Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 10 No. 4 (2021), pp. 247-252.

www.toc.ui.ac.ir
www.ui.ac.ir

ON FINITE GROUPS ALL OF WHOSE BI-CAYLEY GRAPHS OF BOUNDED VALENCY ARE INTEGRAL

MAJID AREZOOMAND

ABSTRACT. Let $k \geq 1$ be an integer and $\mathcal{I}_k$ be the set of all finite groups G such that every bi-Cayley graph $\mathrm{BCay}(G,S)$ of G with respect to subset S of length $1 \leq |S| \leq k$ is integral. Let $k \geq 3$. We prove that a finite group G belongs to $\mathcal{I}_k$ if and only if $G \cong \mathbb{Z}_3$, $\mathbb{Z}_2^r$ for some integer r, or S_3.

1. Introduction

A finite simple graph is called *integral* if all its adjacency eigenvalues are integers. In 1974, Harary and Schwenk [10] introduced the concept of integral graphs, and proposed the problem of classifying integral graphs. This problem has been attacked during last forty years and it is still open, for a survey see [1, 6].

Let G be a finite group and $S \subseteq G$. The bi-Cayley graph of G with respect to S, which is denoted by $\mathrm{BCay}(G,S)$, is the graph with vertex set $G \times \{1,2\}$ and edge set $\{\{(g,1),(sg,2)\} \mid g \in G, s \in S\}$. $\mathrm{BCay}(G,S)$ is an undirected $|S|$-regular graph. The present author started the problem of classifying integral bi-Cayley graphs in [3]. In [3], it is proved that all bi-Cayley graphs of a group G is integral if and only if G is a cyclic group of order 3, an elementary abelian 2-group or the symmetric group on three symbols. Also, in [5], finite groups admitting a connected cubic bi-Cayley graph classified. In this paper, we are going to study finite groups G for which $\mathrm{BCay}(G,S)$ is integral whenever $|S| \leq k$ for some fixed integer k. This problem is the natural analogue of the problem of the group classes defined in [7] and later studied in [8, 12].

Communicated by Ali Reza Ashrafi.
MSC(2010): Primary: 05C50; Secondary: 20C10, 05C25, 05E10.
Keywords: Bi-Cayley graph, Integer eigenvalues, Irreducible representation.
Received: 01 December 2020, Accepted: 02 May 2021.
Manuscript Type: Research Paper.

2. **Main Results**

Let $k \geq 1$ be an integer and $\mathcal{I}_k$ be the set of all finite groups G such that every bi-Cayley graph $\mathrm{BCay}(G,S)$ of G with respect to subset S of length $1 \leq |S| \leq k$ is integral. Since for any element $a \in G$, $\mathrm{BCay}(G,\{a\}) \cong |G|K_2$ is integral, it is obvious that $\mathcal{I}_1$ is the set of all finite groups. Let us start with the following lemma.

Lemma 2.1. *Let $G \in \mathcal{I}_k$. Then every undirected Cayley graph of G of valency at most k is integral.*

Proof. Let S be an inverse-closed subset of $G \setminus \{1\}$ and $\Gamma = \mathrm{Cay}(G,S)$. Then, by [3, Lemma 3.2], $\mathrm{BCay}(G,S)$ is integral if and only if Γ is integral. Since $G \in \mathcal{I}_k$, $\mathrm{BCay}(G,S)$ is integral and so Γ is integral. This completes the proof. □

Lemma 2.2. *Let $k \geq 2$ and $G \in \mathcal{I}_k$. Then*

(*i*) *the order of any non-identity element of G is 2 or 3,*
(*ii*) *every subgroup of G is an element of $\mathcal{I}_k$,*
(*iii*) *for every normal subgroup N of G such that $|N| \mid k$, we have $G/N \in \mathcal{I}_l$, where $l = k/|N|$.*

Proof. Let $a \in G$ be of order $n > 1$ and $S = \{1, a\}$. Then $\mathrm{BCay}(G,S) \cong |G : \langle a\rangle|\mathrm{BCay}(\langle a\rangle, S)$ is integral. On the other hand, $\mathrm{BCay}(\langle a\rangle, S) \cong C_{2n}$, which implies that $2n \in \{3,4,6\}$. Hence $n = 2$ or 3. This proves (i).

Let $H \leq G$ and $S \subset H$ with $2 \leq |S| \leq k$. Then, by [3, Lemma 3.5], $\mathrm{BCay}(G,S) \cong |G : H|\mathrm{BCay}(H,S)$. Since $G \in \mathcal{I}_k$, $\mathrm{BCay}(H,S)$ is integral. This implies that $H \in \mathcal{I}_k$, which proves (ii).

Let N be a normal subgroup of G such that $|N| \mid k$. Put $l = k/|N|$. Let $\overline{S} = \{Ns_1, \ldots, Ns_m\}$ be a subset of G/N with $2 \leq m \leq l$. Then, by [4, proof of Lemma 2.2], $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G/N, \overline{S})[mK_1]$, where $S = \bigcup_{i=1}^{m} Ns_i$. On the other hand, $|S| \leq m|N| \leq l|N| = k$. Since $G \in \mathcal{I}_k$, $\mathrm{BCay}(G,S)$ is integral and so $\mathrm{BCay}(G/N, \overline{S})$ is integral. This proves (iii). □

Corollary 2.3. *A finite group G belongs to $\mathcal{I}_2$ if and only if the order of any non-identity element of G is 2 or 3.*

Proof. One direction is clear by Lemma 2.2(i). Conversely, suppose that G is a finite group in which the order of any non-identity element of G is 2 or 3. Let $S \subseteq G$ and $|S| = 2$. Since $\mathrm{BCay}(G,S) \cong \mathrm{BCay}(G,Sg)$ for all $g \in G$ [11], we may assume that $1 \in S$. Hence $S = \{1, s\}$, where the order of s is 2 or 3. Now, by [3, Lemma 3.5], $\mathrm{BCay}(G,S) \cong |G : \langle s\rangle|\mathrm{BCay}(\langle s\rangle, S)$. Since $\mathrm{BCay}(\langle s\rangle, S)$ is a cycle of length 4 or 6, it is integral. This means that $\mathrm{BCay}(G,S)$ is integral i.e $G \in \mathcal{I}_k$. □

From now on, we focus on classifying elements of $\mathcal{I}_k$, where $k \geq 3$. First we determine finite p-groups contained in $\mathcal{I}_k$.

Lemma 2.4. *Let $k \geq 3$ and $G \in \mathcal{I}_k$ be a p-group. Then $G \cong \mathbb{Z}_2^r$ or $\mathbb{Z}_3$ for some integer r.*

Proof. Since $G \in \mathcal{I}_k$, by Lemma 2.2(i), every element of G is of order p and $p \in \{2,3\}$. Let $G \ncong \mathbb{Z}_2^r$. Then there exists an element $a \in G$ of order 3. Then $\langle a \rangle$ is the unique subgroup of G of order 3, since if there exists $a, a^2 \neq b \in G$ then, putting $S = \{1, a, b\}$, since $\mathrm{BCay}(G,S)$ is integral, $\mathrm{BCay}(\langle S \rangle, S)$ is also integral and so $\langle S \rangle$ admits a connected cubic integral bi-Cayley graph and therefore $\langle S \rangle \cong \mathbb{Z}_3$ by [5, Theorem A], a contradiction. Since the exponent of G is 3, we have $G \cong \mathbb{Z}_3$, as desired. ☐

Corollary 2.5. *Let $G \in \mathcal{I}_k$, $k \geq 3$, be nilpotent. Then $G \cong \mathbb{Z}_2^r$ or $\mathbb{Z}_3$ for some integer r.*

Proof. By Lemma 2.4, it is sufficent to prove that G is a p-group. Suppose, towards a contradiction, that G is not a p-group. Then, by Lemmas 2.2(ii) and 2.4, $G = P \times Q$, where $P \cong \mathbb{Z}_2^r$ for some r and $Q \cong \mathbb{Z}_3$. Let $a \in P$ and $b \in Q$. Then, by Lemma 2.2(i), ab is of order 2 or 3. Hence $ab \in P$ or $ab \in Q$, which implies that $b \in P$ or $a \in Q$, a contradiction. ☐

It remains to classifying non-nilpotent elements of $\mathcal{I}_k$. To this aim, we need the following lemma. First let us recall n-Cayley graphs: a graph Γ is called an n-Cayley graph over a group G if $\mathrm{Aut}(\Gamma)$ admits a semiregular subgroup on $V(\Gamma)$ isomorphic to G with n orbits. Equivalently, a graph Γ is an n-Cayley graph over a group G if and only if there exists n^2 subsets T_{ij}, $1 \leq imj \leq n$ of G such that Γ is isomorphic to the graph $\mathrm{Cay}(G; T_{ij} \mid 1 \leq i,j \leq n)$ with vertex set $G \times \{1, \ldots, n\}$ where (g,i) and (h,j) are adjacent whenever $hg^{-1} \in T_{ij}$, for more details see [2].

Lemma 2.6. *Let $G \in \mathcal{I}_k$ and $N \trianglelefteq G$. Then $G/N \in \mathcal{I}_k$.*

Proof. Let $\Gamma_{G/N} = \mathrm{BCay}(G/N, \mathcal{S})$ be a bi-Cayley graph over G/N of valency at most k. We will prove that $\Gamma_{G/N}$ is integral. Let $|G:N| = m$, $|N| = n$ and $T = \{t_1, \ldots, t_m\}$ be a right transversal of N in G. Let $\mathcal{S} = \{Ns \mid s \in S\}$ for some $S \subseteq T$. Let $\Gamma_G = \mathrm{BCay}(G,S)$. Since N also acts on the vertex set of Γ_G semiregularly, Γ_G is a $2m$-Cayley graph over N with orbits $(t_iN, 1)$ and $(t_iN, 2)$ where $i = 1, \ldots, m$. More exactly, by a suitable ordering of vertices of Γ_G, we have $\Gamma_G = \mathrm{Cay}(N, R_{ij} \mid 1 \leq i,j \leq 2m)$, where

$$R_{ij} = \begin{cases} \emptyset & 1 \leq i,j \leq m \text{ or } m+1 \leq i,j \leq 2m \\ \{x \in N \mid t_j x t_i^{-1} \in S\} & 1 \leq i \leq m,\ m+1 \leq j \leq 2m \\ \{x \in N \mid t_j x t_i^{-1} \in S^{-1}\} & m+1 \leq i \leq 2m,\ 1 \leq j \leq m \end{cases}$$

Let $S_{ij} = R_{ij}$ where $1 \leq i \leq m,\ m+1 \leq j \leq 2m$ and $S_{ji} = R_{ij}$ where $m+1 \leq i \leq 2m,\ 1 \leq j \leq m$. By considering the trivial character of N, by [2, Theorem 6], each eigenvalue of the $2m \times 2m$ block matrix

$$A = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix},$$

where 0 is the $m \times m$ zero matrix and $B = [|S_{ji}|]_{1 \leq i,j \leq m}$, is an eigenvalue of Γ_G. Since $G \in \mathcal{I}_k$ and $|S| \leq k$, Γ_G is integral. Hence eigenvalues of A are integers.

We note that $|S_{ij}| = |S_{ji}|$ and we claim that $|S_{ij}| \leq 1$. To see this, suppose, towards a contradiction, that x and y are two distinct elements of S_{ij}. Then $x, y \in N$ and since $S \subseteq T$, $t_j x t_i^{-1} = t$ and

$t_jyt_i^{-1} = t'$ for some $t, t' \in T$. Hence $Nt_jxt_i^{-1} = Nt$ and $Nt_jyt_i^{-1} = Nt'$. Since $N \trianglelefteq G$, we get $Nt_jt_i^{-1} = Nt = Nt'$, which implies that $t = t'$. Hence $x = y$, a contradiction. On the other hand, $|S_{ij}| = 1$ if and only if $Nt_jt_i^{-1} = Ns$ for some $s \in S$. This implies that $|S_{ij}| = 1$ if and only if $(Nt_i, 1)$ is adjacent to $(Nt_j, 1)$ in $\Gamma_{G/N}$. This means that the adjacency matrix of $\Gamma_{G/N}$ is similar to the matrix A and so $\Gamma_{G/N}$ is integral. This completes the proof. □

The proof of following lemma is similar to the proof of [7, Lemma 13], however in the case $k \geq 4$ it is a direct consequence of [7, Lemma 13] and Lemma 2.1.

Lemma 2.7. *Let $G \in \mathcal{I}_k$, $k \geq 3$, and $3 \mid |G|$. Then G has a normal Sylow 3-subgroup.*

Proof. By Lemma 2.2(i), $|G| = 2^m3^n$ for some integers $m \geq 0$ and $n \geq 1$. There is nothing to prove if G is a 3-group. Hence, we may assume that $m \geq 1$. Let P be a Sylow 3-subgroup and Q be a Sylow 2-subgroup of G. Then Lemma 2.4 implies that $P \cong \mathbb{Z}_3$ and $Q \cong \mathbb{Z}_2^m$. Furthermore, $G = PQ$ and $P \cap Q = 1$.

We prove the result by induction. If $|G| = 6$ then clearly G has a normal Sylow 3-subgroup. Suppose that every group $H \in \mathcal{I}_k$, $k \geq 3$, which 3 divides $|H|$ and $|H| < |G|$, has a normal Sylow 3-subgroup (induction hypothesis). We will prove that G has a normal Sylow 3-subgroup.

By Burnside's pq-Theorem, G is a solvable group and so every minimal normal subgroup of G is elementary abelian. Let K be a minimal normal subgroup of G and M be a maximal normal subgroup of G containing K. Then Lemma 2.2 and Corollary 2.5 imply that $K \cong \mathbb{Z}_2^r$ for some integer r or $\mathbb{Z}_3$. Furthermore, $M \in \mathcal{I}_k$ and $G/M \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$.

Let $G/M \cong \mathbb{Z}_2$. Then 3 divides $|M|$. Hence, by induction hypothesis, M has a normal Sylow 3-subgroup, say N. Since N is a characteristic subgroup of M and M is a normal subgroup of G, N is also a normal subgroup of G. Furthermore, N is a Sylow 3-subgroup of G.

Now let $G/M = \langle Mx \rangle \cong \mathbb{Z}_3$ for some $x \in G \setminus M$. Then $M = Q$ and since by Lemma 2.2(i), x has order 2 or 3, $x^3 \in M$ implies that x has order 3. So $\langle x \rangle = P$ and $G = M \rtimes \langle x \rangle$. Clearly G is not abelian, as then G has an element of order 6. Hence there exists $a \in M$ such that $xa \neq ax$. Let $U = \langle a, a^x, a^{x^2} \rangle$ and $V = \langle a, x \rangle$. Then $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Clearly, $V = U \rtimes \langle x \rangle$. If $U \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ then $V \cong A_4$. If $U \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ then $W = \langle aa^x, aa^{x^2}, x \rangle$ is a subgroup of V isomorphic to A_4. Hence, in either case V (and so G) contains a subgroup $L = \langle y, x \mid y^3 = x^3 = 1, xyx = y^2 \rangle \cong A_4$. It is well-known that A_4 has four conjugacy classes with representatives 1, y, x and yx and

$$\begin{aligned} \chi : G &\to \mathbb{C} \\ 1 &\mapsto 1 \\ y &\mapsto \exp(2\pi i/3) \\ x &\mapsto \exp(4\pi i/3) \\ yx &\mapsto 1 \end{aligned}$$

http://dx.doi.org/10.22108/toc.2020.125478.1774

is an irreducible character of G of dimension 1. Let $S = \{1, y, yx\}$. Then $|\chi(S)| = |\chi(1) + \chi(y) + \chi(yx)| = \sqrt{5 - 2\sqrt{3}}$ is an eigenvalue of G by [3, Lemma 3.5], a contradiction. Thus the case $3 \nmid |M|$ is impossible. This completes the proof. □

Now we are ready to determine non-nilpotent elements of $\mathcal{I}_k$, where $k \geq 3$.

Lemma 2.8. *Let $G \in \mathcal{I}_k$, $k \geq 3$, be a finite non-nilpotent group. Then $G \cong S_3$.*

Proof. By Lemma 2.2(1), $|G| = 2^m 3^n$, where $m, n \geq 1$. Let P and Q be Sylow 3-subgroup and 2-subgroup of G, respectively. Then, by Lemma 2.2(ii) and Corollary 2.5, $P = \langle a \rangle \cong \mathbb{Z}_3$ and $Q \cong \mathbb{Z}_2^r$ for some positive integer r. Furthermore, by Lemma 2.7, $P \trianglelefteq G$ and so $G = P \rtimes Q$. Since G is non-nilpotent, it is not abelian and so $Q \nleq C_G(P)$, which means that there exists $b \in Q$ such that $ab \neq ba$. Let $C = C_G(P)$. Then, by the Normalizer-centralizer Theorem, G/C is isomorphic to a subgroup of $\mathrm{Aut}(P) \cong \mathbb{Z}_2$. Hence $G = C$ or $|G : C| = 2$. Since $Q \nleq C$ the first case is impossible. So $|G : C| = 2$. Since G has no element of order 6, $C = P$, $|G| = 6$ and then it is clear that $G \cong S_3$. □

Now we combine Corollary 2.5 and Lemma 2.8 to get our main result.

Theorem 2.9. *Let G be a finite group and $k \geq 3$ be an integer. Then $G \in \mathcal{I}_k$ if and only if $G \cong \mathbb{Z}_3$, $\mathbb{Z}_2^r$ for some positive integer r, or S_3.*

Proof. Suppose that $G \in \mathcal{I}_k$. Then Corollary 2.5 and Lemma 2.8 imply that $G \cong \mathbb{Z}_3$, $\mathbb{Z}_2^r$ for some positive integer r, or S_3.

Let $G = \langle a \rangle \cong \mathbb{Z}_3$ and $S \subseteq G$, where $|S| \leq k$. Then $|S| \leq 3$. Since K_2, C_6 and $K_{3,3}$ are integral, we find that $\mathrm{BCay}(G, S)$ is integral. Hence $G \in \mathcal{I}_k$.

Let $G \cong \mathbb{Z}_2^r$ and $S \subseteq G$, where $|S| \leq k$. Let χ be an irreducible character of G and $s \in S$. Then $\chi(s) = 1$ or -1. So $|\sum_{s \in S} \chi(s)|$ is an integer. Hence $\mathrm{BCay}(G, S)$ is integral, which implies that $G \in \mathcal{I}_k$.

Now let $G \cong S_3$ and $S \subseteq G$, where $|S| \leq k$. By a simple GAP program [9], one can see that $\mathrm{BCay}(G, S)$ is integral. Hence $G \in \mathcal{I}_k$. This completes the proof. □

Acknowledgments

The author gratefully appreciate an anonymous referee for constructive comments and recommendations which definitely helped to improve the readability and quality of the article.

References

[1] O. Ahmadi, N. Alon, L. F. Blake and I. E. Shparlinski, Graphs with integral spectrum, *Linear Algebra Appl.*, **430** (2009) 547–552.

[2] M. Arezoomand and B. Taeri, On the characteristic polynomial of n-Cayley digraphs, *Electron. J. Combin.*, **20** (2013) pp. 14.

[3] M. Arezoomand and B. Taeri, A classification of finite groups with integral bi-Cayley graphs, *Trans. Comb.*, **4** (2015) 55–61.

[4] M. Arezoomand and B. Taeri, Finite BCI-groups are solvable, *Int. J. Group Theory*, **5** (2016) 1–6.

[5] M. Arezoomand and B. Taeri, Finite groups admitting a connected cubic integral bi-Cayley graph, *Alg. Struc. Appl.*, **5** (2018) 35–43.

[6] K. Baliňska, D. Cvetkovič, Z. Radosavljevič, S. Simič and D. Stevanovič, A survey on integral graphs, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat.*, **13** (2002) 42–65.

[7] I. Estélyi and I. Kovács, On groups all of whose undirected Cayley graphs of bounded valency are integral, *Electron. J. Combin.*, **21** no.4 (2014) pp. 11.

[8] Y.-Q Feng and I. Kovács, On finite groups with prescribed two-generator subgroups and integral Cayley graphs, *J. Group Theory*, https://www.degruyter.com/document/doi/10.1515/jgth-2020-0094/html.

[9] GAP - Groups, Algorithms, Programming, a System for Computational Discrete Algebra, Version 4.6.5, 2013, http://gap-system.org.

[10] F. Harary and A. J. Schwenk, *Which Graphs Have Integral Spectra?*, Lecture Notes in Mathematics, **406** Springer, Berlin, (1974) 45-51.

[11] Z. P. Lu, C. Q. Wang and M. Y. Xu, Semisymmetric cubic graphs constructed from bi-Cayley graphs of An, *Ars Combin.*, **80** (2006) 177–187.

[12] X. Ma and K. Wang, On finite groups all of whose cubic Cayley graphs are integral, *J. Algebra Appl.*, **15** (2016) pp. 10.

Majid Arezoomand

University of Larestan, 74317-16137, Lar, Iran

Email: arezoomand@lar.ac.ir

http://dx.doi.org/10.22108/toc.2020.125478.1774