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CERTAIN CLASSES OF COMPLEMENTARY EQUIENERGETIC GRAPHS

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Abstract. The energy of a graph is the sum of the absolute values of the eigenvalues of a graph. Two graphs are said to be equienergetic if they have same energy. A graph is said to be complementary equienergetic if it is equienergetic with its complement. Recently several complementary equienergetic graphs have been identified. In this paper, we characterize the cycles, paths, complete bipartite regular graphs and iterated line graphs of regular graphs, which are complementary equienergetic.

1. Introduction

Let G be a simple, undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *degree* of a vertex v_i in G, denoted by d_i is the number of edges incident to it. A graph G is said to be a *regular graph* of degree r if all its vertices have same degree equal to r. If the vertex set of G can be partitioned into two non-empty sets so that no edge joins two vertices in the same set, then G is called *bipartite graph* and the two sets are called *partite sets*. A *semi-regular bipartite* graph with parameters (n_1, n_2, r_1, r_2) is a bipartite graph with partite sets V_1 and V_2 with $|V_1| = n_1$ and $|V_2| = n_2$ and vertices in V_1 has degree r_1 and vertices in V_2 has degree r_2. The *complement* of a graph G is the graph $\overline{G}$ with vertex set $V(\overline{G}) = V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph G is *self-complementary* if it is isomorphic to its complement. The *line graph* $L(G)$ of a graph G is the graph whose vertex set has one-to-one correspondence with the set of edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent

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in G. The k-th iterated line graph of G is defined as $L^k(G) = L\big(L^{k-1}(G)\big)$, for $k = 1, 2, \ldots$, where $L^0(G) = G$ and $L^1(G) = L(G)$.

Let C_n denotes the cycle on n vertices, P_n denotes the path on n vertices and $K_{p,q}$ be the complete bipartite graph on $p+q$ vertices.

The adjacency matrix of a graph G of order n is the $n \times n$ matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$, otherwise. The eigenvalues of $A(G)$ labeled as $\lambda_1, \lambda_2, \ldots, \lambda_n$ are said to be the *eigenvalues* of a graph G. Two different graphs are said to be *cospectral* if they have same eigenvalues [6].

The *energy* of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of the absolute values of the eigenvalues of G [9]. That is,

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i| \,.$$

More details about graph energy can be found in [10, 11, 14, 15]. Two graphs G_1 and G_2 of the same order are said to be *equienergetic* if $\mathcal{E}(G_1) = \mathcal{E}(G_2)$. The concept of equienergetic graphs trace back to 2004. Since from its origin, tremendous work on equienergetic graphs has been carried out [2, 7, 8, 12, 13, 17, 18, 20, 21, 24, 25].

A non-self-complementary graph G is said to be *complementary equienergetic* if $\mathcal{E}(G) = \mathcal{E}(\overline{G})$. Recently Ramane et al. [19] constructed the complementary equienergetic graphs and in [16] characterized the complementary equienergetic strongly regular graphs. Ali et al. [1] determined all complementary equienergetic graphs with at most 10 vertices. In this paper we characterize the complementary equienergetic cycles, paths, complete bipartite regular graphs and iterated line graphs of regular graphs.

We require following results to prove the results of this paper.

Theorem 1.1. [23] *If G is an r-regular graph of order n with eigenvalues $r, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of $\overline{G}$ are, $n-r-1, -1-\lambda_2, \ldots, -1-\lambda_n$.*

Theorem 1.2. [3] *For $n \geq 3$, the eigenvalues of the cycle C_n are,*

$$2\cos\left(\frac{2\pi k}{n}\right) \quad \text{for } k = 0, 1, 2, \ldots, n-1.$$

Theorem 1.3. [4, 5] *Let G be a connected graph of order n with vertices $v_1, v_2, \ldots, v_n$ and m edges. If d_i is the degree of the vertex v_i for $i = 1, 2, \ldots, n$, then*

$$\frac{2m}{n} \leq \lambda_{max} \leq \max_{(v_i,v_j)\in E(G)} \{\sqrt{d_i d_j}\ \},$$

where λ_{max} is the maximum eigenvalue. Moreover, the equality holds if and only if G is regular or semi-regular bipartite.

Theorem 1.4. [22] *Let G be a regular graph of order n_0, degree $r_0 \geq 3$, and for $k \geq 2$ the k-th iterated line graph of G be of degree r_k and of order n_k. Then*

$$\mathcal{E}\left(L^k(G)\right) = 4n_k\left(\frac{r_k - 2}{r_k + 2}\right).$$

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Theorem 1.5. [26] *Let G be a regular graph of order n_0, of degree $r_0 \geq 3$, and for $k \geq 2$ the k-th iterated line graph of G be of degree r_k and of order n_k. Then*

$$\mathcal{E}\left(\overline{L^k(G)}\right) = \frac{4n_k r_k}{r_k+2} - 2(r_k+1).$$

2. Complementary equienergetic graphs

For the energy of cycle C_n and of its complement $\overline{C_n}$, the absolute values of their eigenvalues are taken based on the nature of Cosine function.

Theorem 2.1. *For $n \geq 3$, the cycle C_n is complementary equienergetic if and only if $n = 4, 5$, and 6.*

Proof. Let $\mathcal{E}(C_n) = \mathcal{E}(\overline{C_n})$. We need to prove that $n = 4, 5$ and 6. We prove this by considering two cases: when n is odd and when n is even.
Case (i) When n is odd.

Using Theorem 1.2, we have

$$\begin{aligned} \mathcal{E}(C_n) &= \sum_{k=0}^{n-1} \left|2\cos\left(\frac{2\pi k}{n}\right)\right| \\ &= 2 + 2\left[\sum_{1\leq k\leq \lfloor \frac{n-1}{4}\rfloor} 2\cos\left(\frac{2\pi k}{n}\right) - \sum_{\lfloor \frac{n-1}{4}\rfloor < k \leq \frac{n-1}{2}} 2\cos\left(\frac{2\pi k}{n}\right)\right]. \end{aligned} \tag{2.1}$$

Now by using Theorem 1.1 and Theorem 1.2 we have

$$\begin{aligned} \mathcal{E}\left(\overline{C_n}\right) &= |n-2-1| + \sum_{k=1}^{n-1}\left|-1-2\cos\left(\frac{2\pi k}{n}\right)\right| \\ &= (n-3) + 2\left[\sum_{1\leq k\leq \lfloor \frac{n}{3}\rfloor}\left(1+2\cos\left(\frac{2\pi k}{n}\right)\right) - \sum_{\lfloor \frac{n}{3}\rfloor < k < \frac{n+1}{2}}\left(1+2\cos\left(\frac{2\pi k}{n}\right)\right)\right]. \end{aligned} \tag{2.2}$$

From Eqs. (2.1) and (2.2), $\mathcal{E}(C_n) = \mathcal{E}\left(\overline{C_n}\right)$ only when $n = 5$, which can be verified easily.

Case (ii) When n is even.

Using Theorem 1.2 we have

$$\begin{aligned} \mathcal{E}(C_n) &= \sum_{k=0}^{n-1} \left|2\cos\left(\frac{2\pi k}{n}\right)\right| \\ &= 4 + 2\left[\sum_{1\leq k\leq \lfloor \frac{n}{4}\rfloor} 2\cos\left(\frac{2\pi k}{n}\right) - \sum_{\lfloor \frac{n}{4}\rfloor < k \leq \frac{n}{2}} 2\cos\left(\frac{2\pi k}{n}\right)\right] \end{aligned} \tag{2.3}$$

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and by using Theorem 1.1 and Theorem 1.2 we have

$$\begin{aligned}(2.4)\quad \mathcal{E}(\overline{C_n}) &= |n-2-1| + \sum_{k=1}^{n-1}\left|-1-2\cos\left(\frac{2\pi k}{n}\right)\right| \\ &= (n-2) + 2\left[\sum_{1\le k\le \lfloor \frac{n}{3}\rfloor}\left(1+2\cos\left(\frac{2\pi k}{n}\right)\right) - \sum_{\lfloor \frac{n}{3}\rfloor < k < \frac{n+1}{2}}\left(1+2\cos\left(\frac{2\pi k}{n}\right)\right)\right].\end{aligned}$$

From Eqs. (2.3) and (2.4), $\mathcal{E}(C_n) = \mathcal{E}(\overline{C_n})$ only when $n = 4$ and 6, which can be verified easily. □

The path P_n being a non-regular graph, we can not directly write the spectrum of $\overline{P_n}$ and hence its energy. Due to this fact we make use of bounds on eigenvalues.

Theorem 2.2. *For $n \ge 2$, the path P_n is complementary equienergetic if and only if $n = 4$.*

Proof. For $n = 2, 3$, the result is obvious. Let $n \ge 4$. As, P_n is a path on n vertices and having $n-1$ edges, it is neither regular nor semi-regular bipartite graph for $n \ge 4$. Therefore by Theorem 1.3,

$$(2.5)\qquad \frac{2(n-1)}{n} < \lambda_{\max}(P_n) < 2,$$

where $\lambda_{\max}(P_n)$ is the maximum eigenvalue of P_n.
We observe that, $\left|2 - \frac{2(n-1)}{n}\right| = \frac{2}{n}$, which is less than one. Thus,

$$\lambda_{\max}(P_n) = \frac{2(n-1)}{n} + a,$$

where $0 < a < 1$, $a \in \mathbb{R}$. Now, applying Theorem 1.3 on $\overline{P_n}$, we have

$$(2.6)\qquad \frac{(n-1)(n-2)}{n} < \lambda_{\max}(\overline{P_n}) < (n-2),$$

with $\left|(n-2) - \frac{(n-1)(n-2)}{n}\right| = 1 - \frac{2}{n}$, which is less than one. Thus,

$$\lambda_{\max}(\overline{P_n}) = \frac{(n-1)(n-2)}{n} + b,$$

where $0 < b < 1$, $b \in \mathbb{R}$. Therefore,

$$\mathcal{E}(P_n) < n\left[\frac{2(n-1)}{n} + a\right]$$

and

$$\mathcal{E}(\overline{P_n}) < n\left[\frac{(n-1)(n-2)}{n} + b\right].$$

For $\mathcal{E}(P_n) = \mathcal{E}(\overline{P_n})$, we have

$$(2.7)\qquad \frac{2(n-1)}{n} + a = \frac{(n-1)(n-2)}{n} + b$$

with $\left|\frac{(n-1)(n-2)}{n} - \frac{2(n-1)}{n}\right| = \frac{(n-1)(n-4)}{n}$ which is more than one and LHS, RHS of Eq. (2.7) can not be equal for all $n \ge 6$.

For $n = 4$, P_4 and $\overline{P_4}$ are isomorphic, have equal energy and for $n = 5$, $\mathcal{E}(P_5) = 5.4641$ and $\mathcal{E}(\overline{P_5}) = 6.3402$.

The converse is obvious due to the fact that P_4 is a self complementary graph. □

Theorem 2.3. *For $p \geq 1$, the complete bipartite graph $K_{p,p}$ is complementary equienergetic if and only if $p = 2$.*

Proof. The eigenvalues of $K_{p,p}$ are p, 0 $(2p-2$ times) and $-p$. Hence $\mathcal{E}(K_{p,p}) = 2p$. By Theorem 1.1, the eigenvalues of $\overline{K_{p,p}}$ are $p-1$, -1 $(2p-2$ times) and $p-1$. Hence $\mathcal{E}\left(\overline{K_{p,p}}\right) = 4p - 4$.

Now $\mathcal{E}(K_{p,p}) = \mathcal{E}\left(\overline{K_{p,p}}\right)$ implies $2p = 4p - 4$. This is true for $p = 2$. Converse is obvious. □

In [18] it was shown that if G is a regular graph on n vertices and of degree $r \geq 3$, then $L^2(G)$ is complementary equienergetic if and only if G is a complete graph on 6 vertices.

Theorem 2.4. *Let G be a regular graph of order n_0, of degree $r_0 \geq 3$, and let for $k \geq 2$ the k-th iterated line graph of G be of degree r_k and of order n_k. Then $\mathcal{E}\left(L^k(G)\right) = \mathcal{E}\left(\overline{L^k(G)}\right)$ if and only if*

$$r_k \neq \begin{cases} 4t, & t = 1, 2, \ldots \\ 4t+1, & t = 1, 2, \ldots \end{cases} \qquad \textit{where} \quad r_k \geq 6.$$

Proof. For $r_k \geq 6$, from Theorem 1.4 and Theorem 1.5, $\mathcal{E}\left(L^k(G)\right) = \mathcal{E}\left(\overline{L^k(G)}\right)$ if and only if

$$4n_k\left(\frac{r_k - 2}{r_k + 2}\right) = \frac{4n_k\, r_k}{r_k + 2} - 2(r_k + 1).$$

Which is possible if and only if

$$r_k^2 + 3r_k - 4n_k + 2 = 0. \tag{2.8}$$

The Eq. (2.8) holds if and only if $r_k \neq \begin{cases} 4t, & t = 1, 2, \ldots \\ 4t+1, & t = 1, 2, \ldots . \end{cases}$

As, for $r_k = 4t$, Eq. (2.8) becomes $8t^2 + 6t + 1 = 2n_k$, which is not possible. For $r_k = 4t+1$, Eq. (2.8) becomes $8t^2 + 10t + 3 = 2n_k$, which is also not possible. Hence, the result follows. □

3. Conclusion

The present work focuses on the necessary and sufficient conditions for certain classes of graphs for they being equienergetic with their complements, which include some class of regular (cycle, complete bipartite regular graphs and iterated line graphs of regular graphs) and non-regular graphs (path). One can think on necessary and sufficient condition for graphs of any other class to be equienergetic with their complements.

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