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ON THE EXTREMAL CONNECTIVE ECCENTRICITY INDEX AMONG TREES WITH MAXIMUM DEGREE

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Abstract. The connective eccentricity index (CEI) of a graph G is defined as $\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d_G(v)}{\varepsilon_G(v)}$, where $d_G(v)$ is the degree of v and $\varepsilon_G(v)$ is the eccentricity of v. In this paper, we characterize the unique trees with the maximum and minimum CEI among all n-vertex trees and n-vertex conjugated trees with fixed maximum degree, respectively.

1. Introduction

All graphs considered in this paper are simple and connected. Chemical compounds can be represented by graphs, known as chemical graphs, in which vertices represent atoms and edges represent the bonds of the considered chemical compound. Let G be graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ be the set of all neighbors of v in G. The degree of $v \in V(G)$, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. A vertex with degree one is called a pendent vertex , and an edge is said to be a pendent edge if one of its end vertices is a pendent vertex. A vertex of degree greater than two is called a branching vertex. A path in which one of the end vertices is pendent and the other is branching, and all the internal vertices (if exist) have degree two is called pendent path. The maximum degree of G denoted by $\Delta(G)$ is the the maximum degree of its vertices. The graph formed from G by deleting any vertex $v \in V(G)$ (resp. edge $uv \in E(G)$) is denoted by $G - v$ (resp. $G - uv$). Similarly, the graph formed from G by adding an edge uv is denoted by $G + uv$, where u and v are non-adjacent vertices of G. For a vertex subset A of $V(G)$, denote by $G[A]$ the subgraph induced by A. The distance between vertices u and v of G, denoted by $d_G(u, v)$, is the length of a shortest path connecting u and

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v in G. For $v \in V(G)$, the eccentricity of v in G, denoted $\varepsilon_G(v)$, is the maximum distance from v to all other vertices of G. The diameter of a graph G is the maximum eccentricity of all vertices in G. A connected graph G that has n vertices and $n-1$ edges is known as a tree. As usual, by P_n and S_n we denote the path and the star on n vertices, respectively. The subset M of edges of G is said to be matching in G if no two edges in M share a common vertex. A vertex v is called M-saturated if an edge of M is incident with v. If each vertex in G is M-saturated then M is called perfect matching. A graph which contains a perfect matching is called conjugated graph.

Topological indices are graph invariants used in theoretical chemistry to encode molecules for the design of chemical compounds with given physicochemical properties or given pharmacological and biological activities [9]. In this paper we consider eccentricity-based topological index.

The eccentric connectivity index is one of the eccentricity-based topological indices, defined as

$$\xi^c(G) = \sum_{v \in V(G)} d_G(v)\varepsilon_G(v).$$

It has been studied extensively, see [9, 6, 13, 14, 7].

Gupta et al. in 2000 [1], proposed the connective eccentricity index, defined as

$$\xi^{ce}(G) = \sum_{v \in V(G)} \frac{d_G(v)}{\varepsilon_G(v)}.$$

From experiments for treating hypertension of chemical compounds like non-peptide N-benzylimidazole derivatives, the results obtained using the connective eccentricity index were better than the corresponding values obtained using Balabans mean square distance index. Therefore it is worth studying mathematical properties of connective eccentricity index.

Mathematical properties of connective eccentricity index have been studied extensively for trees, unicyclic and general graph. In particular, Yu and Feng [11] obtained upper or lower bounds for connective eccentricity index of graphs in terms of many graph parameters such as radius, maximum degree, independence number, vertex connectivity, minimum degree, number of pendent vertices and number of cut edges. Li and Zhao [5] studied the extremal properties of connective eccentricity index among n-vertex trees with given graph parameters such as, number of pendent vertices, matching number, domination number, diameter, vertex bipartition. Xu et al. [10] characterized the extremal graph for connective eccentricity index among all connected graph with fixed order and fixed matching number. Zhao et al. [15] identified the unique graph with maximum CEI among all n-vertex graphs with given number of cut vertices, they also determined the maximum CEI of k-connected bipartite graph of order n with fixed diameter having maximum CEI, and obtained the unique tree with minimum connective eccentricity index among all trees with given degree sequence. Hayat [2] characterized the unique graphs with maximum CEI among k-connected graphs with fixed parameters such as diameter, independence number and minimum degree.

For more studies on connective eccentricity index of graphs we refer [4, 8, 12] and the references cited therein.

Let $\mathbb{T}(n,\Delta)$ and $\mathbb{T}(2m,\Delta)$ be the classes of all n-vertex trees and conjugated trees with fixed maximum degree Δ, respectively. First, we characterize the unique trees with the maximum and minimum CEI from $\mathbb{T}(n,\Delta)$ and $\mathbb{T}(2m,\Delta)$, respectively.

2. Preliminaries

In this section, we give some transformations which will be used in the forth coming section.

Lemma 2.1. [11] *Let G be a connected graph with a cut edge $e = uv$, and let G/e be the graph obtained from G by contracting the edge e into a new vertex w_e (such that it is adjacent to each vertex in $N_G(u) \cup N_G(v) \setminus \{u, v\}$ and then attaching a pendent edge at w_e. If e is not a pendent edge, then $\xi^{ce}(G_1) < \xi^{ce}(G_2)$.*

Lemma 2.2. [5] *Let H be a nontrivial connected graph with $u \in V(H)$. For two nonnegative integers ℓ and m, let $G_{u;\ell,m}$ be the graph obtained from H by attaching two pendent paths of length ℓ and m, respectively at u. If $\ell \geq m \geq 1$, then $\xi^{ce}(G_{u;\ell,m}) > \xi^{ce}(G_{u;\ell+1,m-1})$.*

Lemma 2.3. *Let T be a tree and uw, uv be two non-pendent edges of T such that $d_T(v) = k+1$, and $v_1, \ldots, v_k$ are the pendent neighbors of v. Let $T' = T - vv_i + wv_i$ for $i = 1, 2, \ldots, k$. Then $\xi^{ce}(T) < \xi^{ce}(T')$.*

Proof. Assume that T_1 and T_2 be the components of $T - uw - uv$ containing u and w, respectively. Let $\varepsilon_{T_1}(u) = q_1$ and $\varepsilon_{T_2}(w) = q_2$. Since uw is a non-pendent edge of T, we have $d_T(w) \geq 2$ and $q_2 \geq 1$.

From the construction of T', we have the following claims.

Claim 1. For $x \in V(T_1)$, $d_T(x) = d_{T'}(x)$ and $\varepsilon_T(x) = \varepsilon_{T'}(x)$.

It is obvious that $d_T(x) = d_{T'}(x)$ for $x \in V(T_1)$. Note that $\varepsilon_T(u) = \max\{q_1, 1+q_2\} = \varepsilon_{T'}(u)$. If $x \in N_{T_1}(u)$, then $\varepsilon_T(x) = \max\{\varepsilon_{T_1}(x), 2+q_2\} = \varepsilon_{T'}(x)$. If $x \in V(T_1) \setminus N_{T_1}[u]$, then $\varepsilon_T(x) = \max\{\varepsilon_{T_1}(x), d_T(x,w)+1+q_2\} = \varepsilon_{T'}(x)$. This proves Claim 1.

Claim 2. For $x \in V(T_2)$, $d_T(x) = d_{T'}(x)$ and $\varepsilon_T(x) \geq \varepsilon_{T'}(x)$.

For $x \in V(T_2)$, it is obvious that $d_T(x) = d_{T'}(x)$, and thus Claim 2 follows by noting that $\varepsilon_T(x) = \max\{\varepsilon_{T_2}(x), d_T(x,w)+1+q_1, d_T(x,w)+3\} \geq \max\{\varepsilon_{T_2}(x), d_T(x,w)+1+q_1, 2\} = \varepsilon_{T'}(x)$.

By Claims 1 and 2, we have

$$\begin{aligned}\xi^{ce}(T) - \xi^{ce}(T') \quad &\leq \quad \frac{d_T(w)}{\varepsilon_T(w)} - \frac{d_{T'}(w)}{\varepsilon_{T'}(w)} + \frac{d_T(v)}{\varepsilon_T(v)} - \frac{d_{T'}(v)}{\varepsilon_{T'}(v)} \\ &\quad + \sum_{i=1}^{k}\left(\frac{d_T(v_i)}{\varepsilon_T(v_i)} - \frac{d_{T'}(v_i)}{\varepsilon_{T'}(v_i)}\right).\end{aligned}$$

Note that for $i = 1, \ldots, k$, $d_T(v_i) = d_{T'}(v_i)$ and $\varepsilon_T(v_i) = \max\{2+q_1, 3+q_2\} \geq \max\{1+q_2, 2+q_1, 3\} = \varepsilon_{T'}(v_i)$. Thus

$$\sum_{i=1}^{k}\left(\frac{d_T(v_i)}{\varepsilon_T(v_i)} - \frac{d_{T'}(v_i)}{\varepsilon_{T'}(v_i)}\right) < 0.$$

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It follows that

$$\xi^{ce}(T)-\xi^{ce}(T')<\frac{d_T(w)}{\varepsilon_T(w)}-\frac{d_{T'}(w)}{\varepsilon_{T'}(w)}+\frac{d_T(v)}{\varepsilon_T(v)}-\frac{d_{T'}(v)}{\varepsilon_{T'}(v)}.$$

By the facts that $d_T(w)=d_{T'}(w)-k$, $d_T(v)=d_{T'}(v)+k$, $\varepsilon_T(w)=\max\{q_2,1+q_1,3\}\geq\max\{q_2,1+q_1,2\}=\varepsilon_{T'}(w)$, $\varepsilon_T(v)=\max\{1+q_1,2+q_2\}=\varepsilon_{T'}(v)$, and $\varepsilon_T(w)\leq\varepsilon_T(v)$, we have

$$\begin{aligned}\xi^{ce}(T)-\xi^{ce}(T') &< \frac{d_T(w)}{\varepsilon_T(w)}-\frac{d_T(w)+k}{\varepsilon_T(w)}+\frac{d_T(v)}{\varepsilon_T(v)}-\frac{d_T(v)-k}{\varepsilon_T(v)}\\ &= k\left(\frac{1}{\varepsilon_T(v)}-\frac{1}{\varepsilon_T(w)}\right)\\ &\leq 0,\end{aligned}$$

implying that $\xi^{ce}(T)<\xi^{ce}(T')$, as desired. □

3. Trees with extremal CEI in $\mathbb{T}(n,\Delta)$

Let $S_{n,a}$ be the tree formed by attaching $a-1$ and $n-a-1$ pendent vertices to the two end vertices of P_2, respectively, where $\lceil\frac{n}{2}\rceil\leq a\leq n-1$. In particular, $S_{n,n-1}=S_n$.

Theorem 3.1. *Among all trees in $\mathbb{T}(n,\Delta)$, $S_{n,\Delta}$ is the unique tree with maximum CEI, where $\lceil\frac{n}{2}\rceil\leq\Delta\leq n-1$.*

Proof. Let $T\in\mathbb{T}(n,\Delta)$ such that $\xi^{ce}(T)$ is as large as possible. For $\Delta=n-1$ it is trivial as $\mathbb{T}(n,n-1)=\{S_n\}$.

We suppose that $\Delta\leq n-2$. Let $\ell(T)$ be the number of non-pendent vertices of T, then $\ell(T)\geq 2$. We claim that $\ell(T)=2$. Suppose that $\ell(T)\geq 3$, and u be a vertex of T such that $d_T(u)=\Delta$. We consider the following two possible cases.

Case 1. All non-pendent edges of T are incident with u. Let v_1,v_2 be two non-pendent distinct vertices different from u. Let $T'=T-\{v_2x:x\in N_T(v_2)\backslash\{u\}\}+\{v_1x:x\in N_T(v_2)\backslash\{u\}\}$. Clearly, $T'\in\mathbb{T}(n,\Delta)$. Then by Lemma 2.3, $\xi^{ce}(T)<\xi^{ad}(T')$, a contradiction to the choice of T.

Case 2. There is a non-pendent edge vw of T, where v and w are different from u. Assume that $d_T(u,v)<d_T(u,w)$. Let $T'=T-\{wy:y\in N_T(w)\backslash\{v\}\}+\{vy:y\in N_T(w)\backslash\{v\}\}$. Clearly, $T'\in\mathbb{T}(n,\Delta)$. By Lemma 2.1, $\xi^{ce}(T)<\xi^{ce}(T')$, a contradiction. So $\ell(T)=2$, i.e., $\xi^{ce}(T)\cong\xi^{ce}(S_{n,\Delta})$. □

Let $F_{n,a}$ be the tree formed by adding an edge between the pendent vertex of S_{a+1} and one terminal vertex of P_{n-a-1}, where $2\leq a\leq n-1$. In particular, $F_{n,2}=P_n$.

Theorem 3.2. *Among all trees in $\mathbb{T}(n,\Delta)$, $F_{n,\Delta}$ is the unique tree with minimum CEI, where $2\leq\Delta\leq n-1$.*

Proof. Let $T\in\mathbb{T}(n,\Delta)$ such that $\xi^{ce}(T)$ is as small as possible. We only need to show that $T\cong F_{n,\Delta}$. For $\Delta=2,n-1$, it is trivial as $\mathbb{T}(n,2)=\{P_n\}$ and $\mathbb{T}(n,n-1)=\{S_n\}$. We suppose that $3\leq\Delta\leq n-2$.

Choose a vertex $v\in V(T)$ with degree Δ. Let $N_T(v)=\{v_1,\ldots,v_\Delta\}$. Let T_i be the component of $T-v$ containing v_i, where $i=1,\ldots,\Delta$. Suppose that for some i, T_i is not a path with one terminal vertex v_i. Then there is a vertex in T_i such that its degree in T is at least three. So there is a vertex w in

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T_i such that $d_T(v,w)$ is as large as possible. That is to say, there are two pendent paths, say with lengths ℓ and m respectively at w in T. Assume that $\ell \geq m$. So $T \cong G_{w;\ell,m}$, where G is the graph obtained from T by deleting the vertices of degree two and one in the two pendent paths. By Lemma 2.2, we have $\xi^{ce}(T) = \xi^{ce}(G_{w;\ell,m}) > \xi^{ce}(G_{w;\ell+1,m-1})$, a contradiction. Therefore, for each $i = 1,\ldots,\Delta$, T_i is a path with one terminal vertex v_i. So T consist of Δ pendent paths at v. By Lemma 2.2, $T \cong F_{n,\Delta}$. $\square$

4. Trees with extremal CEI in $\mathbb{T}(2m,\Delta)$

Let $A_{2m,a}$ be the tree obtained by attaching a pendent edge to each vertex of $S_{m,a-1}$, where $\lceil \frac{m}{2} \rceil + 1 \leq a \leq m$.

Theorem 4.1. *Among all trees in $\mathbb{T}(2m,\Delta)$, $A_{2m,\Delta}$ is the unique tree with maximum CEI, where $\lceil \frac{m}{2} \rceil + 1 \leq \Delta \leq m$.*

Proof. Let $T \in \mathbb{T}(2m,\Delta)$ such that $\xi^{ce}(T)$ is as large as possible. Let $M(T)$ be the unique perfect matching of T, and u be a vertex of T such that $d_T(u) = \Delta$. We proceed with the following claims.

Claim 1. $M(T)$ does not contain a non-pendent edge of T.

Suppose that $M(T)$ contains at least one non-pendent edge of T. Then we consider the following two possible cases.

Case 1. u is incident with some non-pendent edge in $M(T)$, say uw. Let v be a non-pendent neighbour of u different from w. Let $T' = T - \{wx : x \in N_T(w) \setminus \{u\}\} + \{vx : x \in N_T(v) \setminus \{u\}\}$. It is obvious that $T \in \mathbb{T}(2m,\Delta)$ as $\lceil \frac{m}{2} \rceil + 1 \leq \Delta$. Then by Lemma 2.3, $\xi^{ce}(T) < \xi^{ad}(T')$, a contradiction.

Case 1. u is not incident with any non-pendent edge in $M(T)$. Let vw be a non-pendent edge in $M(T)$. Let $T' = T - \{wy : y \in N_T(w) \setminus \{v\}\} + \{vy : y \in N_T(w) \setminus \{v\}\}$. It is obvious that $T' \in \mathbb{T}(2m,\Delta)$. Then by Lemma 2.1, $\xi^{ce}(T) < \xi^{ce}(T')$, a contradiction.

Thus, $M(T)$ does not contain a non-pendent edge of T. Therefore, each vertex in T except the branch vertices is either a pendent or adjacent to a pendent vertex. If $\Delta = m$, then $T \cong A_{2m,\Delta}$, and the result holds trivially. In what follows, we suppose that $\Delta \leq m-1$. Then T contains at least two branch vertices. Let $b_1, b_2, \ldots, b_r$ be the branch vertices of T, where $r \geq 2$. Let $\bar{T}$ be the tree obtained from T with $V(\bar{T}) = \{b_1, b_2, \ldots, b_r\}$ such that $b_ib_j \in \bar{T}$ if and only if $b_ib_j \in T$.

Claim 2. $|\bar{T}| = 2$.

Suppose that $|\bar{T}| \geq 3$, then $\bar{T}$ have at least two pendent edges.

Case 1. $\bar{T}$ has a pendent edge xy, where $x \neq u$ and $d_{\bar{T}}(y) = 1$. Let z be another neighbour of x in $\bar{T}$, and xx_1, yy_1, zz_1 be the pendent edges of T. Assume that T_1 and T_2 be the components of $T - xy$ containing x and y, respectively. Let $\varepsilon_{T_1}(x) = a$. Since $y \in V(\bar{T})$ we have $d_T(y) \geq 3$ and $|N_T(y) \setminus \{x, y_1\}| = k \geq 1$. Let $T' = T - \{yw : w \in N_T(y) \setminus \{x, y_1\}\} + \{xw : w \in N_T(y) \setminus \{x, y_1\}\}$. Obviously, $T' \in \mathbb{T}(2m,\Delta)$.

From the construction of T', it is easy to see that for $v \in V(T) \setminus \{x, y\}$ we have $d_T(v) = d_{T'}(v)$ and $d_T(x) = d_{T'}(x) - k$, $d_T(y) = d_{T'}(y) + k$, $\varepsilon_T(x) = \max\{a, 3\} \geq a = \varepsilon_{T'}(x)$, $\varepsilon_T(y) = a + 1 = \varepsilon_{T'}(y)$, and $\varepsilon_T(x) \leq \varepsilon_T(y)$.

Moreover, for $v \in V(T_1) \setminus \{x\}$, $\varepsilon_T(v) = \max\{\varepsilon_{T_1}(v), d_T(v,x)+3\} \geq \varepsilon_{T'}(v) = \max\{\varepsilon_{T_1}(v), d_T(v,x)+2\}$. For $v \in V(T_2) \setminus \{y, y_1\}$, $\varepsilon_T(v) = d_T(v,y) + 1 + a > \varepsilon_{T'}(v) = d_T(v,y) + a$.

By the definition of CEI, we have

$$\begin{aligned}\xi^{ce}(T)-\xi^{ce}(T') &= \sum_{v\in V(T)\setminus\{x,y\}}\left(\frac{d_T(v)}{\varepsilon_T(v)}-\frac{d_{T'}(v)}{\varepsilon_{T'}(v)}\right)+\frac{d_T(x)}{\varepsilon_T(x)}-\frac{d_{T'}(x)}{\varepsilon_{T'}(x)}\\ &+ \frac{d_T(y)}{\varepsilon_T(y)}-\frac{d_{T'}(y)}{\varepsilon_{T'}(y)}\\ &< \frac{d_T(x)}{\varepsilon_T(x)}-\frac{d_T(x)+k}{\varepsilon_T(x)}+\frac{d_T(y)}{\varepsilon_T(y)}-\frac{d_T(y)-k}{\varepsilon_T(y)}\\ &= k\left(\frac{1}{\varepsilon_T(y)}-\frac{1}{\varepsilon_T(x)}\right)\\ &\leq 0,\end{aligned}$$

implying that $\xi^{ce}(T)<\xi^{ce}(T')$, a contradiction.

Case 2. All pendent edges of $\bar{T}$ are incident with u. Obviously, $\bar{T}=S_r$. Let uu_1, uu_2, zz_1 be two edges in $\bar{T}$. Let z be a non-pendent neighbour of u_1 different from u in T, and u_1z_1, u_2z_2, zz_3 be pendent edges in T. Assume that T_1, T_2 and T_3 be the components of $T-uu_1-uu_2$ containing u_1, u_2 and u, respectively. Since $u_2\in V(\bar{T})$ we have $d_T(u_2)\geq 3$ and $|N_T(y)\setminus\{u,z_2\}|=k\geq 1$. Let $T'=T-\{u_2w : w\in N_T(u_2)\setminus\{u,z_2\}\}+\{u_1w : w\in N_T(u_2)\setminus\{u,z_2\}\}$. Obviously, $T'\in\mathbb{T}(2m,\Delta)$.

From the construction of T', it is easy to see that for $v\in V(T)\setminus\{u_1,u_2\}$ we have $d_T(v)=d_{T'}(v)$ and $d_T(u_1)=d_{T'}(u_1)-k$, $d_T(u_2)=d_{T'}(u_2)+k$, $\varepsilon_T(u_1)=\varepsilon_{T'}(u_1)$, $\varepsilon_T(u_2)=\varepsilon_{T'}(u_2)$. Moreover, for $v\in V(T_1)$, $\varepsilon_T(v)=d_T(v,u_1)+4>\varepsilon_{T'}(v)=d_T(v,u_1)+3$. For $v\in V(T_2)\setminus\{u_2,z_2\}$, $\varepsilon_T(v)=d_T(v,w)+5>\varepsilon_{T'}(v)=d_T(v,w)+4$. For $v\in V(T_3)$, $\varepsilon_T(v)=\varepsilon_{T'}(v)$.

By the definition of CEI, we have

$$\begin{aligned}\xi^{ce}(T)-\xi^{ce}(T') &= \sum_{v\in V(T)\setminus\{u_1,u_2\}}\left(\frac{d_T(v)}{\varepsilon_T(v)}-\frac{d_{T'}(v)}{\varepsilon_{T'}(v)}\right)+\frac{d_T(u_1)}{\varepsilon_T(u_1)}-\frac{d_{T'}(u_1)}{\varepsilon_{T'}(u_1)}\\ &+ \frac{d_T(u_2)}{\varepsilon_T(u_2)}-\frac{d_{T'}(u_2)}{\varepsilon_{T'}(u_2)}\\ &< \frac{d_T(u_1)}{\varepsilon_T(u_1)}-\frac{d_T(u_1)-k}{\varepsilon_T(u_1)}+\frac{d_T(u_2)}{\varepsilon_T(u_2)}-\frac{d_T(u_2)+k}{\varepsilon_T(u_2)}\\ &= k\left(\frac{1}{\varepsilon_T(u_1)}-\frac{1}{\varepsilon_T(u_2)}\right)\\ &= 0,\end{aligned}$$

implying that $\xi^{ce}(T)<\xi^{ce}(T')$, a contradiction. So, $|\bar{T}|=2$, and thus $\bar{T}=P_2$, i.e., $T\cong A_{2m,\Delta}$. □

Let $B_{2m,a}$ be the tree formed by adding an edge between the pendent vertex of $P_{2(m-a+1)}$ and the center of $A_{2(a-1),a-1}$,, where $2\leq a\leq m$. In particular, $B_{2m,2}=P_{2m}$.

Theorem 4.2. *Among all trees in $\mathbb{T}(2m,\Delta)$, $B_{2m,\Delta}$ is the unique tree with minimum CEI, where $2\leq\Delta\leq m$.*

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Proof. Let $T \in \mathbb{T}(2m, \Delta)$ such that $\xi^{ce}(T)$ is as small as possible. We only need to show that $T \cong B_{2m,\Delta}$. For $\Delta = 2$ it is trivial. We suppose that $\Delta \geq 3$.

Choose a vertex $v \in V(T)$ with degree Δ. Let $N_T(v) = \{v_1, \dots, v_\Delta\}$. Let T_i be the component of $T - v$ containing v_i, where $i = 1, \dots, \Delta$. Suppose that for some i, T_i is not a path with one terminal vertex v_i. Then there is a vertex in T_i such that its degree in T is at least three. So there is a vertex w in T_i such that $d_T(v, w)$ is as large as possible. That is to say, there are two pendent paths, say $P_1 = wu_1 \cdots u_\ell$ and $P_2 = wv_1 \cdots v_m$, where $\ell \geq m \geq 1$. So $T \cong G_{w;\ell,m}$, where $G = T[V(T) \setminus \{u_1, \dots, lu_\ell, v_1, \dots, u_m\}]$. Let $T' = T - wu_1 + v_1u_1$ if $m = 1$ and $T' = T - v_{m-2}v_{m-1} + u_\ell v_{m-1}$ if $m \geq 2$. Clearly, $T' \in \mathbb{T}(2m, \Delta)$. Note that $T' \cong G_{w;\ell+1,0}$ if $m = 1$ and $T' \cong G_{w;\ell+2,m-2}$ if $m \geq 2$. By Lemma 2.2, we have $\xi^{ce}(T') < \xi^{ce}(T)$, a contradiction. Therefore, for each $i = 1, \dots, \Delta$, T_i is a path with one terminal vertex v_i. So T consist of Δ pendent paths at v.

We claim that there is exactly one pendent path at v in T with length at least three. Otherwise, then there are at least two pendent paths at v with length at least three, say $R_1 = ws_1 \cdots s_\ell$ and $R_2 = wz_1 \cdots z_m$, where $\ell \geq m \geq 3$. Then $T \cong G_{w;\ell,m}$, where $G = T[V(T) \setminus \{s_1, \dots, s_\ell, z_1, \dots, z_m\}]$. Let $T'' = T - z_{m-2}z_{m-1} + s_\ell z_{m-1}$. Clearly, $T'' \in \mathbb{T}(2m, \Delta)$. Note that $T'' \cong G_{w;\ell+2,m-2}$ if $m \geq 2$. By Lemma 2.2, we have $\xi^{ce}(T'') < \xi^{ce}(T)$, a contradiction. Hence, $T \cong B_{2m,\Delta}$. □

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