text{if } m = n \end{cases}$$

Proof. We prove this theorem by considering two cases. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the bipartition of G, where $m \leq n$.

Case 1. $m = n$. Then by Theorem 1.5, U and W are the edge detour monophonic sets of size $edm(G)$ and so by Theorem 2.5(ii), $f_{edm}(G) = 1$.

Case 2. $m < n$. Then by Theorem 1.5, U is the unique edge detour monophonic set of size $edm(G)$ and so by Theorem 2.5(i), $f_{edm}(G) = 0$. □

Theorem 2.19. *For every pair a, b of positive integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $f_{edm}(G) = a$ and $edm(G) = b$.*

Proof. If $a = 0$, let $G = K_b$. Then by Theorem 2.17, $f_{edm}(G) = 0$ and by Theorem 1.2, $edm(G) = b$. Thus we assume that $0 < a < b$. We consider four cases.

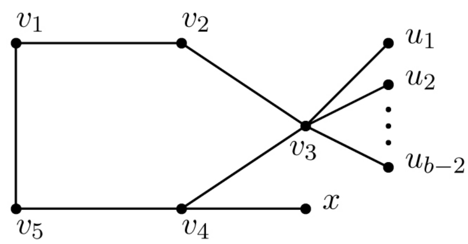

Figure 4. G

Case 1. $a = 1$. If $b = 2$, then take $G = C_{2n}$. By Theorem 2.16, $f_{edm}(G) = a$ and Theorem 1.6, $edm(G) = b$. So, we assume that $b \geq 3$. Let G be the graph obtained from the cycle C_5 : $v_1, v_2, v_3, v_4, v_5, v_1$ of order 5 by adding $b-1$ new vertices $u_1, u_2, \ldots, u_{b-2}, x$ and joining each $u_i (1 \leq i \leq b-2)$ to v_3; and joining x to v_4. The graph G is shown in Figure 4. Let $S = \{u_1, u_2, \ldots, u_{b-2}, x\}$ be the set of all semi-extreme vertices of G. By Theorem 1.1, every edge detour monophonic set of G contains S. It is clear that S is not an edge detour monophonic set of G. It is easily verified that $S_1 = S \cup \{v_1\}$, $S_2 = S \cup \{v_2\}$ and $S_3 = S \cup \{v_5\}$ are the edge detour monophonic sets of size $edm(G)$. Hence $edm(G) = b$. Moreover, since S_1 is the unique edge detour monophonic set of size $edm(G)$ containing the vertex v_1, it follows that $f_{edm}(S_1) = 1$ and so $f_{edm}(G) = 1$.

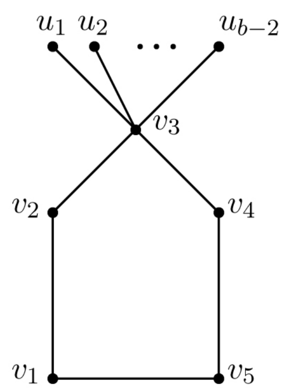

Figure 5. G

Case 2. $a = 2$. Then $b \geq 3$. Let G be the graph obtained from the cycle $C_5 : v_1, v_2, v_3, v_4, v_5, v_1$ of order 5 by adding $b-2$ new vertices $u_1, u_2, \ldots, u_{b-2}$ and joining each $u_i (1 \leq i \leq b-2)$ to v_3. The graph G is shown in Figure 5. Let $S = \{u_1, u_2, \ldots, u_{b-2}\}$ be the set of all semi-extreme vertices of G. By Theorem 1.1, every edge detour monophonic set of G contains S. Clearly, S is not an edge detour monophonic set of G. Also $S \cup \{x\}$, where $x \in V(G) - S$, is not an edge detour monophonic set of G. It is easily verified that $S_1 = S \cup \{v_1, v_5\}$, $S_2 = S \cup \{v_1, v_4\}$, $S_3 = S \cup \{v_2, v_4\}$, $S_4 = S \cup \{v_1, v_2\}$, $S_5 = S \cup \{v_4, v_5\}$ and $S_6 = S \cup \{v_2, v_5\}$ are the edge detour monophonic sets of size $edm(G)$. Hence $edm(G) = b$. If x is an element of $S_i (1 \leq i \leq 6)$, then $\{x\}$ is a subset of at least two edge detour monophonic sets of size $edm(G)$. Hence it follows from Theorem 2.5(i) and (ii) that $f_{edm}(G) \geq 2$. Since S_1 is the unique edge detour monophonic set of size $edm(G)$ containing $\{v_1, v_5\}$, we have $f_{edm}(G) = 2$.

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Case 3. $a \geq 3$ and $b = a + 1$. We consider two subcases.

Subcase (i). a is even. Let $a = 2m$. Let $C_i : x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i1} (1 \leq i \leq m)$ be m copies of a cycle of order 5 and let $P : x, y$ be a path of order 2. Let G be the graph obtained from $C_i\,(1 \leq i \leq m)$ and P by identifying the vertices x_{i3} from C_i and y from P. The graph G is shown in Figure 6. Now, x is the unique semi-extreme vertex of G. Then by Theorem 1.1, every edge detour monophonic set of G contains x. Clearly, $\{x\}$ is not an edge detour monophonic set of G. Let $M_i = \{\{x_{i1}, x_{i2}\}, \{x_{i1}, x_{i4}\}, \{x_{i1}, x_{i5}\}, \{x_{i2}, x_{i4}\}, \{x_{i2}, x_{i5}\}, \{x_{i4}, x_{i5}\}\}$ $(1 \leq i \leq m)$. We observe that every edge detour monophonic set of G contains exactly one element from $M_i\,(1 \leq i \leq m)$. Thus $edm(G) \geq 2m + 1$. Since $S = \{x, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, \ldots, x_{m1}, x_{m2}\}$ is an edge detour monophonic set of G, it follows that $edm(G) = 2m + 1 = b$.

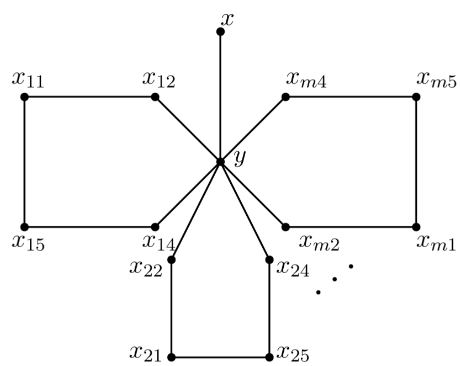

Figure 6. G

Next we show that $f_{edm}(G) = a$. Since every edge detour monophonic set of size $edm(G)$ contains $\{x\}$, it follows from Theorem 2.9 that $f_{edm}(G) \leq edm(G) - |\{x\}| = b - 1 = a$. Let T be a minimum forcing edge detour monophonic subset of S. If $|T| < a$, then there exist an element $y \in S - \{x\}$ such that $y \notin T$, say $y = x_{11}$. Let $S' = (S - \{x_{11}\}) \cup \{x_{14}\}$. Then $S' \neq S$ and S' is also an edge detour monophonic set of size $edm(G)$ such that it contains T, which is a contradiction to T a forcing edge detour monophonic subset of S. Thus $|T| = a$ and so $f_{edm}(G) = a$.

Subcase (ii). a is odd. Let $a = 2m+1$. Let $C_i : x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i1} (1 \leq i \leq m)$ be m copies of a cycle of order 5 and let $C : u, v, w, x, y, z, u$ be a cycle of order 6. Let G be the graph obtained from $C_i\,(1 \leq i \leq m)$ and C by identifying the vertices x_{i3} from C_i and u from C_6; and joining the vertices w and y. The graph G is shown in Figure 7. Now, x is the only semi-extreme vertex of G. Then by Theorem 1.1, every edge detour monophonic set of G contains x. Clearly, $\{x\}$ is not an edge detour monophonic set of G. Let $M_i = \{\{x_{i1}, x_{i2}\}, \{x_{i1}, x_{i4}\}, \{x_{i1}, x_{i5}\}, \{x_{i2}, x_{i4}\}, \{x_{i2}, x_{i5}\}, \{x_{i4}, x_{i5}\}\}\,(1 \leq i \leq m)$ and $M_{m+1} = \{\{w, x\}, \{x, y\}\}$. We observe that every edge detour monophonic set of G contains exactly one element from $M_i\,(1 \leq i \leq m + 1)$. Thus $edm(G) \geq 2m + 2$. Since $S = \{x, w, x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, \ldots, x_{m1}, x_{m2}\}$ is an edge detour monophonic set of G, it follows that $edm(G) = 2m + 2 = b$.

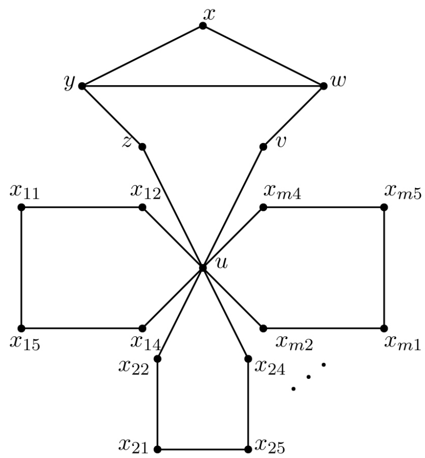

FIGURE 7. G

Next we show that $f_{edm}(G) = a$. Since every edge detour monophonic set of size $edm(G)$ contains $\{x\}$, it follows from Theorem 2.9 that $f_{edm}(G) \leq edm(G) - |\{x\}| = b - 1 = a$. Then by an argument similar to Subcase (i), we have $f_{edm}(G) = a$.

Case 4. $a \geq 3$ and $b \geq a + 2$. Let $F_i : l_i, m_i, n_i, o_i, p_i, q_i, l_i\,(1 \leq i \leq a)$ be "a" number of copies of C_6. Let H be the graph obtained from $F_i(1 \leq i \leq a)$ by identifying the vertices o_{i-1} of F_{i-1} and l_i of $F_i(2 \leq i \leq a)$. Let G be the graph obtained from H by adding $b - a$ new vertices $z_1, z_2, \ldots z_{b-a-1}, z$ and joining each $z_i(1 \leq i \leq b - a - 1)$ to o_a; joining the vertex z to l_1 and joining the vertices m_i and $q_i(1 \leq i \leq a)$. The graph G is shown in Figure 8. Let $S = \{z_1, z_2, \ldots z_{b-a-1}, z\}$ be the set of all semi-extreme vertices of G. Then by Theorem 1.1, every edge detour monophonic set of G contains S. Clearly, S is not an edge detour monophonic set of G. We observe that every edge detour monophonic set of size $edm(G)$ contains exactly one vertex from $\{m_i, q_i\}$ for every $i(1 \leq i \leq a)$. Thus $edm(G) \geq b$. Since $S' = S \cup \{m_1, m_2, \ldots, m_a\}$ is an edge detour monophonic set of G, it follows that $edm(G) = b$.

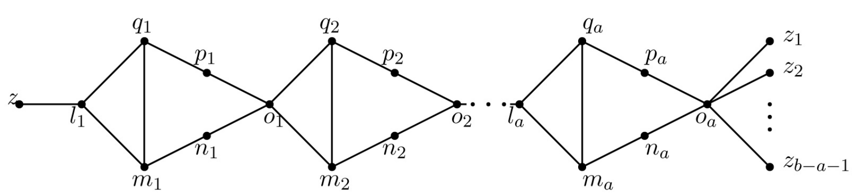

FIGURE 8. G

Next we show that $f_{edm}(G) = a$. Since every edge detour monophonic set of size $edm(G)$ contains S, it follows from Theorem 2.9 that $f_{edm}(G) \leq edm(G) - |S| = a$. Let T be a minimum forcing edge detour monophonic subset of S'. If $|T| < a$, then there exist an element $y \in S' - S$ such that $y \notin T$, say $y = m_1$. Let $S'' = (S' - \{m_1\}) \cup \{q_1\}$. Then $S'' \neq S'$ and S'' is also an edge detour monophonic set of size $edm(G)$ such that it contains T, which is a contradiction to T a forcing edge detour monophonic subset of S'. Thus $|T| = a$ and so $f_{edm}(G) = a$. □

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P. Titus

Department of Mathematics, University College of Engineering Nagercoil, Nagercoil-629 004, India

Email: titusvino@yahoo.com

K. Ganesamoorthy

Department of Mathematics, Coimbatore Institute of Technology, Coimbatore - 641 014, India

Email: kvgm_2005@yahoo.co.in

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