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VERTEX DECOMPOSABILITY OF COMPLEXES ASSOCIATED TO FORESTS

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ABSTRACT. In this article, we discuss the vertex decomposability of three well-studied simplicial complexes associated to forests. In particular, we show that the bounded degree complex of a forest and the complex of directed trees of a multidiforest is vertex decomposable. We then prove that the non-cover complex of a forest is either contractible or homotopy equivalent to a sphere. Finally we provide a complete characterization of forests whose non-cover complexes are vertex decomposable.

1. Introduction

The concept of a (pure) vertex decomposable simplicial complex was introduced by Provan and Billera [19] in order to study the diameter problems. Later, in [4], Björner and Wachs extended this notion to non-pure simplicial complexes. Defined in a recursive way (see Definition 2), the notion of vertex decomposability enjoys a very rich literature. In [27], Wachs showed that a vertex decomposable simplicial complex is shellable (see Definition 3) and hence sequentially Cohen-Macaulay¹ [12, Theorem 3.33]. In this direction, we have the following strict implications:

$$(1.1) \quad \text{vertex decomposable} \implies \text{shellable} \implies \text{sequentially Cohen-Macaulay}$$

Let K be a simplicial complex on vertex set $\{x_1, \dots, x_n\}$ and let $R = \mathbb{F}[x_1, \dots, x_n]$ denote the polynomial ring on n variables over some field \mathbb{F} . The monomial ideal associated to K , denoted I_K , is the ideal in R generated by all monomials $x_{i_1} \cdots x_{i_s}$ whenever $\{x_{i_1}, \dots, x_{i_s}\} \notin K$. The *Stanley-Reisner*

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¹See [12, Definition 3.22] for the definition of sequentially Cohen-Macaulay complex.

ring of K is the quotient ring $\mathbb{F}[K] := R/I_K$. It is known (see for example [2]) that a complex K is sequentially Cohen-Macaulay if and only if $\mathbb{F}[K]$ is sequentially Cohen-Macaulay in the algebraic sense (see [25, Definition III 2.9] for the definition of the later). This connection between algebra and topology motivated researchers in the last decade to explore the vertex decomposability of a complex K in order to study the algebraic properties of $\mathbb{F}[K]$ (see for instance [7, 29, 18, 5]).

Note that a vertex decomposable simplicial complex is homotopy equivalent to a wedge of spheres since it is shellable [3, Theorem 4.1]. However, the converse is not true in general (for example the disjoint union of two simplices of dimension 1 is homotopy equivalent to a 0-sphere but not shellable and hence not vertex decomposable). Also the boundary complexes of simplicial polytopes are shellable [30, Chapter 8] but many of them are not vertex decomposable [13, Section 6]. Recently, in [6], Coleman et al. showed that a vertex decomposable complex is shelling completable (see [6, Definition 1.2]) and hence satisfy the Simon's Conjecture [21].

Inspired by the importance of vertex decomposable complexes, in this article we study the vertex decomposability of various simplicial complexes associated to forests. The article is organized as follows. In Section 2 we recall all the important definitions and relevant tools. In Section 3, we prove that the bounded degree complex (a generalization of matching complexes) of a forest is vertex decomposable (cf. Theorem 3.1). Section 4 is devoted towards the study of non-cover complexes of graphs. Here, we show that the non-cover complex of a forest is either contractible or homotopy equivalent to a sphere. We also give a complete list of forests whose non-cover complexes are vertex decomposable (cf. Theorem 4.5). In the final section we show that the complex of directed trees of a multidiforest is vertex decomposable (cf. Theorem 5.1).

2. Preliminaries

An (*undirected*) graph is an ordered pair $G = (V(G), E(G))$ where $V(G)$ is called the set of vertices and $E(G) \subseteq V(G) \times V(G)$, the set of (unordered) edges of G . The graph G is called *simple* if $(v, v) \notin E$ for any $v \in V(G)$. The vertices $v_1, v_2 \in V$ are said to be adjacent, if $(v_1, v_2) \in E$. A vertex v is said to be *adjacent* to an edge e (and vice versa), if v is an end point of e , i.e., $e = (v, w)$. Two edges $e, f \in E(G)$ are said to be adjacent if both are adjacent to a common vertex. The number of vertices adjacent to a vertex v in G is called the *degree* of v in G , denoted $\deg_G(v)$. If $\deg_G(v) = 1$, then v is called a *leaf vertex* of G and the edge adjacent to v is called a *leaf edge* of G . Two graphs G and H are called *isomorphic*, denoted $G \cong H$, if there exists a bijection, $f : V(G) \rightarrow V(H)$ such that $(v, w) \in E(G)$ if and only if $(f(v), f(w)) \in E(H)$.

A graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a *subgraph* of the graph G . For a nonempty subset S of $E(G)$, the induced subgraph $G[S]$, is the subgraph of G with $E(G[S]) = S$ and $V(G[S]) = \{a \in V(G) : a \text{ is adjacent to } e \text{ for some } e \in S\}$. For a nonempty subset U of $V(G)$, the induced subgraph $G[U]$, is the subgraph of G with vertices $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) \mid a, b \in U\}$. For $U \subsetneq V(G)$ and $S \subsetneq E(G)$, the graphs $G[V(G) \setminus U]$ and $G[E(G) \setminus S]$ will be denoted by $G - U$ and $G - S$ respectively.

A *tree* is a graph in which any two vertices are connected by exactly one path and a *forest* is a family of disjoint trees. Let T be a tree. Vertex $v \in V(T)$ is called an *internal vertex* if $\deg_T(v) > 1$. An internal vertex is called a *corner vertex*, if it is adjacent to at most one internal vertex. For example: in Figure 1, v_2 is an internal vertex but not a corner vertex, and v_1, v_3 are corner vertices.

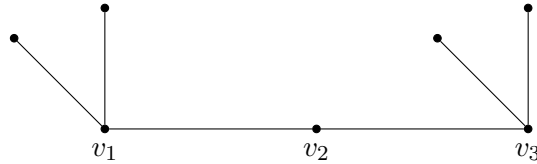


FIGURE 1

Proposition 2.1. *Every tree with more than one edge has a corner vertex.*

Proof. We prove this by labelling all the vertices of T . Start with a leaf v and label it 1. Next we give label 2 to all the vertices adjacent to v . We then give label 3 to all those vertices which are adjacent to at least one vertex labelled 2 and is not already labelled. We continue labelling vertices of T with this argument. Since T is finite and has no cycle, this labelling stops at certain stage say at ℓ . Since $|V(T)| \geq 3, \ell \geq 3$. Therefore, observe that, any vertex labelled $\ell - 1$ will be a corner vertex. \square

The following graphs are a special class of trees.

Definition 1. A *caterpillar graph* is a tree in which every vertex is on a central path or only one edge away from the path (see Figure 2 for examples).

A caterpillar graph of length n is denoted by $G_n(m_1, \dots, m_n)$, where n represents the number of vertices of the central path and m_i denotes the number of leaves adjacent to i^{th} vertex of the central path.



FIGURE 2. Caterpillar graphs

An (*abstract*) *simplicial complex* K is a collection of finite sets such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$. The elements of K are called *simplices* of K . If $\sigma \in K$ and $|\sigma| = k + 1$, then σ is said to be k -*dimensional*. The *dimension* of K , denoted $\dim(K)$, is the maximum of the dimensions of its simplices. Further, if $\sigma \in K$ and $\tau \subseteq \sigma$ then τ is called a *face* of σ and if $\tau \neq \sigma$ then τ is called a *proper face* of σ . The set of 0-dimensional simplices of K is denoted by $V(K)$, and its elements are

called *vertices* of K . Maximal simplices of K are called *facets* and K is called *pure* if all its facets have the same dimension.

A *subcomplex* of a simplicial complex K is a simplicial complex whose simplices are contained in K . For $\sigma \in K$, the boundary of simplex σ , denoted $\partial(\sigma)$ is collection of all proper faces of σ . If σ is a vertex, then $\partial(\sigma) = \emptyset$. For $k \geq 0$, the k -*skeleton* of a simplicial complex K is the collection of all those simplices of K whose dimension is at most k .

For a simplex $\sigma \in K$, define

$$\begin{aligned} \text{lk}(\sigma, K) &:= \{\tau \in K : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in K\}, \\ \text{del}(\sigma, K) &:= \{\tau \in K : \sigma \not\subseteq \tau\}. \end{aligned}$$

The simplicial complexes $\text{lk}(\sigma, K)$ and $\text{del}(\sigma, K)$ are called *link* of σ in K and (*face*) *deletion* of σ in K respectively. The join of two simplicial complexes K_1 and K_2 , denoted as $K_1 * K_2$, is a simplicial complex whose simplices are disjoint union of simplices of K_1 and of K_2 . Let Δ^S denotes a $(|S| - 1)$ -dimensional simplex with vertex set S and $\partial(\Delta^S)$ denotes the boundary of simplex Δ^S , i.e., $(|S| - 2)$ -skeleton of Δ^S . Then, the *cone* on K with apex a , denoted as $C_a(K)$, is defined as

$$C_a(K) := K * \Delta^{\{a\}}.$$

For $a, b \notin V(K)$, the suspension of K , denoted as $\Sigma(K)$, is defined as

$$\Sigma(K) := K * \partial(\Delta^{\{a,b\}}).$$

Observe that, for any vertex $v \in V(K)$, we have

$$K = C_v(\text{lk}(v, K)) \cup \text{del}(v, K) \text{ and } C_v(\text{lk}(v, K)) \cap \text{del}(v, K) = \text{lk}(v, K).$$

Clearly, $C_v(\text{lk}(v, K))$ is contractible. Therefore, if $\text{lk}(v, K)$ is contractible in $\text{del}(v, K)$ then from [10, Remark 2] we get the following homotopy equivalence

$$(2.1) \quad K \simeq \text{del}(v, K) \vee \Sigma(\text{lk}(v, K))$$

where \vee denotes the wedge of topological spaces. In this article, empty wedge will mean that the space is contractible.

Definition 2. A simplicial complex K is called *vertex decomposable* if K is a simplex, or K contains a vertex v such that

- (i). both $\text{lk}(v, K)$ and $\text{del}(v, K)$ are vertex decomposable, and
- (ii). any facet of $\text{del}(v, K)$ is a facet of K .

A vertex v which satisfies condition (ii) is called a *shedding vertex* of K . We call vertex v a *decomposing vertex* of K if v satisfies both the conditions (i) and (ii).

The following can be easily inferred from Definition 2.

Proposition 2.2. For two simplicial complexes K_1 and K_2 , the join $K_1 * K_2$ is vertex decomposable if and only if both K_1 and K_2 are vertex decomposable.

Definition 3. A simplicial complex K is called *shellable* if its facets can be arranged in linear order F_1, F_2, \dots, F_t in such a way that the subcomplex $(\bigcup_{1 \leq j < r} \Delta^{F_j}) \cap \Delta^{F_r}$ is pure and $(\dim(\Delta^{F_k}) - 1)$ -dimensional for all $k = 2, \dots, t$. Such an ordering of facets is called a shelling order of K .

Lemma 2.3. [1, Lemma 3.3] Let K_1 and K_2 be two contractible subcomplexes of a simplicial complex K such that $K = K_1 \cup K_2$. Then $K \simeq \Sigma(K_1 \cap K_2)$, where $\Sigma(X)$ denotes the suspension of space X .

For a space X , let $\Sigma^r(X)$ denote its r -fold suspension, where $r \geq 1$ is a natural number. Recall that, there is a homotopy equivalence

$$(2.2) \quad \begin{aligned} \mathbb{S}^{r-1} * X &\simeq \Sigma^r(X), \text{ and} \\ \Sigma^r(\mathbb{S}^t) &\simeq \mathbb{S}^{t+r+1}. \end{aligned}$$

3. The bounded degree complex

Let G be a graph and $\vec{\lambda} : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be a labelling of the vertices of G with non-negative integers. The *bounded degree complex*, denoted $\text{BD}^{\vec{\lambda}}(G)$, is a simplicial complex whose vertices are the edges of G and faces are subsets $\sigma \subseteq E(G)$ such that for each $v \in V(G)$, the degree of vertex v in the induced subgraph $G[\sigma]$ is at most $\vec{\lambda}(v)$ (see Figure 3 for example). When $\vec{\lambda}(v) = k$ for all $v \in V(G)$, the bounded degree complex $\text{BD}^{\vec{\lambda}}(G)$ is called the *k-matching complex* of graph G and denoted by $M_k(G)$.

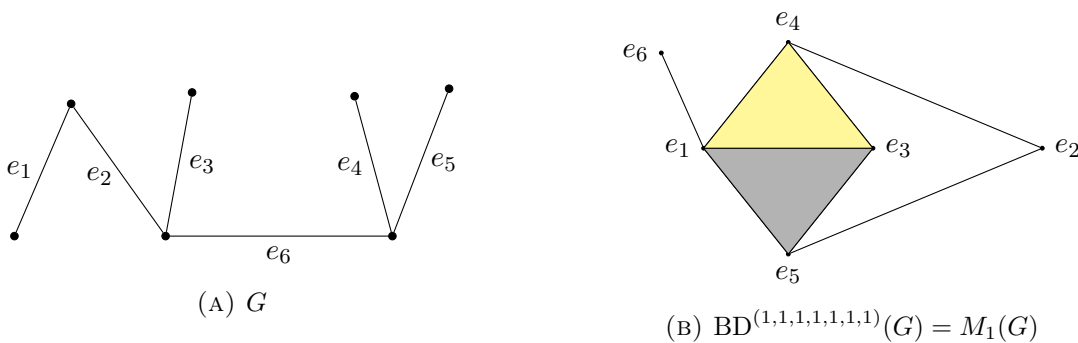


FIGURE 3

Bounded degree complexes were introduced by Reiner and Roberts in [20] and further studied by Jonsson in [12]. For more on these complexes, interested reader is referred to [12, 23, 28].

In [15, Theorem 4.13], Marietti and Testa proved that the 1- matching complexes of forests are homotopy equivalent to a wedge of spheres. In [26], Vega studied the homotopy type of 2-matching complexes of caterpillar graphs (see Definition 1) and conjectured [26, Conjecture 7.3] that the k -matching complex of caterpillar graphs are either contractible or homotopy equivalent to a wedge of

spheres. The author, in [22, Theorem 1.2], proved this conjecture by showing that the bounded degree complexes of forests are homotopy equivalent to wedge of spheres. Recently, Matsushita [16] showed that these complexes are shellable. Here, we strengthen his result by showing that the bounded degree complexes of forests are in fact vertex decomposable.

We first introduce a few notations. For $\vec{\lambda} : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ a labelling of G and $v \in V(G)$, the induced labelling $\vec{\lambda}_{G,v}$ of graph $G - v$ is given by

$$(3.1) \quad \vec{\lambda}_{G,v}(u) = \vec{\lambda}(u) \quad \forall u \in V(G - v)$$

and for $e = (v, w) \in E(G)$, the labelling $\vec{\lambda}_{G,e}$ of graph $G - e$ is given by

$$(3.2) \quad \vec{\lambda}_{G,e}(u) = \begin{cases} \vec{\lambda}(u), & \text{if } u \notin \{v, w\}, \\ \vec{\lambda}(u) - 1, & \text{if } u \in \{v, w\}. \end{cases}$$

Theorem 3.1. *Let \mathcal{F} be a forest and $\vec{\lambda} : V(\mathcal{F}) \rightarrow \mathbb{Z}_{\geq 0}$ be a labelling of its vertices. Then, $\text{BD}^{\vec{\lambda}}(\mathcal{F})$ is vertex decomposable.*

Proof. We prove this by induction on the number n of edges in the forest. If \mathcal{F} has only one edge then the result is clear. Let \mathcal{F} has $n \geq 2$ edges and assume that all bounded degree complexes of forests with at most $n - 1$ edges are vertex decomposable.

If \mathcal{F} has an isolated vertex v , then $\text{BD}^{\vec{\lambda}}(\mathcal{F}) = \text{BD}^{\vec{\lambda}_{\mathcal{F},v}}(\mathcal{F} - v)$. Moreover, if $\vec{\lambda}(u) = 0$ for a vertex $u \in V(\mathcal{F})$ then also $\text{BD}^{\vec{\lambda}}(\mathcal{F}) = \text{BD}^{\vec{\lambda}_{\mathcal{F},u}}(\mathcal{F} - u)$. Therefore, we can assume that \mathcal{F} does not have any isolated vertex and $\vec{\lambda}(v) \neq 0$ for any $v \in V(\mathcal{F})$.

If \mathcal{F} has an isolated edge e then the result is clear from Proposition 2.2 as $\text{BD}^{\vec{\lambda}}(\mathcal{F})$ is a cone over $\text{BD}^{\vec{\lambda}_{\mathcal{F},e}}(\mathcal{F} - e)$ with apex e . Otherwise, \mathcal{F} will have a corner vertex, say w . Further, if $\deg_{\mathcal{F}}(w) \leq \vec{\lambda}(w)$ then $\text{BD}^{\vec{\lambda}}(\mathcal{F})$ is a cone over $\text{BD}^{\vec{\lambda}_{\mathcal{F},\ell}}(\mathcal{F} - \ell)$ with apex ℓ for each leaf edge ℓ adjacent to w . In both cases the result follows from induction and Proposition 2.2.

Now consider $\deg_{\mathcal{F}}(w) > \vec{\lambda}(w)$. If w is adjacent to an internal vertex v then take $e = (w, v)$, otherwise choose $e = (w, v)$ for some $v \in N(w)$.

Observe that,

$$\begin{aligned} \text{lk}(e, \text{BD}^{\vec{\lambda}}(\mathcal{F})) &= \text{BD}^{\vec{\lambda}_{\mathcal{F},e}}(\mathcal{F} - e), \text{ and} \\ \text{del}(e, \text{BD}^{\vec{\lambda}}(\mathcal{F})) &= \text{BD}^{\vec{\lambda}}(\mathcal{F} - e). \end{aligned}$$

Induction implies that both $\text{lk}(e, \text{BD}^{\vec{\lambda}}(\mathcal{F}))$ and $\text{del}(e, \text{BD}^{\vec{\lambda}}(\mathcal{F}))$ are vertex decomposable. Thus, it is now enough to show that $e = (w, v)$ is a shedding vertex of $\text{BD}^{\vec{\lambda}}(\mathcal{F})$.

Let σ be a facet of $\text{del}(e, \text{BD}^{\vec{\lambda}}(\mathcal{F})) = \text{BD}^{\vec{\lambda}}(\mathcal{F} - e)$ and $E_e = \{e' \in E(\mathcal{F} - e) : w \in e'\}$. Since w is a corner vertex and $\deg_{\mathcal{F}}(w) > \vec{\lambda}(w)$, $|E_e| \geq \vec{\lambda}(w)$ and all edges in E_e are leaf edges of \mathcal{F} . Clearly, $E_e \subseteq E(\mathcal{F} - e)$ which implies that $\deg_{(\mathcal{F}-e)[\sigma]}(w) = \sigma \cap E_e = \vec{\lambda}(w)$. Therefore, $\sigma \cup \{e\} \notin \text{BD}^{\vec{\lambda}}(\mathcal{F})$ implying that σ is a facet of $\text{BD}^{\vec{\lambda}}(\mathcal{F})$. \square

The technique used in the above proof can be applied to a bigger class of graphs. A subgraph H of G will be called *cycle subgraph*, if $H \cong C_n$ for some $n \geq 3$. For $v \in V(G)$, $L_G(v)$ will denote the number of leaves adjacent to v in G .

Proposition 3.2. *Let G be a graph and let $\vec{\lambda}$ be a labelling of G . If for each cycle subgraph H of G there is a vertex $v \in V(H)$ such that $L_G(v) \geq \vec{\lambda}(v)$, then $BD^{\vec{\lambda}}(G)$ is vertex decomposable.*

Proof. Proceed by induction on the number n of cycle subgraphs of G . If $n = 0$, i.e., G has no cycle then the result follows from Theorem 3.1.

Now consider, G is a graph with n cycles and H is a cycle subgraph of G . Let $v, w \in V(H)$ such that $L_G(v) \geq \vec{\lambda}(v)$ and $e = (v, w) \in E(H)$. It is easy to observe that, $lk(e, BD^{\vec{\lambda}}(G)) = BD^{\vec{\lambda}_{G,e}}(G - e)$ and the number of cycles in $G - e$ is less than n . Since e is a non-leaf edge, $G - e$ and the labelling $\vec{\lambda}_{G,e}$ satisfy the hypothesis of Proposition 3.2. Thus, by induction, $lk(e, BD^{\vec{\lambda}}(G))$ is vertex decomposable. Using similar arguments we get that $del(e, BD^{\vec{\lambda}}(G)) = BD^{\vec{\lambda}}(G - e)$ is also vertex decomposable. Moreover, for any facet σ of $del(e, BD^{\vec{\lambda}}(G))$, $deg_{G[\sigma]}(v) = \vec{\lambda}(v)$ (since $L_{G-e}(v) = L_G(v) \geq \vec{\lambda}(v)$) which implies that σ is a facet of $BD^{\vec{\lambda}}(G)$. This completes the proof of Proposition 3.2. \square

A *fully whiskered graph* is a graph in which every non-leaf vertex is adjacent to at least one leaf vertex. The following is an immediate corollary of Proposition 3.2.

Corollary 3.3. *The 1-matching complex of any fully whiskered graph is vertex decomposable.*

4. The non-cover complex

A subset $I \subseteq V(G)$ is called an *independent set* of graph G if the induced subgraph $G[I]$ does not have any edge. A subset $S \subseteq V(G)$ is called a *cover* of G if $V(G) \setminus S$ is an independent set of G .

The *independence complex* of a graph G , denoted as $Ind(G)$, is a simplicial complex whose simplices are all independent sets of G . The *non-cover complex* of graph G , denoted $\mathcal{NC}(G)$, is a simplicial complex whose simplices are non-covers of G .

Example: Figure 4 consists of graph $G_2(2, 1)$ and $\mathcal{NC}(G_2(2, 1))$. The complex $\mathcal{NC}(G_2(2, 1))$ has 4 faces, namely $\{v_1, v_3, e_4\}$, $\{v_2, v_3, v_5\}$, $\{v_2, v_4, v_5\}$ and $\{v_3, v_4, v_5\}$.

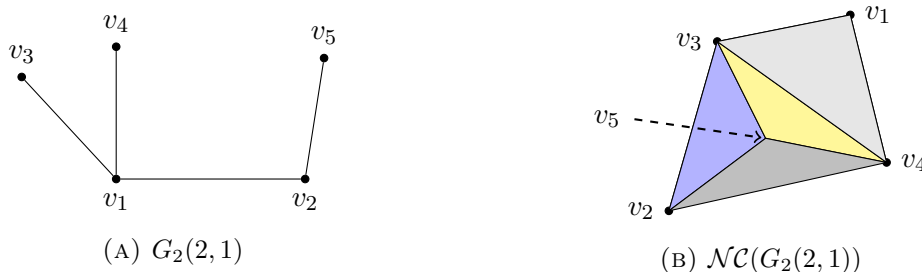


FIGURE 4

The (*combinatorial*) Alexander dual $AD(K)$ of a simplicial complex K is the simplicial complex

$$AD(K) = \{\sigma \subseteq V(K) : V(K) \setminus \sigma \notin K\}.$$

It is easy to see that $\mathcal{NC}(G)$ is the Alexander dual of $\text{Ind}(G)$. Independence complexes have been studied extensively in last few decades and the homotopy type of these complexes have been computed for various classes of graphs (for instance see [1, 9, 29]). Even though the reduced homology of $\mathcal{NC}(G)$ is related to that of $\text{Ind}(G)$ due to the Alexander duality theorem², the homotopy type of non-cover complexes remains mysterious. See [17] for results related to the topology of the Alexander dual.

It is easy to observe that the facets of $\mathcal{NC}(G)$ are in one to one correspondence with the edges of G . In particular, any edge $(u, v) \in E(G)$ gives the unique facet $V(G) \setminus \{u, v\}$ and vice versa.

Lemma 4.1. *Let G_1 and G_2 be two disjoint connected graphs such that $|E(G_1)|, |E(G_2)| \geq 1$. Then, $\mathcal{NC}(G_1 \sqcup G_2) \simeq \Sigma(\mathcal{NC}(G_1) * \mathcal{NC}(G_2))$.*

Proof. Let $K_1 = \Delta^{V(G_1)} * \mathcal{NC}(G_2)$ and $K_2 = \mathcal{NC}(G_1) * \Delta^{V(G_2)}$. Since $|E(G_1)|, |E(G_2)| \geq 1$, both K_1 and K_2 are contractible and $\mathcal{NC}(G_1 \sqcup G_2) = K_1 \cup K_2$. Thus, from Lemma 2.3, we get that $\mathcal{NC}(G_1 \sqcup G_2) \simeq \Sigma(K_1 \cap K_2)$. Now the proof follows from the observation that $K_1 \cap K_2 = \mathcal{NC}(G_1) * \mathcal{NC}(G_2)$. \square

For a vertex $w \in V(G)$, let $\text{st}_w(G)$ denote the graph on vertex set $N_G[w]$ and $E(\text{st}_w(G)) = \{(v, w) : v \in N_G(w)\}$.

Theorem 4.2. *Let v be a leaf vertex of graph G , $(v, w) \in E(G)$ and $\deg_G(w) > 1$. Then,*

$$\mathcal{NC}(G) \simeq \Sigma^{|N_G(w)|-1} \mathcal{NC}(G - N_G[w]).$$

Proof. Since v is a leaf vertex, $\text{lk}(w, \mathcal{NC}(G))$ is a cone with an apex a , hence contractible. Thus, from (2.1), we have

$$\mathcal{NC}(G) \simeq \text{del}(w, \mathcal{NC}(G)) \bigvee \Sigma(\text{lk}(w, \mathcal{NC}(G))) \simeq \text{del}(w, \mathcal{NC}(G)).$$

Claim 1. $\text{del}(w, \mathcal{NC}(G)) = \text{del}(w, \mathcal{NC}(\text{st}_w(G) \sqcup G - N_G[w]))$

Proof of Claim 1. Clearly, $\text{del}(w, \mathcal{NC}(\text{st}_w(G) \sqcup G - N_G[w])) \subseteq \text{del}(w, \mathcal{NC}(G))$. To show the other way inclusion, let σ be a facet of $\text{del}(w, \mathcal{NC}(G))$. We know that there exists $(a, b) \in E(G)$ such that $\sigma = V(G) \setminus \{a, b, w\}$. If (a, b) is an edge of $\text{st}_w(G)$ or of $G - N_G[w]$ then the result follows. Otherwise, without loss of generality assume that $a \in V(\text{st}_w(G))$. In this case $\sigma \subseteq V(G) \setminus \{a, w\}$ which is a facet of $\text{del}(w, \mathcal{NC}(\text{st}_w(G) \sqcup G - N_G[w]))$. \square

From Lemma 4.1, it is easy to see that $\text{del}(w, \mathcal{NC}(\text{st}_w(G) \sqcup G - N_G[w])) \simeq \mathcal{NC}(\text{st}_w(G) \sqcup G - N_G[w])$ (since $w \notin \mathcal{NC}(\text{st}_w(G))$). Therefore, Theorem 4.2 follows from Lemma 4.1, Equation (2.2) and the observation that $\mathcal{NC}(\text{st}_w(G)) = \partial(\Delta^{N_G(w)})$. \square

²**Alexander duality theorem ([24]):** Let K be a simplicial complex and $V(K) \notin K$. Then for all $-1 \leq i \leq |V(K)| - 2$, $\tilde{H}_i(AD(K)) = \tilde{H}_{|V(K)|-i-3}(K)$. Here, $\tilde{H}_i(K)$ denotes the i^{th} reduced homology group of K .

Ehrenborg and Hetyei [8] showed that, for any forest \mathcal{F} , the complex $\text{Ind}(\mathcal{F})$ is either contractible or homotopy equivalent to a sphere. The following result is a direct consequence of Lemma 2.3 and Theorem 4.2, it says that the Alexander dual of independence complexes of forests also have the same homotopy type.

Corollary 4.3. *For any forest \mathcal{F} , the complex $\mathcal{NC}(\mathcal{F})$ is either contractible or homotopy equivalent to a sphere.*

We now discuss the vertex decomposability of non-cover complexes. If G contains an isolated vertex v and at least one edge then $\mathcal{NC}(G) = C_v(\mathcal{NC}(G - v))$. Thus, Proposition 2.2 implies that $\mathcal{NC}(G)$ is vertex decomposable if and only if $\mathcal{NC}(G - v)$ is vertex decomposable.

Proposition 4.4. *Let G_1 and G_2 be two disjoint graphs such that $|E(G_1)|, |E(G_2)| \geq 1$. Then, $\mathcal{NC}(G_1 \sqcup G_2)$ is not shellable hence not vertex decomposable.*

Proof. On contrary, assume that $\mathcal{NC}(G_1 \sqcup G_2)$ is shellable with the shelling order F_1, \dots, F_t . Without loss of generality, assume that $F_1 = V(G_1 \sqcup G_2) \setminus \{a_1, b_1\}$ where $(a_1, b_1) \in E(G_1)$. Define $r = \min\{i \in [t] : V(G_2) \not\subseteq F_i\}$. Clearly, $1 < r \leq t$. It is easy to see that, for each $1 \leq j < r$, $|F_r \cap F_j| = |F_r| - 2$. Thus, $\dim\left(\left(\bigcup_{1 \leq j < r} \Delta^{F_j}\right) \cap \Delta^{F_r}\right) < \dim(\Delta^{F_r}) - 1$ which contradicts the fact that F_1, \dots, F_t is a shelling order. □

Recall that v is a decomposing vertex of $\mathcal{NC}(G)$ if v is a shedding vertex of $\mathcal{NC}(G)$, and both $\text{lk}(v, \mathcal{NC}(G))$ and $\text{del}(v, \mathcal{NC}(G))$ are vertex decomposable. Thus, $\mathcal{NC}(G)$ is vertex decomposable if and only if $\mathcal{NC}(G)$ has a decomposing vertex.

Theorem 4.5. *Let \mathcal{F} be a forest without any isolated vertex. Then, the complex $\mathcal{NC}(\mathcal{F})$ is vertex decomposable if and only if \mathcal{F} is connected and has at most two internal vertices.*

Proof. Let \mathcal{F} be connected. If \mathcal{F} has at most one internal vertex then $\mathcal{NC}(\mathcal{F})$ is the boundary of a simplex, hence vertex decomposable. Let \mathcal{F} have exactly two internal vertices, i.e., \mathcal{F} is the caterpillar graph $G_2(m, n)$ for some $m, n \geq 1$. Let v_1 and v_2 be the internal vertices (cf. Figure 4a). It is easy to see that,

$$\begin{aligned} \text{lk}(v_1, \mathcal{NC}(G_2(m, n))) &= \Delta^{[m]} * \mathcal{NC}(G_1(n)), \text{ and} \\ \text{del}(v_1, \mathcal{NC}(G_2(m, n))) &= \text{del}(v_1, \mathcal{NC}(\text{st}_{v_1}(G_2(m, n)) \sqcup (G_2(m, n) - N_{G_2(m, n)}[v_1]))) \\ &= \text{del}(v_1, \mathcal{NC}(G_1(m + 1))) * \Delta^{[n]} \\ &= \partial(\Delta^{[m+1]}) * \Delta^{[n]} \end{aligned}$$

Thus, both $\text{lk}(v_1, \mathcal{NC}(G_2(m, n)))$ and $\text{del}(v_1, \mathcal{NC}(G_2(m, n)))$ are vertex decomposable from induction and Proposition 2.2. To show that v_1 is a decomposing vertex, it is now enough to show that v_1 is a shedding vertex. Let $F = V(G_2(m, n)) \setminus \{a, b, v_1\}$ be a facet of $\text{del}(v_1, \mathcal{NC}(G_2(m, n)))$ such that $(a, b) \in E(G_2(m, n))$. If $v_1 \in \{a, b\}$ then clearly F is facet of $\mathcal{NC}(G_2(m, n))$. Let $v_1 \notin \{a, b\}$. In

this case $v_2 \in \{a, b\}$ and $F \subsetneq V(\mathcal{F}) \setminus \{v_1, v_2\}$, which contradicts the fact that F is a facet of $\text{del}(v_1, \mathcal{NC}(G_2(m, n)))$. Hence, $\mathcal{NC}(G_2(m, n))$ is vertex decomposable.

If \mathcal{F} is not connected then $\mathcal{NC}(\mathcal{F})$ is not vertex decomposable from Proposition 4.4. Therefore, consider that \mathcal{F} is a tree and has at least 3 internal vertices. In this case, we show that any vertex of \mathcal{F} is not a decomposing vertex of $\mathcal{NC}(\mathcal{F})$. We prove this in three parts.

(1) **v is a non-corner internal vertex**

In this case, $\text{lk}(v, \mathcal{NC}(\mathcal{F})) = \mathcal{NC}(\mathcal{F} - v)$ which is not shellable by Proposition 4.4 (since $\mathcal{F} - v$ is disjoint union of two graphs with at least one edge each) and hence not vertex decomposable. Therefore, any non-corner internal vertex is not a decomposing vertex.

(2) **v is a corner vertex**

Since \mathcal{F} has at least 3 internal vertices, we can choose a corner vertex $w \in V(\mathcal{F})$ such that $v \notin N_{\mathcal{F}}[w]$. Let x be a leaf vertex adjacent to w . Observe that, $F = V(\mathcal{F}) \setminus \{v, w, x\}$ is facet of $\text{del}(v, \mathcal{NC}(\mathcal{F}))$ but F is not a facet of $\mathcal{NC}(\mathcal{F})$. Thus, v is not a shedding vertex hence not a decomposing vertex.

(3) **v is a leaf vertex**

The proof is similar to that of part (2). Choose vertices $w, x \in V(\mathcal{F})$ such that $v \notin N_{\mathcal{F}}[w]$ and x is a leaf vertex adjacent to w . Then, $V(\mathcal{F}) \setminus \{v, w, x\}$ is facet of $\text{del}(v, \mathcal{NC}(\mathcal{F}))$ but not a facet of $\mathcal{NC}(\mathcal{F})$ implying that v is not a decomposing vertex.

This completes the proof of Theorem 4.5. □

5. Complexes of directed trees

A multidigraph G is a pair (V, E) of finite sets V and E with two maps $s, t : E \rightarrow V$. The sets V and E are called *vertex set* and *edge set* of G respectively. An edge $e \in E$ will be denoted as $(s(e) \rightarrow t(e))$, where $s(e)$ is called the *source* of e and $t(e)$ is called the *target* of e . If for every two distinct edges $e, e' \in E$ either $s(e) \neq s(e')$ or $t(e) \neq t(e')$, then G is called a *digraph*. With every multidigraph $G = (V, E)$, we can associate an (undirected) graph G^{un} as follows: the vertex of G^{un} is V and two vertices u and v are adjacent in G^{un} if and only if $(u \rightarrow v) \in E$ or $(v \rightarrow u) \in E$. A multidigraph \mathcal{F} is called *multidiforest* if its underlying graph \mathcal{F}^{un} is a forest.

A *directed cycle* of G is a connected subgraph C of G such that each vertex of C is the source of exactly one edge and target of exactly one edge. A *directed forest* is a multidigraph \mathcal{F} such that \mathcal{F} does not contain any directed cycle and different edges of \mathcal{F} have distinct targets.

Definition 4. For a multidigraph $G = (V, E)$, the *complex of directed trees* is a simplicial complex, denoted as $DT(G)$, whose simplices are the subsets $\sigma \subseteq E$ such that the induced subgraph $G[\sigma]$ is a directed forest.

The study of complexes of directed trees of digraphs was initiated by Kozlov [14]. Later, in [15], Marietti and Testa generalized these complexes for multidigraphs and showed that these complexes for multidiforsts are homotopy equivalent to a wedge of spheres. In [11, Lemma 3.2], Jojić showed

that these complexes are in fact vertex decomposable for those directed graphs G such that G^{un} is a forest. His proof was dependent on another result [29, Theorem 1] due to Woodroffe. Here we give an independent proof of the fact that complexes of directed trees of multidigraphs are vertex decomposable. Observe that if G^{un} is a forest for any directed graph G then G is a multidigraph. Hence, Theorem 5.1 is an improvement on the previously known results.

Theorem 5.1. *Let \mathcal{F} be a multidigraph. Then the complex $DT(\mathcal{F})$ is vertex decomposable.*

For a multidigraph $G = (V, E)$ and $e \in E$, let $G_{\downarrow e}$ denotes the multidigraph obtained from G by first removing the edges with target $t(e)$, and then identifying the vertex $s(e)$ with the vertex $t(e)$ (see Figure 5). The multidigraph $G_{\downarrow e}$ was introduced by Marietti and Testa in [15].



FIGURE 5

Observe that,

$$(5.1) \quad \begin{aligned} \text{lk}(e, DT(G)) &= DT(G_{\downarrow e}), \text{ and} \\ \text{del}(e, DT(G)) &= DT(G - e). \end{aligned}$$

Proof of Theorem 5.1. Proof is by induction on the number n of edges of \mathcal{F} . The result is trivially true if \mathcal{F} has only one edge. Let the result be true for any forest with at most $n - 1$ edges and let \mathcal{F} be a forest with n edges.

Since both the multidigraphs $\mathcal{F}_{\downarrow e}$ and $\mathcal{F} - e$ have less number of edges, both the complexes $\text{lk}(e, DT(\mathcal{F}))$ and $\text{del}(e, DT(\mathcal{F}))$ are vertex decomposable from induction for each edge e of \mathcal{F} . Therefore, to show that $DT(\mathcal{F})$ is vertex decomposable, it is enough to find a shedding vertex of $DT(\mathcal{F})$.

If $e, f \in E(\mathcal{F})$ such that $s(e) = s(f)$ and $t(e) = t(f)$ then e is a shedding vertex. If not, then there exists a facet $\sigma \in \text{del}(e, DT(\mathcal{F}))$ such that $\sigma \cup \{e\} \in DT(\mathcal{F})$ but then $\sigma \cup \{f\} \in \text{del}(e, DT(\mathcal{F}))$ which is a contradiction to the assumption that σ is a facet of $\text{del}(e, DT(G))$.

Therefore, we can assume that $\mathcal{F} = (V, E)$ is a directed graph such that \mathcal{F}^{un} is a forest. Let u be a leaf vertex of \mathcal{F}^{un} and $(u, v) \in E(\mathcal{F}^{un})$. Then there are following four possible cases:

- (1) $(v \rightarrow u) \in E$ and $(u \rightarrow v) \notin E$:

In this case, observe that $DT(\mathcal{F}) = C_e(DT(\mathcal{F} - e))$ where $e = (v \rightarrow u)$. Therefore, $DT(\mathcal{F})$ is vertex decomposable from induction and Proposition 2.2.

- (2) $(v \rightarrow u), (u \rightarrow v) \in E$:

Let $e = (u \rightarrow v)$ and $f = (v \rightarrow u)$. Here we show that e is a shedding vertex of $DT(\mathcal{F})$. Let σ be a facet of $\text{del}(e, DT(\mathcal{F}))$. Since $e \notin \sigma$ and σ is a facet, $f \in \sigma$ (because $\sigma \cup \{f\} \in \text{del}(e, DT(\mathcal{F}))$). Therefore, $\sigma \cup \{e\} \notin DT(\mathcal{F})$ implying that σ is a facet of $DT(\mathcal{F})$.

- (3) $(v \rightarrow u) \notin E$, $(u \rightarrow v) \in E$ and there is no $w (\neq u) \in V$ such that $(w \rightarrow v) \in E$:

This case is similar to the case (1). Here, $DT(\mathcal{F}) = C_f(DT(\mathcal{F} - f))$ with $f = (u \rightarrow v)$.

- (4) $(v \rightarrow u) \notin E$, $(u \rightarrow v) \in E$ and there is an edge $(w \rightarrow v) \in E$ such that $w \neq u$:

In this case, we show that the edge $g = (w \rightarrow v)$, where $w \neq u$, is a shedding vertex. Let σ be a facet of $\text{del}(g, DT(\mathcal{F}))$. If there exists $y \in V$ such that $y \neq w, u$ and $(y \rightarrow v) \in \sigma$ then $\sigma \cup \{g\} \notin DT(\mathcal{F})$ implying that σ is a facet of $DT(\mathcal{F})$. Otherwise, $(u \rightarrow v) \in \sigma$ (because σ is a facet of $\text{del}(g, DT(\mathcal{F}))$ and $\sigma \cup \{(u \rightarrow v)\} \in \text{del}(g, DT(\mathcal{F}))$) which again implies $\sigma \cup \{g\} \notin DT(\mathcal{F})$.

This completes the proof of Theorem 5.1. □

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