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ON THE VC-DIMENSION, COVERING AND SEPARATING PROPERTIES OF THE CYCLE AND SPANNING TREE HYPERGRAPHS OF GRAPHS

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Abstract. In this paper, we delve into studying some relations between the structure of the cycles and spanning trees of a graph through the lens of its cycle and spanning tree hypergraphs which are hypergraphs with the edge set of the graph as their vertices and the edge sets of the cycles and spanning trees as their hyperedges respectively. In particular, we investigate relations between these hypergraphs from the perspective of the VC-dimension and some important separating and covering features of hypergraph theory and amongst the results, for example show that the VC-dimension of the cycle hypergraph is less than or equal to the VC-dimension of the spanning tree hypergraph and their gap can be arbitrary large. Note that VC-dimension is an important measure of complexity and a fundamental notion in numerous fields such as extremal combinatorics, graph theory, statistics and the theory of machine learning. Also we compare the separating and covering features of the mentioned hypergraphs and for instance show that the separating number of the cycle hypergraph is less than or equal to that of the spanning tree hypergraph. These hypergraphs help us to make several connections between cycles and spanning trees of graphs and compare their complexities.

1. Introduction

Investigation of the cycle structure and spanning tree structure of graphs and specifically, relationships between them has been always among important aspects of the study of graphs and networks. The family of all cycles (spanning trees) of a graph forms a hypergraph, called the cycle (spanning tree) hypergraph of the graph, with the edge set of the graph as its vertices and the edge sets of the

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cycles (spanning trees) as its hyperedges. The general theme of this paper is to study relation between cycle and spanning tree structure of graphs through the lens of such hypergraphs and compare some essential hypergraph-theoretic features of these hypergraphs together, such as their VC-dimensions and some separating and covering properties.

The study of graphs and hypergraphs are some of the main lines of research in combinatorics (see [3], [4], [6], [9], [12], [16] as some classical references). These two subjects have many interactions to each other. For example, one of the important instances of interactions between graph theory and hypergraph theory is to consider hypergraphs associated to a given graph and use parameters such as the VC-dimension and also separating and covering properties of those hypergraphs for studying that original graph. There are several works in the literature in this direction. For example, in papers such as [13] and [14], the VC-dimensions of the hypergraph of connected sets of the vertices and hypergraphs of paths, neighborhoods, stars, and some other ones associated to a given graph were considered from different viewpoints. In [2], several graph-theoretic parameters were studied by using the notions of the VC-dimension and test cover (which is an important separating-covering property) of certain hypergraphs associated to a graph such as the hypergraph of balls. In the papers [5] and [7], the VC-dimension of the hypergraph of balls was used for studying the problems of the coverings of graphs by balls. As some other examples, in paper [18], certain VC-dimension aspects of the neighborhood hypergraphs of graphs, and in [17] some connections between VC-dimension and model theory, were studied by the author of the present paper. Note that VC-dimension, separating number (or witness number) and transversal number are among the most important invariants and parameters associated to a given hypergraph in hypergraph theory and encode many information about that hypergraph. VC-theory was first initiated in the works of Vapnic and Chervonenkis. Then, the theory was developed by discovering important results such as the celebrated Sauer-Shelah lemma. Nowadays, VC-dimension has appeared in many research fields and is regarded as a core concept in numerous areas of mathematics, computer science and statistics, in particular, extremal combinatorics and machine learning. Moreover, separating and covering features and also transversal-theoretic concepts have been widely used in combinatorics. The interested reader can see [1], [2], [12] and [16] for some more information about these notions.

During the course of our investigation in this paper, we pursue a few goals. One is to study certain aspects of the VC-dimension of the cycle and spanning tree hypergraphs and relations between them. In particular, as one of the highlights of the paper, we compare those VC-dimensions and show that the VC-dimension of the cycle hypergraph of a connected graph is always less than or equal to the VC-dimension of the spanning tree hypergraph and their gap can be arbitrary large. Indeed, from the point of view of the VC-dimension as a measure of complexity, we show that spanning trees have higher complexity. On the way of proving this result, we investigate the structure of the shattered sets by those two hypergraphs, give some bounds for their VC-dimensions and compute those dimensions in some cases. In addition, some relationships between transversal number of the cycle hypergraph and VC-dimension of the spanning tree hypergraph are obtained. From another perspective, we study and compare several separating and covering properties of the cycle and spanning tree hypergraphs and for example show that the separating number of the cycle hypergraph is less than or equal to that of the spanning tree hypergraph. We also present the relations between test covers and twin-freeness in two hypergraphs. The above results compare the complexities of the cycle and spanning tree hypergraphs from the point of view of VC-theoretic and separating and covering parameters. On the way of the above investigations, in different parts of the paper we try to make connections between cycle structure and spanning trees of connected graphs by frequently relating the structures of the cycle and spanning tree hypergraphs to each other.

Organization of the rest of the paper is as follows. After reviewing necessary preliminaries in Section 2, in Section 3 we study certain aspects of the VC-dimensions and separating and covering features of the cycle and spanning tree hypergraphs of graphs and several connections between them.

2. Preliminaries

By a hypergraph $(X, \mathcal{F})$ in this paper we mean a finite set X, which is called the vertex set (or domain) of the hypergraph, and a family $\mathcal{F}$ of subsets of X called the hyperedges. For a subset $Y \subseteq X$ we define $\mathcal{F} \cap Y := \{A \cap Y : A \in \mathcal{F}\}$. We call the hypergraph $(Y, \mathcal{F} \cap Y)$ the *trace* of the hypergraph $(X, \mathcal{F})$ on the set Y. Also for each $A \in \mathcal{F}$, we call $A \cap Y$ the trace of the set A on Y. In a hypergraph $(X, \mathcal{F})$, a subset $Y \subseteq X$ is called *shattered* if we have $\mathcal{F} \cap Y = \mathcal{P}(Y)$ where $\mathcal{P}(Y)$ is as usual the notation for the power set of Y. The *VC-dimension* of the hypergraph $(X, \mathcal{F})$ (which we denote by $VC(X, \mathcal{F})$), is the largest $n \in \mathbb{N}$ such that there exists some subset of size n of X which is shattered by $(X, \mathcal{F})$.

In the following, we define some important notions associated to a hypergraph.

Definition 2.1. *Let $(X, \mathcal{F})$ be a hypergraph. By a* test cover *of $(X, \mathcal{F})$ we mean a family $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ is a covering of $(X, \mathcal{F})$, which means that $X \subseteq \bigcup_{A \in \mathcal{G}} A$, and also for every $x, y \in X$, there is $A \in \mathcal{G}$ that contains exactly one of x and y.*

Definition 2.2. *Two different elements $x, y \in X$ in a hypergraph $(X, \mathcal{F})$ are called twins if each $A \in \mathcal{F}$ contains either both of x and y or none of them. We call $(X, \mathcal{F})$ twin-free if such twins x and y in X do not exist.*

Definition 2.3. *(see [12]) By a blocking set of a hypergraph $(X, \mathcal{F})$ we mean some $B \subseteq X$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{F}$. A blocking set whose proper subsets are not blocking sets is called a transversal. The size of the smallest transversal of $(X, \mathcal{F})$ is denoted by $\tau(X, \mathcal{F})$ and is called the transversal number (or blocking number) of $(X, \mathcal{F})$.*

We start to review some notions from graph theory. Through the paper, we use the notation $G = (V(G), E(G))$ for representing a graph with the vertex set $V(G)$ and the edge set $E(G)$. Also if G' is a subgraph of G, then by $V(G')$ and $E(G')$ we mean the vertex set and edge set of G' respectively. For a subset U of the edges of G, by $V(U)$ we mean the set of the vertices appeared in U. Also by $[U]$

we mean the graph $(V(U), U)$. When it is clear from the context, we may use U instead of $[U]$. If each of two arbitrary sets A and B is a subgraph or subset of the edges of G, then by the notation $A[B]$ we mean the subgraph $[C]$ where C is the set of the edges of A with both ending vertices belonging to $V(A) \cap V(B)$. The maximum and minimum degrees of the vertices of G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. By the circumference of G, denoted by $circ(G)$, we mean the size of the longest cycle in G. For any $A \subseteq E(G)$, by $G \setminus A$ we mean the graph $(V(G), E(G) \setminus A)$. Also if A is a subgraph of G, then by $G \setminus A$ we mean the graph $(V(G), E(G) \setminus E(A))$. For any subgraph A and edge e, by $A \setminus e$ and $A - e$ we mean $(V(A), E(A) \setminus \{e\})$. Similarly, by by the notations $A + e$ and $A \cup \{e\}$ we mean the subgraph $(V(A) \cup V(\{e\}), E(A) \cup \{e\})$. By a *cut-set* in a connected graph we mean a subset of the edges such that if they are removed from the graph then the graph becomes disconnected. In a connected graph, we call an edge of the graph a *bridge* if the set containing that single edge is a cut-set. We say that a graph is *bridgeless* if it has no bridge. By a minimal cut-set we mean a cut-set whose strict subsets are not cut-sets. For $k \in \mathbb{N}$, we call a connected graph k-connected (k-edge-connected) if removing less than k many vertices (edges) from the graph does not make it disconnected. By $edgeCon(G)$ we mean the maximum number k for which G is k-edge-connected. By a cut-vertex in a graph we mean a vertex such that if it is removed, then the number of the connected components of the graph increases. Let A and B be two subsets of the vertices of a graph. Then, by the notation $E(A, B)$ we mean the set of the edges of the graph with one end in A and the other in B.

For the following definition, one can see the Subsection 3.1 of the book [9].

Definition 2.4. *Let G be a graph. A maximal connected subgraph without a cut-vertex in G is called a block of G.*

By the definition of blocks, it is straightforward to see that each block in a graph is either a maximal 2-connected subgraph or a bridge (with its ends), or an isolated vertex (see [9]). Also two different blocks overlap in at most one common vertex. Furthermore, it is not hard to verify that every cycle of the graph belongs to exactly one of its 2-connected blocks. Indeed, the set of the cycles of the graph is exactly the same as the union of the sets of the cycles of its blocks.

The following statement is a known fact and is straightforward to be verified.

Remark 2.5. *In every connected graph G, any subset of the edges of G which does not contain any cycle can be extended to a spanning tree of G.*

By a spanning forest of a graph G with connected components $G_1, \ldots, G_t$, we mean a subgraph T of G such that the graph $T[G_i]$ is a spanning tree of the graph G_i for every $i = 1, \ldots, t$.

Let T be a spanning tree of a connected graph G. We remind that for every edge $e \in E(G) \setminus E(T)$, the graph $T + e$ contains a unique cycle denoted by C_e. Each such cycle C_e is called a *fundamental cycle* of graph G with respect to the spanning tree T and the edge e. Also, for every $e \in E(T)$, the graph $T - e$ has exactly two connected components. The set of the edges of G between vertices of these connected components is usually denoted by D_e and called the *fundamental cut* of G with respect to

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the spanning tree T and edge e. Recall that the cospanning tree which is corresponding to a spanning tree T of G is the graph $(V(G), E(G) \setminus E(T))$.

Below, we recall the definition of two well-known classes of graphs, namely, the classes of "wheel graphs" and "friendship graphs". We will use them later in the paper.

Definition 2.6. *By the wheel graph W_n of order $n \geqslant 4$, we mean a graph consisting of a cycle of length $n-1$ plus one other vertex which is connected to all vertices of that cycle and is called the center of the wheel graph.*

Definition 2.7. *Fix some $t \in \mathbb{N}$. The friendship graph F_t is defined to be the graph formed by union of t copies of C_3 in a common vertex.*

By a *unicyclic* graph with n vertices and m edges we mean a connected graph with $m = n$. It is easy to see that a connected graph is unicyclic if and only if it contains exactly one cycle.

The following statement is easy to be verified.

Remark 2.8. *Assume that G is a connected graph and $B_1, \ldots, B_t$ are the blocks of G. Also for each $i \leqslant t$, let T_i be a spanning tree of B_i. Then, $\bigcup_{i \leqslant t} T_i$ is a spanning tree for G. Moreover, if T is a spanning tree for G, then $T_i := T[B_i]$ is a spanning tree for B_i for each i and furthermore, $T = \bigcup_{i \leqslant t} T_i$.*

The following theorem can be found in the book [9, Theorem 2.3.2].

Theorem 2.9. *(Erdos, Posa 1965) There exist a function $f : \mathbb{N} \to \mathbb{R}$ such that for every $k \in \mathbb{N}$ and every graph G, the graph either contains k vertex-disjoint cycles or contains a set A of vertices with $|A| \leqslant f(k)$ such that every cycle of G contains at least one of the vertices of A.*

3. Cycle and spanning tree hypergraphs and VC-dimension

Hypergraph theory is one of the central areas of research in combinatorics. In this theory, there are many notions associated to a given hypergraph and the hypergraph is studied by them. VC-dimension, separating and covering properties and transversals are among the most important aspects of a hypergraph. In this section, we delve into studying the cycles and spanning trees of graphs and some of their relations through the lens of the cycle and spanning tree hypergraphs (as will be defined below) and above mentioned parameters of them. Indeed, we provide some results regarding the relations between VC-dimensions and separating-covering features of the cycle and spanning tree hypergraphs. The main results of this section are statements 3.3 and 3.4 in which the structure of these two hypergraphs and shattered sets by them are investigated and also they are compared for example by showing that for any connected graph, the VC-dimension of its cycle hypergraph is less than or equal to the VC-dimension of its spanning tree hypergraph and their gap can be arbitrary large. Also in the results of the statements 3.6, 3.7 and 3.8, separating and covering features of two hypergraphs are compared. Results of this section make several connections between cycle structure and spanning tree structure of connected graphs by means of the cycle and spanning tree hypergraphs.

Note that we have mentioned a brief history and importance of the notion of the VC-dimension and its connections to graphs in the introduction of the paper. Nowadays, VC-dimension has appeared in many areas such as mathematics, computer science and statistics, in particular, extremal combinatorics and the theory of machine learning as a fundamental notion. Also separating and covering features and transversal-theoretic concepts are widely used in combinatorics.

We define the following hypergraphs which are some of essential notions we consider in this paper.

Definition 3.1. *By the* cycle hypergraph *of a graph G, we mean the hypergraph $(E(G), \mathcal{CYC}(G))$ where we define $\mathcal{CYC}(G) := \{E(C) :\ C \text{ is a cycle in } G\}$. Moreover, if the graph is connected, then by the* spanning tree hypergraph *of G, we mean the hypergraph $(E(G), \mathcal{SPNT}(G))$ where we define $\mathcal{SPNT}(G) := \{E(T) :\ T \text{ is a spanning tree of } G\}$.*

We start to study the VC-dimension of the cycle hypergraph of graphs. As an example, it is easy to see that for a graph G, we have $VC(E(G), \mathcal{CYC}(G)) = 0$ if and only if G has at most one cycle (or equivalently speaking, G is either a forest or an unicyclic graph). Also, we will give an example of computing the VC-dimension of the cycle hypergraph in a particular class of graphs, namely the class of wheel graphs, in the proof of Theorem 3.4(2). In the following, we define a notion very similar but slightly different from the notion of shattering and will use it frequently as a technical tool.

Definition 3.2. *In a hypergraph $(X, \mathcal{F})$, a subset $Y \subseteq X$ is called almost shattered by $(X, \mathcal{F})$ if $\mathcal{P}(Y) \setminus \{\emptyset\} \subseteq \mathcal{F} \cap Y$. The almost VC-dimension of $(X, \mathcal{F})$, denoted by $VC_{al}(X, \mathcal{F})$, is the largest integer n such that there exists some subset of X of size n which is almost shattered by $(X, \mathcal{F})$.*

It is easily seen from the above definition that if a subset of X is shattered by $(X, \mathcal{F})$, then it is almost shattered. Moreover, $VC(X, \mathcal{F}) \leqslant VC_{al}(X, \mathcal{F})$.

The following theorem which is one of the main results of the paper studies the VC-dimensions of the cycle and spanning tree hypergraphs and the structure of the shattered sets by them.

Theorem 3.3. *(Structure of the shattered sets) Let G be a graph with n vertices. Then, the following hold.*

1. *If G is connected, then a subset $A \subseteq E(G)$ is shattered by the spanning tree hypergraph of G if and only if A is not a cut-set of G and does not have any cycle. In particular,*

$$VC(E(G), \mathcal{SPNT}(G)) = \max\{|A| : A \text{ has no cycle and is not a cut-set}\} \leqslant n-1. \quad (\star)$$

   Moreover, $VC(E(G), \mathcal{SPNT}(G)) = n-1$ if and only if G has two spanning trees with no common edges.

2. *Let $d := VC(E(G), \mathcal{CYC}(G))$ and $d_{al} := VC_{al}(E(G), \mathcal{CYC}(G))$. Then, for every $\emptyset \neq A \subseteq E(G)$ almost shattered by the cycle hypergraph (in particular, if A is shattered by the cycle hypergraph), the subgraph $[A]$ is either a cycle or a vertex disjoint union of some paths of G. Moreover, if G has at least one cycle, then we have $d \leqslant d_{al} \leqslant circ(G) - 1 \leqslant n-1$. If $d = n-1$, then G is Hamiltonian. If G is the complete graph K_n for some $n \geqslant 5$, then $d = n-1$.*

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Moreover, a subset of edges of K_n of size $n-1$ which is shattered by the cycle hypergraph is either the edge set of a path or a cycle.

(3) *Assume that G is connected and $A \subseteq E(G)$ is almost shattered by the cycle hypergraph of G (in particular, if A is shattered by the cycle hypergraph). Then, the set A is not a cut-set of G. Moreover, if A is the edge set of a cycle, then for every $e \in A$, the set $A \setminus \{e\}$ is shattered by $(E(G), \mathcal{SPNT}(G))$. Furthermore, if A is not the edge set of any cycle, then it is shattered by $(E(G), \mathcal{SPNT}(G))$.*

Proof 1) Let $A \subseteq E(G)$ be shattered. Assume for contradiction that A is a cut-set. Then, A has common edge with every spanning tree of G. It follows that there is no spanning tree T with $E(T) \cap A = \emptyset$. Therefore, A is not shattered by $(E(G), \mathcal{SPNT}(G))$ which is a contradiction. Hence, A is not a cut-set. Similarly, A does not have any cycle since otherwise no spanning tree contains A which again contradicts the assumption that A is shattered.

Now we prove the converse. Assume that $A \subseteq E(G)$ is not a cut-set and also does not have any cycle. Take some arbitrary $X \subseteq A$ and let $Y := A \setminus X$. Then, Y is not a cut-set since A is not a cut-set. Therefore, $G' := (V(G), E(G) \setminus Y)$ is connected. Moreover, X has no cycle since A contains no cycle. So X is a subset of the edge set of G' and contains no cycle of G'. Now by Remark 2.5, G' has some spanning tree T that contains X. Note that T contains no edge of Y. Obviously, T is also a spanning tree of G. Moreover, $E(T) \cap A = X$. Thus, the subset X of A is obtained as the trace of some hyperedge of the spanning tree hypergraph on A. Therefore, since X was an arbitrary subset of A, the set A is shattered by the spanning tree hypergraph.

The proof of the equality in $(\star)$ is clear by what just proved. Also if $A \subseteq E(G)$ is shattered by $(E(G), \mathcal{SPNT}(G))$, then $|A| \leqslant n-1$ since A does not contain any cycle. So $(\star)$ is established.

If G has two edge-disjoint spanning trees, say T_1 and T_2, then $E(T_1)$ is not a cut-set since the graph $G \setminus E(T_1)$ is a connected graph. Also T_1 has no cycle. Hence, by what proved above, $E(T_1)$ is shattered by $(E(G), \mathcal{SPNT}(G))$. It follows that $VC(E(G), \mathcal{SPNT}(G)) \geqslant n-1$. Therefore, $VC(E(G), \mathcal{SPNT}(G)) = n-1$.

Now we show the converse. Assume that $VC(E(G), \mathcal{SPNT}(G)) = n-1$ and $A \subseteq E(G)$ is shattered by $(E(G), \mathcal{SPNT}(G))$ where $|A| = n-1$. As proved above, A does not contain any cycle. Thus, $[A]$ is a spanning tree. Also A is not a cut-set of G which follows that $G \setminus A$ is connected and has some spanning tree T. Now $[A]$ and T are two spanning trees of G with no common edges.

2) Assume that $\emptyset \neq A \subseteq E(G)$ is almost shattered by the cycle hypergraph. So, there exists some cycle C such that $E(C) \cap A = A$. Thus, $A \subseteq E(C)$. It follows that $[A]$ is either a cycle or a vertex disjoint union of some paths. Now, for the second part, clearly $d \leqslant d_{al}$. So we only show the second inequality. We assume for contradiction that $d_{al} \geqslant k$ where $k := circ(G)$. Note that $k \geqslant 3$. Now, there exists some almost shattered set $A \subseteq E(G)$ with $|A| \geqslant k$. So, there must exist some cycle C such that $E(C) \cap A = A$. Since the size of the longest cycle of G is k, we have $|E(C)| \leqslant k$. So we have $k \leqslant |A| = |E(C) \cap A| \leqslant |E(C)| \leqslant k$. Combining these follows that $A = E(C)$ and $|A| = |E(C)| = k$.

Let $B := A \setminus \{e\}$ for some $e \in A$. Obviously, $B \neq \emptyset$. Since A is almost shattered, there must exist some cycle C' such that $E(C') \cap A = B$. Thus, $B = E(C) \cap E(C')$ since was shown that $A = E(C)$. Hence, two cycles C and C' have $k-1$ edges in common. Now, since each of C and C' has at most k edges, it is not hard to see that C and C' must be the same. It follows that $B = E(C') \cap A = E(C) \cap A = A$ which is a contradiction. It follows that $d_{al} \leqslant circ(G) - 1$.

If $d = n-1$, then by using of what just proved, we have $n - 1 = d \leqslant circ(G) - 1 \leqslant n - 1$. So $circ(G) = n$ which follows that G is Hamiltonian.

Now assume that G is the complete graph K_n for some $n \geqslant 5$. In order to show $d = n-1$, by using what we just proved in above, we only need to show that there is a set of $n-1$ edges of K_n which is shattered by the cycle hypergraph. Let P be any path with $n-1$ edges in K_n. Then, it is not very difficult to see that for every subset A of the edges of P, there exists some cycle C of K_n such that $A = E(C) \cap E(P)$. It follows that $E(P)$ is shattered by $(E(K_n), \mathcal{CYC}(K_n))$. Hence, $d = n-1$.

Now, assume that $A \subseteq E(K_n)$ is shattered by $(E(K_n), \mathcal{CYC}(K_n))$ and $|A| = n-1$. Thus, as proved above A is either the edge set of a vertex disjoint union of some paths of K_n or the edge set of a cycle. In the former case, it is not hard to see that combining with $|A| = n-1$ follows that A is the edge set of a single path with $n-1$ edges. It completes the proof.

3) We first show that the set A is not a cut-set of G. Assume for contradiction that $G' := G \setminus A$ is disconnected. Take and fix some edge $e \in A$ which has its ends in two different connected components of G', say X and Y. Now every cycle of G containing e must also contain some other edge, say e', with one end in X and the other end in some other connected component of G' different from X. But every edge with ends in two different connected components of G' belongs to A. Therefore, every cycle of G containing e contains some other edge of A too. Thus, the set $\{e\}$ can not be the trace of the edge set of any cycle on A. It follows that A is not almost shattered by the cycle hypergraph which is a contradiction.

Now assume that A is the edge set of a cycle. Hence, for every $e \in A$ the set $A \setminus \{e\}$ does not contain any cycle. Also by what we proved, A is not a cut-set which follows that $A \setminus \{e\}$ is not a cut-set too. So, by Part 1, $A \setminus \{e\}$ is shattered by the hypergraph $(E(G), \mathcal{SPNT}(G))$.

Now assume that A is not the edge set of any cycle. Hence, since A is almost shattered by the cycle hypergraph, by using Part 2, $[A]$ is the vertex disjoint union of some paths. So A does not contain any cycle. Also by what we showed, A is not a cut-set. So, by Part 1, A is shattered by $(E(G), \mathcal{SPNT}(G))$. □

The following theorem which is one of the main results of this paper studies the relations between cycle and spanning tree hypergraphs and their VC-dimensions. Among such connections, an important one will be showing (in Part 1) that the VC-dimension of the cycle hypergraph of a connected graph is always bounded from above by the VC-dimension of its spanning tree hypergraph and their gap can be arbitrary large. The following Theorem, similar to a few other results mentioned above, indicates several connections between the structure of the cycles and spanning trees in connected graphs. We remind that the notation τ for the transversal number was defined in Section 2.

Theorem 3.4. *(Comparing cycle and spanning tree hypergraphs) The following relations between cycle and spanning tree hypergraphs hold.*

(1) *For every connected graph G with n vertices,*

$$VC(E(G), \mathcal{CYC}(G)) \leqslant VC(E(G), \mathcal{SPNT}(G)) \quad (*).$$

(2) *The gap between $VC(E(G), \mathcal{CYC}(G))$ and $VC(E(G), \mathcal{SPNT}(G))$ can be arbitrary large in different graphs G.*

(3) *If a connected graph G contains k vertex-disjoint cycles, then $VC(E(G), \mathcal{SPNT}(G)) \geqslant k$.*

(4) *There exists a function $f : \mathbb{N} \to \mathbb{R}$ such that the following hold. For every $k \in \mathbb{N}$ and every connected graph G, either $VC(E(G), \mathcal{SPNT}(G)) \geqslant k$ or $\tau(E(G), \mathcal{CYC}(G)) \leqslant f(k)(\Delta(G) - 1)$.*

Proof 1) Assume that the subset $D \subseteq E(G)$ is a shattered set by the hypergraph $(E(G), \mathcal{CYC}(G))$ and $|D| = VC(E(G), \mathcal{CYC}(G))$. If D is not the edge set of any cycle, then by Theorem 3.3 (3), D is shattered by the hypergraph $(E(G), \mathcal{SPNT}(G))$ and we are done. So, we may assume that D is the edge set of a cycle C which means $D = E(C)$.

Claim: There are edges $e \notin E(C)$ and $e' \in E(C)$ such that the set $(E(C) \cup \{e\}) \setminus \{e'\}$ (which has the same size as D) is shattered by $(E(G), \mathcal{SPNT}(G))$.

Proof of Claim: It is enough to consider and analyse the following two cases for G, namely, being and not being 2-edge-connected.

Case I: In this case we assume that G is 2-edge-connected.

Proof of Claim in Case I: Let n be the number of the vertices of G. Since $E(C)$ is shattered by $(E(G), \mathcal{CYC}(G))$, it is almost shattered by that hypergraph and by using Theorem 3.3(2), we have $|E(C)| \leqslant n - 1$ which follows that C is not a Hamiltonian cycle and there is some vertex in G not belonging to $V(C)$. Thus, since G is connected, there is at least one edge connecting some vertex of C and some vertex outside of C. Fix one of such edges and denote it by e. Obviously, $e \notin E(C)$. Note that by Theorem 3.3(3), $G \setminus E(C)$ is connected.

Now if $G \setminus (E(C) \cup \{e\})$ is connected, then for every arbitrary edge $e' \in E(C)$, the set of the edges $U := (E(C) \cup \{e\}) \setminus \{e'\}$ would not be a cut-set and also clearly would not contain any cycle. Thus, by Theorem 3.3(1), U would be shattered by $(E(G), \mathcal{SPNT}(G))$ and we would be done in Case I.

So, we may assume that $G \setminus (E(C) \cup \{e\})$ is not connected. Thus, since $G_1 := G \setminus E(C)$ is connected (as mentioned above), the edge e must be a bridge edge of G_1. Let subgraphs A and B be two connected components of $G_1 - e$. Clearly, every vertex of G belongs to either A or B. Since G_1 is not 2-edge-connected and adding the edges of the cycle C to G_1 creates a 2-edge-connected graph (indeed, $G_1 \cup C = G$), there must exist some edge of C, say e', in $E(A, B)$. So $A \cup B \cup \{e'\}$ is a connected subgraph. Similar to above, we use the notation U for $(E(C) \cup \{e\}) \setminus \{e'\}$. We claim that the edge set U is shattered by $(E(G), \mathcal{SPNT}(G))$. For proving that, by using Theorem 3.3(1), we only need to show that U does not contain any cycle and is not a cut-set. Since $e' \in E(C)$ and also one vertex of e is outside of C, clearly U contains no cycle. On the other hand, it is not hard to see that $A \cup B \cup \{e'\} = G \setminus U$. We recall from a few lines above that $A \cup B \cup \{e'\}$ is connected. So $G \setminus U$ is connected which follows that U is not a cut-set. Therefore, U is shattered by $(E(G), \mathcal{SPNT}(G))$. Now Case I is established. Case I □

Case II: In this case we assume that G is not 2-edge-connected.

Proof of Claim in Case II: Consider the blocks of G (see Definition 2.4 and the explanation after that). The cycle C must be completely included in one of the blocks of G, say the block B. Obviously, B can not be a single vertex or a bridge edge since it contains C. So by definition of a block, B must be a 2-connected subgraph of G and therefore is 2-edge-connected. We remind that every cycle of G belongs to exactly one of the 2-connected blocks of G and does not have edge intersection with any other block. Hence, every cycle which has edge intersection with C is included in the block B. Since $E(C)$ is almost shattered by the cycle hypergraph of G, there is $\mathcal{H} \subseteq \mathcal{CYC}(G)$ such that $\mathcal{H} \cap E(C) = \mathcal{P}(E(C)) \setminus \{\emptyset\}$ were we remind that $\mathcal{H} \cap E(C) = \{Z \cap E(C) : Z \in \mathcal{H}\}$. So, every cycle in $\mathcal{H}$ has nonempty edge intersection with C. Thus, every cycle in $\mathcal{H}$ is included in the block B. It follows that $E(C)$ is almost shattered by $(E(B), \mathcal{CYC}(B))$. Thus, by applying Case I above for the 2-edge-connected graph B, we can find some edge e connecting some vertex of C and some vertex of $V(B) \setminus V(C)$ and moreover, we can also find another edge $e' \in E(C)$ such that the edge set $(E(C) \cup \{e\}) \setminus \{e'\}$ is shattered by the hypergraph $(E(B), \mathcal{SPNT}(B))$. But since every spanning tree of B is extendible to a spanning tree of G (see Remark 2.8), it is not hard to see that the edge set $(E(C) \cup \{e\}) \setminus \{e'\}$ is shattered by $(E(G), \mathcal{SPNT}(G))$ too. Case II □

It finishes the proof of the Claim. **Claim** □

Now, using the above claim, we have

$$VC(E(G), \mathcal{CYC}(G)) = |D| = |(E(C) \cup \{e\}) \setminus \{e'\}| \leqslant VC(E(G), \mathcal{SPNT}(G)).$$

It follows that $VC(E(G), \mathcal{CYC}(G)) \leqslant VC(E(G), \mathcal{SPNT}(G))$.

2) We give two examples of graph classes with arbitrary large gap between the VC-dimensions of the cycle and spanning tree hypergraphs.

The first example is the class of friendship graphs. Consider the friendship graph F_t (as was defined in Definition 2.7) for some $t \in \mathbb{N}$ with $t \geqslant 2$. It is easily seen that $VC(E(F_t), \mathcal{CYC}(F_t)) = 1$. Now we find the VC-dimension of the spanning tree hypergraph $(E(F_t), \mathcal{SPNT}(F_t))$. It is not difficult to observe that every shattered set contains at most one edge from each C_3's in F_t and moreover, every $S \subseteq E(F_t)$ with exactly one edge from each C_3, is shattered by the hypergraph. Note that each such mentioned S has size t. Therefore, we have $VC(E(F_t), \mathcal{SPNT}(F_t)) = t$. It follows that in the class of friendship graphs, the VC-dimension of the spanning tree hypergraph of graphs can be arbitrary larger than the VC-dimension of the cycle hypergraph.

Now we give the second example. Consider the wheel graphs W_n (as was defined in Section 2) for $n \geqslant 5$. First, it is not difficult to observe that the graph W_n is indeed the union of two spanning trees with no common edges. Therefore, by Theorem 3.3(1), it is followed that we have $VC(E(W_n), \mathcal{SPNT}(W_n)) = n - 1$. We claim that for every $n \geqslant 5$ we have $VC(E(W_n), \mathcal{CYC}(W_n)) = 3$.

Fix some $n \geqslant 5$. In W_n, let E_1 and E_2 be the set of the star edges (i.e. edges connected to the center of the wheel) and outer edges (the rest of edges) respectively. Denote the cycle of W_n with edge set E_2 by C_0. Note that any cycle of G is either the cycle C_0 or has exactly two edges from E_1 and its rest of the edges from E_2. Assume that $A \subseteq E(W_n)$ is shattered by $(E(W_n), \mathcal{CYC}(W_n))$. By Theorem 3.3(2), $[A]$ is either a cycle or a vertex disjoint union of some paths. So A contains at most two edges of E_1.

Claim. $|A| \leqslant 3$.

Proof of Claim: It is enough to prove the claim in each of the following cases:

Case 1) In this case we assume that A contains exactly two edges of E_1 say e_1 and e_2. It is easy to see that there are exactly two cycles in W_n containing both e_1 and e_2. Since A is shattered by the cycle hypergraph, for every $X \subseteq A \cap E_2$ there is some cycle C such that $E(C) \cap A = X \cup \{e_1, e_2\}$. So every such C contains $\{e_1, e_2\}$. Hence, there are at most two such C's. It follows that there are at most two $X \subseteq A \cap E_2$. Hence, $|A \cap E_2| \leqslant 1$ which follows that $|A| \leqslant 3$.

Case 2) In this case we assume that A contains exactly one edge of E_1 say e_1. It is enough to show that $|A \cap E_2| \leqslant 2$. Assume for contradiction that $|A \cap E_2| > 2$. Let v_0 be the common vertex of e_1 and the cycle C_0. Also let e_2 and e_3 be the first and the last edges of $A \cap E_2$ appearing once we start moving on the cycle C_0 from v_0 in one direction (clockwise or counter-clockwise) until again getting back to v_0. Since we assumed that $|A \cap E_2| > 2$, $A \cap E_2$ has at least one other edge e_4 distinct from e_2 and e_3. Now it is not hard to see that every cycle of G containing e_1 and e_4 must also contain e_2 or e_3. It follows that $\{e_1, e_2, e_3, e_4\}$ is not shattered by the cycle hypergraph. Hence, since $\{e_1, e_2, e_3, e_4\} \subseteq A$, the set A is not shattered by the cycle hypergraph which is a contradiction. So, $|A \cap E_2| \leqslant 2$. Therefore, in this case $|A| \leqslant 3$.

Case 3) In this case we assume that A contains no edge of E_1. It is enough to show that no subset of E_2 consisting of four edges is shattered by the cycle hypergraph. Let e_1, e_2, e_3 and e_4 be four arbitrary edges of E_2 in the clockwise ordering of the cycle C_0. It is easy to see that there is no cycle C of G with $E(C) \cap \{e_1, e_2, e_3, e_4\} = \{e_2, e_4\}$. So $\{e_1, e_2, e_3, e_4\}$ is not shattered by the cycle hypergraph. It follows that in this case $|A| \leqslant 3$. **Claim** $\square$

It is not hard to see that every subset of E_2 of size 3 is shattered by $(E(W_n), \mathcal{CYC}(W_n))$. It follows that $VC(E(W_n), \mathcal{CYC}(W_n)) \geqslant 3$ (for every $n \geqslant 5$). Combining this fact with the above claim implies that $VC(E(W_n), \mathcal{CYC}(W_n)) = 3$ for every $n \geqslant 5$. Also it is worth to mention and not hard to verify that $VC(E(W_4), \mathcal{CYC}(W_4)) = 2$.

It shows that the VC-dimension of the spanning tree hypergraph of wheel graphs can be arbitrary larger than the VC-dimension of the cycle hypergraph of them. It completes the proof.

3) Assume that G contains k vertex-disjoint cycles. Let $A \subseteq E(G)$ be a set of edges consisting of one edge from each of those disjoint cycles. So $|A| = k$. It is not hard to see that A is not a cut-set and does not contain any cycle. So by Theorem 3.3(1), A is shattered by $(E(G), \mathcal{SPNT}(G))$. It follows that $VC(E(G), \mathcal{SPNT}(G)) \geqslant k$.

4) By Theorem 2.9 (Erdos-Posa Theorem), there is a function f such that for every $k \in \mathbb{N}$ and graph G, the graph either (i): contains k vertex-disjoint cycles or (ii): contains a set of vertices A with $|A| \leqslant f(k)$ such that every cycle of G contains at least one of the vertices of A. Fix some k and a connected graph G. If case (i) occurs for G, then by using Part 3, we have $VC(E(G), \mathcal{SPNT}(G)) \geqslant k$. If case (ii) occurs for G, then for every vertex $v \in A$, we choose $d(v) - 1$ number of edges connected to v and let W to be the set of all those edges for all $v \in A$. Then, W would have at least one edge from every cycle of G. The reason is that every cycle contains some vertex in A and all except possibly one edges connected to that vertex belong to W. So W is a blocking set of the hypergraph $(E(G), \mathcal{CYC}(G))$. Therefore, $\tau(E(G), \mathcal{CYC}(G)) \leqslant |W| \leqslant \sum_{v \in A} d(v) - 1 \leqslant |A|(\Delta(G) - 1) \leqslant f(k)(\Delta(G) - 1)$. It completes the proof. □

Theorem 3.4 above compares the complexities of the cycle and spanning tree hypergraphs from the point of view of VC-dimension and shows that, in a sense, spanning trees have higher complexity in this regard. Moreover, Part 4 of the theorem makes some relationships between the transversal number of the cycle hypergraph and the VC-dimension of the spanning tree hypergraph of graphs and also gives some bounds for them. Roughly speaking, this part implies that in any connected graph, if the VC-dimension of its spanning tree hypergraph is small, then its cycle hypergraph has small transversals. With this interpretation of this statement in mind, one can see Theorem 3.4(4), in a sense, in the same flavor and line of thought of some classical results in VC-theory and transversal theory (such as the celebrated ϵ-net theorem and (p, q)-theorem (see [16])) which usually have the intention of proving the existence of small transversals for hypergraphs that possess certain conditions, in particular the condition of having small VC-dimension.

Comparing separating and covering features of the cycle and spanning tree hypergraphs

Separating and covering features of a hypergraph are among the fundamental aspects of that hypergraph. In particular, the notions of separating sets and number, test cover and twin-freeness are among essential features of hypergraphs. These notions have appeared frequently in the literature. Also they are used for studying graphs. For example in papers such as [2], [5] and [7], many separating and covering properties of the neighborhood hypergraph of a graph were used for studying that graph. In this part, we investigate connections between the cycle and spanning tree hypergraphs of graphs from the point of view of these important notions. These investigations also helps to make more connections between cycle structure and spanning trees of connected graphs.

Below, we first consider the separating set (or witness set) of hypergraphs (defined by Renyi as follows), and also separating number (or witness number) which is an important feature of a hypergraph that encodes certain information about it. In Proposition 3.6 below, we will compare cycle and spanning tree hypergraphs of connected graphs from the point of view of this parameter.

Definition 3.5. *By a* separating set *(which is also called* witness set*) of a hypergraph $(X, \mathcal{F})$, we mean a subset $W \subseteq X$ such that for every $A, B \in \mathcal{F}$ we have $A \cap W \neq B \cap W$. By a minimal separating set we mean a separating set whose strict subsets are not separating sets anymore. We denote the* *smallest size among all minimal separating sets of the hypergraph* $(X,\mathcal{F})$ *by* $sep(X,\mathcal{F})$ *and call it the* separating number *of* $(X,\mathcal{F})$.

Proposition 3.6. *Let* G *be a connected graph. Then, every separating set (witness set) of the hypergraph* $(E(G),\mathcal{SPNT}(G))$ *is also a separating set of the hypergraph* $(E(G),\mathcal{CYC}(G))$. *In particular,*

$$sep(E(G),\mathcal{CYC}(G)) \leqslant sep(E(G),\mathcal{SPNT}(G)).$$

Proof Let S be a separating set for $(E(G),\mathcal{SPNT}(G))$. We first show that in every cycle C of G, there is at most one edge belonging to $S^c := E(G)\setminus S$. Assume for contradiction that there are $e_1,e_2 \in E(C)\cap S^c$. There is a spanning tree T_1 containing $E(C)\setminus e_2$ and not containing e_2. Now it is easy to see that $T_2 := (T_1\setminus\{e_1\})\cup\{e_2\}$ is also a spanning tree different from T_1. But clearly $E(T_1)\cap S = E(T_2)\cap S$ which is a contradiction with the assumption that S is a separating set for the spanning tree hypergraph. It follows that in every cycle of G, there is at most one edge outside S.

Now we show that S separates every two cycles from each other. Let C_1 and C_2 be two cycles of G such that $E(C_1)\cap S = E(C_2)\cap S$. As proved above, there is at most one edge of C_1 (and similarly for C_2) outside of S. It follows that $|E(C_1)\triangle E(C_2)| \leqslant 2$. It is clear that the edge set of no cycle in a graph can be a strict subset of the edge set of another cycle. Also it is easy to see that every two cycles are different in at least 3 edges. Putting above facts together, we conclude that $C_1 = C_2$. It follows that S is a separating set for the hypergraph $(E(G),\mathcal{CYC}(G))$. Now we conclude that $sep(E(G),\mathcal{CYC}(G)) \leqslant sep(E(G),\mathcal{SPNT}(G))$. □

Twin-freeness and existence of test covers are two close features of a given hypergraph. The following two statements compares the cycle and spanning tree hypergraphs from the perspective of these notions.

Proposition 3.7. *Let* G *be a connected graph. If the hypergraph* $(E(G),\mathcal{CYC}(G))$ *is twin-free, then the hypergraph* $(E(G),\mathcal{SPNT}(G))$ *would be twin-free too.*

Proof Since $(E(G),\mathcal{CYC}(G))$ is assumed to be twin-free, then G has at most one bridge edge. The reason is that every two bridge edges are twins in the cycle hypergraph since they both belong to no cycle of the graph. It is easy to see that an edge of a connected graph is a bridge edge if and only if it belongs to every spanning tree of the graph. Take some $e_1 \in E(G)$. If e_1 is a bridge edge, then since there is at most one bridge edge in G, e_1 would be the only edge belonging to all spanning trees of G. So e_1 is not twin (in the spanning tree hypergraph) with any other edges of the graph. Now assume that e_1 is not a bridge edge. Take any $e_2 \in E(G)$ different from e_1. It is enough to show that there exists a spanning tree containing exactly one of the two edges e_1 and e_2. Let T be a spanning tree with $e_1 \in T$. If $e_2 \notin T$, then we are done. Otherwise, choose one of the edges of the fundamental cut with respect to T and e_1 (see the definition of the fundamental cuts in Section 2), different from e_1, and denote it by e_3. Note that such edge exists since this cut contains at least one edge except e_1 (since e_1 is not a bridge edge). Now it is not hard to observe that $(T\cup\{e_3\})\setminus\{e_1\}$ is a spanning tree that contains e_2 but not e_1. So, e_1 and e_2 are not twins in the spanning tree hypergraph. It follows that the hypergraph $(E(G), \mathcal{SPNT}(G))$ is twin-free. □

Proposition 3.8. *Let G be a connected graph. If the hypergraph $(E(G), \mathcal{CYC}(G))$ has test cover, then the hypergraph $(E(G), \mathcal{SPNT}(G))$ has test cover too.*

Proof Assume that the hypergraph $(E(G), \mathcal{CYC}(G))$ has some test cover. So $(E(G), \mathcal{CYC}(G))$ must be a twin-free hypergraph. Therefore, by Proposition 3.7, $(E(G), \mathcal{SPNT}(G))$ is twin-free too. Also note that since every edge of a connected graph appears in at least one spanning tree, the family of the spanning trees of G covers $E(G)$. Combining the above facts follows that the set $\mathcal{SPNT}(G)$ is itself a test cover for the hypergraph $(E(G), \mathcal{SPNT}(G))$. Therefore, $(E(G), \mathcal{SPNT}(G))$ has test cover. □

Remark 3.9. *The converses of the propositions 3.7 and 3.8 do not hold. An easy counter-example for both is the cycle graph C_n (for any $n \geqslant 3$).*

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