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ON THE RELIABILITY OF MODIFIED BUBBLE-SORT GRAPHS

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Abstract. The modified bubble-sort graph MB_n ($n \geq 2$) has been known as a topology structure of interconnection networks. In this paper, we propose simple method for arc-transitivity of MB_n ($n \geq 2$). Also by using this result we investigate some reliability measures, including super-connectivity, cyclic edge connectivity, etc., in the modified bubble-sort graphs.

1. Introduction

Suppose that Γ is a finite, simple and undirected graph. We use the symbols $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ for showing that the vertex set, edge set, arc set and the automorphism group of Γ, respectively. We say that Γ is *vertex-transitive*, *edge-transitive* and *arc-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitive on $V(\Gamma)$, $E(\Gamma)$ or $A(\Gamma)$, respectively.

Given a finite group G and a subset $S \subseteq G - \{1\}$ such that $S = S^{-1} = \{s^{-1} \mid s \in S\}$, where 1 is the identity element of G. The *Cayley graph* $\mathrm{Cay}(G, S)$ on G with respect to S is defined to have vertex set G and edge set $\{\{g, sg\} \mid g \in G, s \in S\}$. Also a Cayley graph $\mathrm{Cay}(G, S)$ is connected if and only if S generates G. Moreover, $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$ is a subgroup of $\mathrm{AutCay}(G, S)_1$.

The modified bubble-sort graph MB_n (Lakshmivarahan et al, 1993) has the vertex set consisting of all $n!$ permutations of $\{1, 2, \cdots, n\}$. Also two vertices u_1 and v_1 in MB_n are adjacent if and only if $v_1 = u_1(i, i+1)$ or $v_1 = u_1(1, n)$ for all $1 \leq i \leq n-1$. The graphs of MB_3 and MB_4 are given in Figure 1. Note that MB_n is a special Cayley graph. In fact, $MB_n = \mathrm{Cay}(S_n, S)$ where S_n is symmetric permutation group and $S = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(1, n)\}$.

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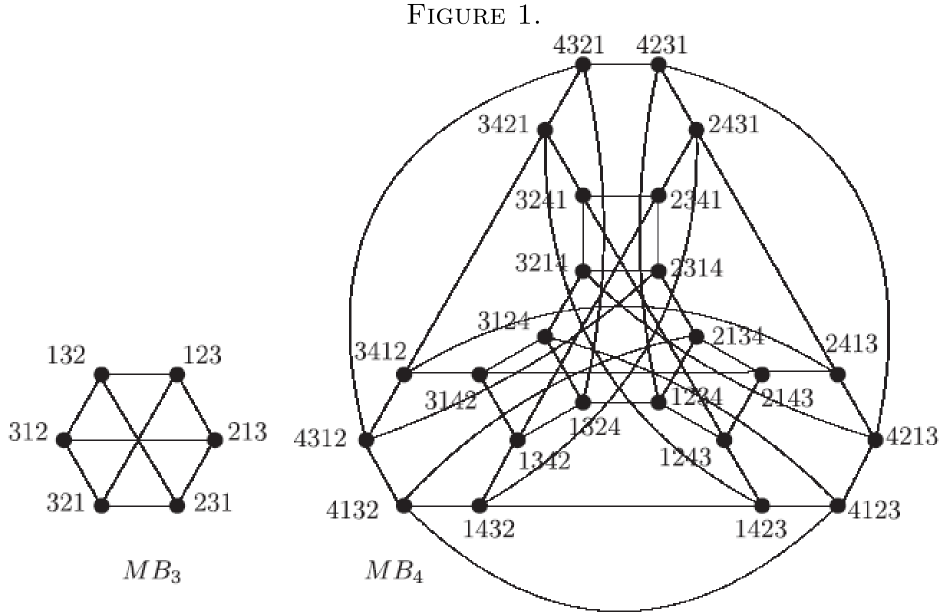
A faulty set $F \subseteq V = V(\Gamma)$ is called a *k-good-neighbour faulty set* if $|N(v) \cap (V - F)| \geq k$ for every vertex $v \in V - F$. A *k-good-neighbour cut* of a graph Γ is a k-good-neighbour faulty set F such that $\Gamma - F$ is disconected. The minimum cardinality of k-good-neighbour cuts is said to be *k-good-neighbour connectivity* of Γ and it is denoted by $\kappa^{(k)}(\Gamma)$. Cheng and Liptak (see [3]) proved that the 1-good-neighbour connectivity of modified bubble-sort graphs is $2n - 2$.

The *vertex-connectivity* of a graph Γ denoted by $\kappa(\Gamma)$, is the minimum number of vertices whose removal results in a disconnected graph or a trivial graph. Similarly, we can define *edge-connectivity* of a graph Γ which is denoted by $\lambda(\Gamma)$. In fact $\kappa(\Gamma)$ and $\lambda(\Gamma)$ are two important factors for measuring the reliability of an interconnection network. Also if $\kappa(\Gamma)$ or $\lambda(\Gamma)$ becomes larger then the network Γ becomes more reliable. It is well known that $\kappa(\Gamma) \leq \lambda(\Gamma) \leq \delta(\Gamma)$, where $\delta(\Gamma)$ is the minimum degree of Γ.

Zhou et al. (see [26]) and the author (see [6]) studied some reliability measures such as super connectivity, cyclically connectivity in the balanced hypercubes and folded hypercubes by using their symmetric properties. In this paper we get the similar results for modified bubble-sort graph MB_n.

FIGURE 1.



2. **Preliminaries**

In this section, we give some terminology and notation. For notation and terminology not define here we follow [2]. A graph is said to be *super-κ*(resp. *super-λ*), if any minimum vertex-cut(resp. edge-cut) isolates a vertex. In order to estimate more precisely the reliability, Esfahanian and Hakimi introduce such a kind of edge cut in [5] that separates a connected graph into a disconnected one without isolated vertices. With the properties of restricted edge connectivity, Li analyzed the reliability of circulant graphs in [9] and improved Bauer's result. For more accurate results, Ou et al. introduce the concepts m-restricted edge cut and m-restricted edge connectivity in [5, 12, 13, 21]. An edge set F is an *m-restricted edge cut* of a connected graph Γ if $\Gamma - F$ is disconnected and each component of $\Gamma - F$ contains at least m vertices (see [5]). Let $\lambda^{(m)}(\Gamma)$ be the minimum size of all m-restricted edge cuts and $\xi_m(\Gamma) = \min\{|\omega(U)| : |U| = m \text{ and } G[U] \text{ is connected}\}$ where $\omega(U)$ is the set of edges with exactly one end vertex in U and $\Gamma[U]$ is the subgraph of Γ induced by U. A graph Γ is *$\lambda^{(m)}$-graph* if $\lambda^{(m)}(\Gamma) = \xi_m(\Gamma)$. Also $\lambda^{(m)}(\Gamma)$ is called *m-restricted edge connectivity* of graph Γ. Moreover, a graph is called *super m-restrict edge connected*, in short, *super-$\lambda^{(m)}$* if every minimum m-restricted edge cut isolates one component $\Gamma[U]$ with $|U| = m$. In the special case, a set F of edges of a connected graph Γ is said to be a restricted edge-cut, if its removal disconnects Γ, and $\Gamma - F$ contains no isolated vertices. If Γ has at least one restricted edge-cut, the restricted edge-connectivity of Γ, denoted by $\lambda^{'}(\Gamma)$, is then defined to be the minimum cardinality over all restricted edge-cuts of Γ. Moreover, a graph Γ is called *super restricted edge-connected*, in short, *super-$\lambda^{'}$* if every minimum restricted edge cut isolates one component of size 2. Similarly, if F is a vertex set then m-restricted cut, m-restricted connectivity and super m-restricted connectivity (in the special case *super-$\kappa^{'}$*) are defined analogously. The super restricted edge-connectivity, restricted edge-connectivity and super connectivity of many interconnection networks has been studied (see [4, 7, 9, 10, 15, 16, 19, 26]).

For a graph Γ, an edge set F is a *cyclic edge-cut* if $\Gamma - F$ is disconnected and at least two of its components contain cycles. Clearly, a graph has a cyclic edge-cut if and only if it has two vertex-disjoint cycles. For a cyclically separable graph G, the *cyclic edge-connectivity* of Γ, denoted by $\lambda_c(\Gamma)$, is defined as the cardinality of a minimum cyclic edge-cut of Γ. Cyclic edge-connectivity plays an important role in many classic fields of graph theory such as measure of network reliability. A graph Γ is said to be *super-λ_c*, if the removal of any minimum cyclic edge-cut of Γ results in a component which is a shortest cycle of Γ. The cyclic edge-connectivity of many interconnection networks has been studied (see [14, 18, 22, 23, 24, 25]).

Let Γ and H be two graphs. The lexicographic product $\Gamma[H]$ is defined as the graph with vertex set $V(\Gamma) \times V(H)$ and for any two vertices (u_1, v_1), $(u_2, v_2) \in V(\Gamma) \times V(H)$, they are adjacent in $\Gamma[H]$ if and only if either u_1 is adjacent to u_2 in Γ, or $u_1 = u_2$ and v_1 is adjacent to v_2 in H. Let n be a positive integer. Denote by C_n the cyclic graph of order n. For a vertex v in a graph Γ, use $N_\Gamma(v)$ to denote the neighborhood of v, that is, the set of vertices adjacent to v. Also for a positive integer m, mK_1 represents the null graph with m vertices and a graph is trivial if it is a vertex.

([11, 17]) Let Γ be a connected graph which is both vertex-transitive and edge-transitive. Then $\kappa(\Gamma) = \delta(\Gamma)$, and moreover, Γ is not super-κ if and only if $\Gamma \cong C_n[mK_1]$ $(n \geq 6)$ or $L(Q_3)[mK_1]$, where $L(Q_3)$ is the line graph of three dimensional hypercube Q_3.

The following results are about the connectivity of edge-transitive graphs. ([1, 9, 16, 23, 24]) Let Γ be a $k(k \geq 3)$-regular edge-transitive graph. Then

1) Γ is super-λ.

2) $\lambda^{'}(\Gamma) = 2k - 2$.

3) Γ is not super-$\lambda^{'}$ if and only if Γ is isomorphic to the three dimensional hypercube Q_3 or to a

four-valent edge-transitive graph of girth 4.

4) If Γ is not isomorphic to K_4, K_5 or $K_{3,3}$ then $\lambda_c(\Gamma) = g(k-2)$, where g is the girth of Γ.

By [25, Theorem 4.8] we have the following result.

Let Γ be a $k(k \geq 3)$-regular edge-transitive graph of girth g. Suppose that Γ is not isomorphic to K_4, K_5 or $K_{3,3}$. If Γ is not super-λ_c, then $(g,k) = (6,3), (4,4), (4,5), (4,6)$ or $(3,6)$. Furthermore, $C_n[2K_1]$ $(n \geq 4)$ is non-super-λ_c, and if $(g,k) = (4,6)$ or $(g,k) = (4,5)$ then $|\Gamma| = 16$ or $|\Gamma| = 12$, respectively.

3. Reliability evaluation of modified bubble-sort graphs

The modified bubble-sort graph is arc-transitive.

Proof. We know that $MB_n = \mathrm{Cay}(S_n, S)$, where $S = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(1,n)\}$. Also it is well known that $\mathrm{Aut}(S_n) \cong S_n$ when $n \neq 2$ and 6. Also for $n = 6$ we have $\mathrm{Aut}(S_n)/S_n \cong \mathbb{Z}_2$. We first prove the following claim.

Claim $\mathrm{Aut}(S_n, S)$ is transitive on S.

Suppose that $(i, i+1), (j, j+1) \in S$. If $i+1 = j$ then $\sigma = (1, 2, \cdots, n) \in \mathrm{Aut}(S_n, S)$ and $(i, i+1)^\sigma = (j, j+1)$. Also if $i+1 \neq j$ then $(1,2)^{\sigma^{i-1}} = (i, i+1)$ and $(1,2)^{\sigma^{j-1}} = (j, j+1)$. Now $(i, i+1)^{(\sigma^{i-1})^{-1}\sigma^{j-1}} = (j, j+1)$ and $(\sigma^{i-1})^{-1}\sigma^{j-1} \in \mathrm{Aut}(S_n, S)$. Now suppose that $(i, i+1), (1, n) \in S$. If $i+1 \neq n$ then $(1,2)^{\sigma^{i-1}} = (i, i+1)$ and $(1,2)^{\sigma^{n-1}} = (1, n)$. Now $(i, i+1)^{(\sigma^{i-1})^{-1}\sigma^{n-1}} = (1, n)$ and $(\sigma^{i-1})^{-1}\sigma^{n-1} \in \mathrm{Aut}(S_n, S)$. Also if $i+1 = n$ then $\sigma = (1, 2, \cdots, n) \in \mathrm{Aut}(S_n, S)$ and $(i, i+1)^\sigma = (1, n)$. Thus the claim holds.

Now for showing that MB_n is arc-transitive, we need to prove that for any pairs of arcs of MB_n there is an automorphism which maps one to other one. Suppose that (a, b) and (c, d) are two arcs of MB_n. We considering the arc $(1, s)$. Since MB_n is vertex-transitive, there is an element $\alpha \in \mathrm{Aut}(\mathrm{MB_n})$ such that $a^\alpha = 1$. Therefore b^α is adjacent to 1. Now by our claim there is an element $\beta \in \mathrm{Aut}(S_n, S)$ such that $(b^\alpha)^\beta = s$. Also $a^\alpha = 1$ and so $(a, b)^{\alpha\beta} = (1, s)$. Also $\alpha\beta \in \mathrm{Aut}(MB_n)$. Similary there exists $\sigma \in \mathrm{Aut}(MB_n)$ such that $(c, d)^\sigma = (1, s)$. Now $\phi = \sigma\beta^{-1}\alpha^{-1} \in \mathrm{Aut}(MB_n)$ and $(c, d)^\phi = (a, b)$. □

$\kappa^{(r)}(C_l[mK_1]) = 2m$ where $l \geq 6$ and $1 \leq r \leq m$.

Proof. Suppose that $V(C_l) = \{1, 2, \cdots, l\}$ and $F = \{1, 4\}$. Clearly for every $v \in V(C_l) - F = \{2, 3, 5, 6, \cdots, l\}$ we see that $|N(v) \cap (V - F)| \geq 1$. Also F is a minimum 1-good-neighbour cut. Thus $\kappa^{(1)}(C_l) = 2$. Now let $V(mK_1) = \{a_1, a_2, \cdots, a_m\}$. We see that

$$F = \{(1, a_1), (4, a_1), (1, a_2), (4, a_2), (1, a_3), (4, a_3), \cdots (1, a_m), (4, a_m)\}$$

is a minimum r-good-neighbour cut, where $1 \leq r \leq m$, and so $\kappa^{(r)}(C_l[mK_1]) = 2m$ MB_n is super-k if and only if $n \geq 2$. □

Proof. If $n = 2$ or $n = 3$ then it is easy to see that MB_n is super-k. In what follows, assume that $n \geq 4$. Suppose to contrary that is MB_n is not super-k. By Theorem 3, we know that MB_n is arc-transitive and so is vertex-transitive and edge-transitive. Now by Proposition 2 it follows that

$MB_n \cong C_l[mK_1]$ $(l \geq 6)$ or $L(Q_3)[mK_1]$. First suppose that $MB_n \cong C_l[mK_1]$ $(l \geq 6)$. We know that $\kappa(MB_n) = n$. Also by [20, Theorems 1], $\kappa(C_l[mK_1]) = 2m$ and so $n = 2m$. Moreover by Lemma 3, $\kappa^{(1)}(C_l[mK_1]) = 2m$. Also we remind that $\kappa^{(1)}(MB_n) = 2n - 2$. Thus $n = 2n - 2$, a contradiction. Now suppose that $MB_n \cong L(Q_3)[mK_1]$. Thus the number of vertices of these graphs are same and so $n! = 12m$. On the other hand we see that $L(Q_3) = K_{4,4,4} - 24K_2$ and $\kappa(L(Q_3)) = 4$. Thus $\kappa(L(Q_3)[mK_1]) = 4m$ and so $n = 4m$. Now by considering $n! = 12m$ we see that $(4m)! = 12m$, a contradiction. □

In the following theorem we show that the delation of any minimum edge-cut of MB_n isolates a vertex.

MB_n is super-λ if and only if $n \geq 2$.

Proof. If $n = 2$ then it is easy to see that MB_2 is super-λ. Now suppose that $n \geq 3$. Thus the valency of MB_n is at least 3. Since MB_n is edge-transitive, the theorem follows from Proposition 2(1). □

From this theorem we immediately have the following corollary. For $n \geq 2$, $\lambda(MB_n) = n$.

In the following theorem we prove that the restricted edge-connectivity of MB_n is $2n - 2$.

For $n \geq 3$, $\lambda^{'}(MB_n) = 2n - 2$.

Proof. Since $n \geq 3$ we see that MB_n has valency $n \geq 3$. Now since MQ_n is edge-transitive, the theorem follows from Proposition 2. □

In the following theorem we show that every minimum restricted edge-cut of MB_n $(n \geq 3)$ isolates an edge.

MB_n is super-$\lambda^{'}$ if and only if $n \geq 3$.

Proof. If $n = 3$ then clearly MB_3 is super-$\lambda^{'}$ (see Figure 1). Also if $n = 4$ then again it is easy to see that MB_4 is super-$\lambda^{'}$ (see Figure 1). Thus we may suppose that $n \geq 5$. Therefore MB_n has valency $n \geq 5$. We know that MB_n is edge-transitive. By Proposition 2, MB_n is not super-$\lambda^{'}$ if and only if it is isomorphic to the three dimensional hypercube Q_3 or a four-valent edge-transitive graph of girth 4. Since MB_n $(n \geq 5)$ has valency at least 5 it implies that $MB_n \cong Q_3$. Also if $MB_n \cong Q_3$ then since the valency of Q_3 is three and the valency of MB_n $(n \geq 5)$ is at least 5, we have a contradiction. Therefore MB_n is super-$\lambda^{'}$ for $n \geq 3$. □

The following theorem shows that for $n \geq 4$, every minimum cyclic edge-cut of MB_n isolates a shortest cycle.

MB_n is super-λ_c if and only if $n \geq 4$.

Proof. If $n = 2$ then clearly MB_2 is not super-λ_c. Also if $n = 3$ then it is easy to see that MB_3 is not super-λ_c. Also if $n = 4$ then clearly MB_4 is super-λ_c. Thus we may suppose that $n \geq 5$. We know that MB_n has order $n!$ and valency n. Thus MB_n is not isomorphic to K_3, K_4 or $K_{3,3}$. Suppose to contrary that MB_n is not super-λ_c. By Proposition 2, $n \in \{3, 4, 5, 6\}$. Now we may suppose that $n = 5$ or $n = 6$. By Proposition 2, MB_4 or MB_5 has 12 or 16 vertices, a contradiction. □

In the following theorem we show that for $n \geq 4$, by removing $4(n-2)$ edges from MB_n we obtain a disconnected graph which has at least two components containing cycle.

For $n \geq 4$, $\lambda_c(MB_n) = 4(n-2)$.

Proof. Since MB_n $(n \geq 4)$ has at least 24 vertices it is not isomorphic to K_3, K_4 and $K_{3,3}$. By [8], we know that MB_n is bipartite and so it does not have odd girth. It is easy to see that the subgraph of MB_n induced by the vertex set $\{i, (1,2), (1,2)(3,4), (3,4)\}$, is a cycle of lenght 4, where i is the identity element of S_n. Thus MB_n has girth 4 and so by Proposition 2, $\lambda_c(MB_n) = 4(n-2)$. □

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