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PERIPHERAL HOSOYA POLYNOMIAL OF COMPOSITE GRAPHS

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ABSTRACT. Peripheral Hosoya polynomial of a graph G is defined as,

$$PH(G,\lambda) = \sum_{k\geq 1} d_P(G,k)\lambda^k,$$

where $d_P(G,k)$ is the number of pairs of peripheral vertices at distance k in G.

Peripheral Hosoya polynomial of composite graphs viz., $G_1 \times G_2$ the Cartesian product, $G_1 + G_2$ the join, $G_1[G_2]$ the composition, $G_1 \circ G_2$ the corona and $G_1\{G_2\}$ the cluster of arbitrary connected graphs G_1 and G_2 are computed and their peripheral Wiener indices are stated as immediate consequences.

1. Introduction

Graphs considered in this paper are *simple* and *connected*. Let G be a graph with n *vertices* and m *edges*. The number of edges incident to a vertex u in graph G is called the *degree*, $deg_G(u)$ of the vertex u. The notation will be simplified to $deg(u)$ if the graph is clear from the context. The *distance*, $d_G(u,v)$ or $d(u,v)$ between vertices u and v is the length of a shortest path between u and v. The furthest distance attained by a vertex u in G is known as the *eccentricity* $e_G(u)$ or $e(u)$ of the vertex u. A vertex v is an *eccentric* vertex of a vertex u if $d(u,v) = e(u)$ and the set containing all eccentric vertices of u is known as an *eccentric set* $E_G(u)$ or $E(u)$ of u. The *diameter* $diam_G$ of G is the maximum eccentricity attained by vertices of G. The *periphery* $P(G)$ is the set of vertices of

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maximum eccentricity. More formally,

$$P(G) = \{u \in V(G) | e(u) = diam_G\}.$$

Vertices in $P(G)$ are known as *peripheral* vertices. For basic definitions and notations we refer [?, ?].

Hosoya polynomial is the most studied polynomial which is related to many distance-based graph invariants mainly with the well known distance-based graph invariant, the Wiener index [?, ?, ?]. It is introduced by Hosoya in his seminal paper [?] in 1988 as,

$$H(G, \lambda) = \sum_{k \geq 1} d(G, k)\lambda^k,$$

where $d(G, k)$ is the number of pairs of vertices at distance k. On the same paper, Hosoya states that $H'(G, 1) = \sum_{\{u,v\} \subset V(G)} d(u, v)$, where H' is the first derivative of H, this sum is known as the Wiener index of the graph G. Motivated by this, terminal Hosoya polynomial is studied in [?], which is defined as,

$$TH(G, \lambda) = \sum_{k \geq 1} d_T(G, k)\lambda^k,$$

where $d_T(G, k)$ is the number of pairs of pendant vertices at distance k in G.

Terminal Wiener index [?] of a graph G is defined as,

$$TW(G) = \sum_{\{u,v\} \subset V_T(G)} d(u, v),$$

where V_T is set of pendant vertices of G. One can observe that $TH'(G, 1) = TW(G)$, where TH' is first derivative of TH. In analogous to terminal Hosoya polynomial Kishori P. Narayankar and D. Shubhalakshimi [?, ?] defined peripheral Hosoya polynomial of a graph G as,

$$PH(G, \lambda) = \sum_{k \geq 1} d_P(G, k)\lambda^k, \tag{1.1}$$

where $d_P(G, k)$ is the number of pairs of peripheral vertices at distance k in G. This polynomial has a direct relationship with a graph theoretical parameter *peripheral Wiener index* $PW(G)$ of G, [?] which is defined as,

$$PW(G) = \sum_{\{u,v\} \subset P(G)} d(u, v),$$

where $P(G)$ is the periphery of G. Again in analogous to H' and TH', $PH'(G, 1) = PW(G)$, where PH' is the first derivative of PH. The study on peripheral Wiener index may be seen in [?, ?, ?, ?] and the references cited there in. Most recent results related to peripheral Hosoya polynomial of graphs are depicted in [?] and peripheral path index polynomial in [?].

In this paper, we consider graph compositions listed in definition ??. The definitions of graph compositions listed from (i) -(iii) below are taken from [?].

Definition 1.1. *Let G_1 and G_2 be two graphs of order n_1 and n_2, respectively. Define graph compositions of G_1 and G_2 as follows:*

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i) The Cartesian product $G_1 \times G_2$*:*

$$V(G_1 \times G_2) = V(G_1) \times V(G_2)$$

for vertices $a = (u_1, v_1), b = (u_2, v_2) \in V(G_1 \times G_2)$,

$$ab \in E(G_1 \times G_2) \textit{ iff } \Big(\big[u_1 = u_2 \textit{ and } v_1v_2 \in E(G_2)\big] \textit{ or } \big[v_1 = v_2 \textit{ and } u_1u_2 \in E(G_1)\big] \Big)$$

ii) The join $G_1 + G_2$*:*

$$V(G_1 + G_2) = V(G_1) \cup V(G_2)$$
$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \textit{ and } v \in V(G_2)\}$$

iii) The composition $G_1[G_2]$*:*

$$V(G_1[G_2]) = V(G_1) \times V(G_2)$$

for vertices $a = (u_1, v_1), b = (u_2, v_2) \in V(G_1 \times G_2)$,

$$ab \in E(G_1[G_2]) \textit{ iff } \Big(\big[u_1 = u_2 \textit{ and } v_1v_2 \in E(G_2)\big] \textit{ or } \big[u_1u_2 \in E(G_1)\big] \Big)$$

iv) The corona product *[?]* $G_1 \circ G_2$*:*
The graph obtained by taking one copy of G_1*, called the center graph,* n_1 *copies of* G_2*, called the outer graph, and making the* i^{th} *vertex of* G_1 *adjacent to every vertex of the* i^{th} *copy of* G_2*, where* $1 \leq i \leq n_1$.

v) The cluster *[?, ?]* $G_1\{G_2\}$ *is obtained by taking one copy of* G_1 *and* n_1 *copies of* G_2 *of a rooted graph* G_2*, and by identifying the root of the* i^{th} *copy of* G_2 *with the* i^{th} *vertex of* G_1*,* $i = 1, 2, \ldots, n_1$.

Yeh and Gutman [**?**] computed the Wiener index of these composite graphs. Motivated by this, Dragan Stevanović [**?**], computed their Hosoya polynomial.

In the following section, the peripheral Hosoya polynomial of composite graphs defined in definition **??** are computed and their peripheral Wiener indices are stated as immediate consequences.

2. Results

Lemma 2.1. *[?, ?] Let* G_1 *and* G_2 *be graphs with vertex sets* $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ *and* $V(G_2) = \{v_1, v_2, \ldots, v_{n_2}\}$*, respectively. Then,*

i) for $a = (u_i, v_j), b = (u_r, v_s) \in V(G_1 \times G_2)$,

$$d_{G_1 \times G_2}(a, b) = d_{G_1}(u_i, u_r) + d_{G_2}(v_j, v_s) \tag{2.1}$$

ii) for $a, b \in V(G_1\{G_2\})$

$$d_{G_1\{G_2\}}(a, b) = \begin{cases} d_{G_2}(a, b), & a, b \in G_2^{(i)} \\ d_{G_2}(a, v) + d_{G_1}(a, b) + d_{G_2}(v, b), & a \in G_2^{(i)} \textit{ and } b \in G_2^{(j)}, i \neq j \end{cases} \tag{2.2}$$

where v *is the root of* G_2 *and* $G_2^{(i)}$ *is a subgraph of* $G_1\{G_2\}$ *corresponding to the* i^{th} *copy of* G_2.

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Also note that:

$$d_{G_1+G_2}(a,b)=\begin{cases}1, & ab\in E(G_1)\cup E(G_2) \text{ or } \big(a\in V(G_1) \text{ and } b\in V(G_2)\big)\\ 2, & \text{otherwise}\end{cases} \tag{2.3}$$

i) for $a=(u_i,v_j), b=(u_r,v_s)\in V(G_1[G_2])$,

$$d_{G_1[G_2]}(a,b)=\begin{cases}0, & u_i=u_r \text{ and } v_j=v_s\\ 1, & u_i=u_r \text{ and } v_jv_s\in E(G_2)\\ 2, & u_i=u_r \text{ and } v_jv_s\notin E(G_2)\\ d_{G_1}(u_i,u_r), & u_i\neq u_r\end{cases} \tag{2.4}$$

ii) for $u,v\in V(G_1\circ G_2)$,

$$d_{G_1\circ G_2}(a,b)=\begin{cases}1, & ab\in E(G_1)\cup\Big(\bigcup_{i=1}^{k_1} E(G_2^{(i)})\Big)\cup X\\ 2, & u,v\in V(G_2^{(i)}) \text{ and } uv\notin E(G_2^{(i)})\\ d_{G_1}(u,v), & u,v\in V(G_1)\\ d_{G_1}(u_i,u_j)+2, & u\in V(G_2^{(i)}) \text{ and } v\in V(G_2^{(j)}), i\neq j\end{cases} \tag{2.5}$$

where $G_2^{(i)}$ is the subgraph of $G_1\circ G_2$ corresponding to the i^{th} copy of G_2 in $G_1\circ G_2$ and $X=\{\{a,b\}\subset V(G_1\circ G_2):[a=u_i \text{ and } b\in V(G^{(i)})] \text{ or } [a\in V(G^{(i)}) \text{ and } b=u_i], 1\leq i\leq n_1\}$

Lemma 2.2. *For the complete graph K_n of order n, $PH(K_n,\lambda)=\binom{n}{2}\lambda$.*

From now on, we shortly write $PH(G)$ instead of $PH(G,\lambda)$.

Theorem 2.3. *Let G_1 and G_2 be graphs with k_1 and k_2 peripheral vertices, respectively. Then,*

$$PH(G_1\times G_2)=2PH(G_1)PH(G_2)+k_2PH(G_1)+k_1PH(G_2).$$

Proof. Let $P(G_1)=\{u_1,u_2,...,u_{k_1}\}$ and $P(G_2)=\{v_1,v_2,...,v_{k_2}\}$ be peripheries of graphs G_1 and G_2, respectively. Then, by equation **??**,

$diam_{G_1\times G_2}=diam_{G_1}+diam_{G_2}$ and $P(G_1\times G_2)=P(G_1)\times P(G_2)$.

$$\begin{aligned}PH(G_1\times G_2)=&\sum_{\{a,b\}\subset P(G_1\times G_2)}\lambda^{d_{G_1\times G_2}(a,b)}\\ =&\frac{1}{2}\sum_{a\in P(G_1\times G_2)}\sum_{\substack{b\in P(G_1\times G_2)\\ b\neq a}}\lambda^{d_{G_1\times G_2}(a,b)}\\ PH(G_1\times G_2)=&\frac{1}{2}\sum_{a\in P(G_1\times G_2)}\Big[\sum_{b\in P(G_1\times G_2)}\lambda^{d_{G_1\times G_2}(a,b)}-1\Big].\end{aligned}$$

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By equation **??**,

$$PH(G_1 \times G_2) = \frac{1}{2} \sum_{1 \le i,j \le k_1} \Big[\sum_{1 \le r,s \le k_2} \lambda^{a_{ir}} \lambda^{b_{js}} - 1 \Big], \qquad \text{where, } a_{ir} = d_{G_1}(u_i, u_r) \text{ and}$$

$$b_{js} = d_{G_1}(v_j, v_s)$$

$$= \frac{1}{2} \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \sum_{r=1}^{k_1} \sum_{s=1}^{k_2} \lambda^{a_{ir}} \lambda^{b_{js}} - \frac{1}{2} \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} 1$$

$$PH(G_1 \times G_2) = \frac{1}{2} \sum_{i=1}^{k_1} \sum_{r=1}^{k_1} \Big(\lambda^{a_{ir}} \sum_{j=1}^{k_2} \sum_{s=1}^{k_2} \lambda^{b_{js}} \Big) - \frac{1}{2} k_1 k_2.$$

But, $PH(G_2) = \sum_{1 \le j < s \le k_2} \lambda^{d_{G_2}(v_j, v_s)} = \frac{1}{2} \sum_{j=1}^{k_2} \sum_{s=1}^{k_2} \lambda^{d_{G_2}(v_j, v_s)} - \frac{1}{2} k_2$

$\implies 2PH(G_2) + k_2 = \sum_{j=1}^{k_2} \sum_{s=1}^{k_2} \lambda^{d_{G_2}(v_j, v_s)}.$

Thus,

$$PH(G_1 \times G_2) = \frac{1}{2} \sum_{i=1}^{k_1} \sum_{r=1}^{k_1} \Big(\lambda^{a_{ir}} (2PH(G_2) + k_2) \Big) - \frac{1}{2} k_1 k_2$$

$$= \frac{1}{2} (2PH(G_1) + k_1)(2PH(G_2) + k_2) - \frac{1}{2} k_1 k_2$$

$$PH(G_1 \times G_2) = 2PH(G_1)PH(G_2) + k_2 PH(G_1) + k_1 PH(G_2).$$

□

Corollary 2.4. *Let G_1 and G_2 be graphs with k_1 and k_2 peripheral vertices, respectively. Then,*

$$PW(G_1 \times G_2) = k_2^2 PW(G_1) + k_1^2 PW(G_2).$$

Theorem 2.5. *Let G_1 and G_2 be (n_1, m_1) and (n_2, m_2) graphs, respectively.*

i) If $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$ then,

$$PH(G_1 + G_2) = \binom{n_1 + n_2}{2} \lambda.$$

ii) If $G_1 \ncong K_{n_1}$ or $G_2 \ncong K_{n_2}$ then,

$$PH(G_1 + G_2) = \Big[m_1 + m_2 - \binom{n_1 + n_2}{2} + \binom{l_1 + l_2}{2} + n_1 n_2) \Big] \lambda$$

$$+ \Big[\binom{n_1 + n_2}{2} - (n_1 n_2 + m_1 + m_2) \Big] \lambda^2,$$

where l_1 and l_2 are cardinalities of sets $S_1 = \{u \in V(G_1) | deg_{G_1}(u) < n_1 - 1\}$ and $S_2 = \{v \in V(G_2) | deg_{G_2}(v) < n_2 - 1\}$, respectively.

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Proof. Let $l_1 = |S_1|$ and $l_2 = |S_2|$, then $0 \leq l_1 \leq n_1$ and $0 \leq l_2 \leq n_2$. The periphery of $G_1 + G_2$ depends on l_1 and l_2.

Case 1: If $l_1 = l_2 = 0$ $\big($**or** $G_1 \cong K_{n_1}$ **and** $G_2 \cong K_{n_2}\big)$ **then** $G_1 + G_2 \cong K_{n_1+n_2}$.
By lemma **??**,

$$PH(G_1 + G_2) = PH(K_{n_1+n_2}) = \binom{n_1 + n_2}{2}\lambda.$$

Case 2: If $l_1 \neq 0$ **or** $l_2 \neq 0$ $\big($**or** $[G_1 \ncong K_{n_1}$ **or** $G_2 \ncong K_{n_2}]\big)$ then $P(G_1 + G_2) = S_1 \cup S_2$.
Short notation: $P_{G_1+G_2} = P(G_1 + G_2)$ and $d_{G_1+G_2}(u, v) = d_{G_1}(u, v) + d_{G_2}(u, v)$.

$$\begin{aligned}
PH(G_1 + G_2) =& \sum_{\{u,v\}\subset P_{G_1+G_2}} \lambda^{d_{G_1+G_2}(u,v)} \\
=& \sum_{\{u,v\}\subset S_1} \lambda^{d_{G_1+G_2}(u,v)} + \sum_{\{u,v\}\subset S_2} \lambda^{d_{G_1+G_2}(u,v)} + \sum_{(u,v)\in S_1\times S_2} \lambda^{d_{G_1+G_2}(u,v)} \\
=& \big[m_1 - \binom{n_1 - l_1}{2} - l_1(n_1 - l_1)\big]\lambda + \Big(\binom{n_1}{2} - m_1\Big)\lambda^2 \\
&+ \big[m_2 - \binom{n_2 - l_2}{2} - l_2(n_2 - l_2)\big]\lambda + \Big(\binom{n_2}{2} - m_2\Big)\lambda^2 + l_1 l_2 \lambda \\
=& \big[m_1 - \binom{n_1 - l_1}{2} - l_1 n_1 + \frac{l_1(l_1 + 1)}{2} + \frac{l_1(l_1 - 1)}{2})\big]\lambda + \Big(\binom{n_1}{2} - m_1\Big)\lambda^2 \\
&+ \big[m_2 - \binom{n_2 - l_2}{2} - l_2 n_2 + \frac{l_2(l_2 + 1)}{2}) + \frac{l_2(l_2 - 1)}{2} + l_1 l_2\big]\lambda \\
&+ \Big(\binom{n_2}{2} - m_2\Big)\lambda^2 \\
=& \big[m_1 - \binom{n_1}{2} + \binom{l_1}{2})\big]\lambda + \Big(\binom{n_1}{2} - m_1\Big)\lambda^2 \\
&+ \big[m_2 - \binom{n_2}{2} + \binom{l_2}{2} + l_1 l_2\big]\lambda + \Big(\binom{n_2}{2} - m_2\Big)\lambda^2 \\
PH(G_1 + G_2) =& \Big[m_1 + m_2 - \binom{n_1 + n_2}{2} + \binom{l_1 + l_2}{2} + n_1 n_2)\Big]\lambda \\
&+ \Big[\binom{n_1 + n_2}{2} - (n_1 n_2 + m_1 + m_2)\Big]\lambda^2
\end{aligned}$$

□

Corollary 2.6. *Let G_1 and G_2 be (n_1, m_1) and (n_2, m_2) graphs, respectively.*

i) If $diam_{G_1} = diam_{G_2} = 1$ then,

$$PW(G_1 + G_2) = \binom{n_1 + n_2}{2}\lambda.$$

ii) If $G_1 \ncong K_{n_1}$ or $G_2 \ncong K_{n_2}$ then,

$$PW(G_1+G_2)=\binom{n_1+n_2}{2}+\binom{l_1+l_2}{2}-(m_1+m_2+n_1n_2)$$

Theorem 2.7. *Let G_1 and G_2 be (n_1,m_1) and (n_2,m_2) graphs, respectively, and k_1 the number of peripheral vertices of G_1, l_2 the cardinality of the set $S_2=\{v\in V(G_2)|deg_{G_2}(v)<n_2-1\}$.*

i) If $diam_{G_1}=diam_{G_2}=1$ then,

$$PH(G_1[G_2])=\binom{n_1n_2}{2}$$

ii) If $diam_{G_1}=1$ and $diam_{G_2}>1$ then,

$$PH(G_1[G_2])=n_1\Big[(m_2-\binom{n_2}{2}+\binom{l_2}{2})\lambda+(\binom{n_2}{2}-m_2)\lambda^2\Big]+l_2^2\binom{n_1}{2}\lambda$$

iii) If $diam_{G_1}=2$ and $diam_{G_2}>1$ then,

$$\begin{aligned}PH(G_1[G_2])=&n_2^2PH(G_1)+\Big[n_1m_2-(n_1-k_1)\Big(\binom{n_2}{2}-\binom{l_2}{2}-k_1l_2n_2\Big)+l_2^2\binom{n_1-k_1}{2}\Big]\lambda\\&+n_1\Big[\binom{n_2}{2}-m_2\Big]\lambda^2\end{aligned}$$

iv) If $diam_{G_1}>2$ or $\big[diam_{G_1}=2$ and $diam_{G_2}>1\big]$ then,

$$PH(G_1[G_2])=k_1\Big[m_2\lambda+\Big(\binom{n_2}{2}-m_2\Big)\lambda^2\Big]+n_2^2PH(G_1)$$

Proof. Let the vertex sets of graphs G_1 and G_2 are labeled as,

$$V(G_1)=\{u_1,u_2,\ldots,u_{k_1},u_{k_1+1},\ldots,u_{n_1}\}\text{ and }V(G_2)=\{v_1,v_2,\ldots,v_{k_2},v_{k_2+1},\ldots,v_{n_2}\},$$

respectively, where $\{u_1,u_2,\ldots,u_{k_1}\}=P(G_1)$ and $\{v_1,v_2,\ldots,v_{k_2}\}=P(G_2)$.

By equation **??**, the periphery of $G_1[G_2]$ depends on the diameter of G_1, d_{G_1}.

Case 1: If $diam_{G_1}=diam_{G_2}=1$ **then** $G_1[G_2]=K_{n_1n_2}$.

$$\begin{aligned}PH(G_1[G_2])=&PH(K_{n_1n_2})=H(K_{n_1n_2})\\=&\binom{n_1n_2}{2},\qquad\text{by lemma ??}.\end{aligned}$$

Case 2: If $diam_{G_1}=1$ **and** $diam_{G_2}>1$ **then,**

$$P(G_1[G_2])=V(G_1)\times S_2\text{ and }d_{G_1[G_2]}=2$$

$$
\begin{aligned}
PH(G_1[G_2]) &= \sum_{\{a,b\}\subset P_{G_1[G_2]}} \lambda^{d_{G_1[G_2]}(a,b)} \\
&= \frac{1}{2}\sum_{a\in P_{G_1[G_2]}}\sum_{b\in P_{G_1[G_2]}} \lambda^{d_{G_1[G_2]}(a,b)} - \frac{1}{2}\sum_{a\in P_{G_1[G_2]}} \lambda^{d_{G_1[G_2]}(a,a)} \\
&= \frac{1}{2}\sum_{i=1}^{n_1}\sum_{j=1}^{l_2}\sum_{r=1}^{n_1}\sum_{s=1}^{l_2} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} - \frac{1}{2}n_1 l_2, \\
&\qquad\qquad \text{where } a_{ij}=(u_i,v_j) \text{ and } b_{rs}=(u_r,v_s)
\end{aligned}
$$

$$
\begin{aligned}
PH(G_1[G_2]) =& \frac{1}{2}\sum_{i=1}^{n_1}\sum_{j=1}^{l_2}\Big[\sum_{b_{rs}\in X_{ij}^{(1)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} + \sum_{b_{rs}\in X_{ij}^{(2)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} \\
&+ \sum_{b_{rs}\in X_{ij}^{(3)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} + \sum_{b_{rs}\in X_{ij}^{(4)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})}\Big] - \frac{1}{2}n_1 l_2
\end{aligned}
$$

$$
\begin{aligned}
\text{where, } X_{ij}^{(1)} &= \{(u_r,u_s)\in P_{G_1[G_2]} | u_r=u_i \text{ and } v_jv_s\in E(G_2)\}, \\
X_{ij}^{(2)} &= \{(u_r,u_s)\in P_{G_1[G_2]} | u_r=u_i \text{ and } v_jv_s\notin E(G_2)\}, \\
X_{ij}^{(3)} &= \{(u_r,u_s)\in P_{G_1[G_2]} | u_r=u_i \text{ and } v_j=v_s\} \text{ and} \\
X_{ij}^{(4)} &= \{(u_r,u_s)\in P_{G_1[G_2]} | u_r\neq u_i\}.
\end{aligned}
$$

$$
\begin{aligned}
PH(G_1[G_2]) =& \frac{1}{2}\sum_{i=1}^{n_1}\sum_{j=1}^{l_2}\Big[[deg_{G_2}(v_j)-(n_2-l_2)]\lambda + [n_2-1-deg_{G_2}(v_j)]\lambda^2+1\Big] \\
&+ \frac{1}{2}\sum_{i=1}^{n_1}\sum_{j=1}^{l_2}\sum_{\substack{r=1\\ r\neq i}}^{n_1}\sum_{s=1}^{l_2} \lambda^{d_{G_1}(u_i,u_r)} - \frac{1}{2}n_1 l_2 \\
=& \frac{1}{2}\sum_{i=1}^{n_1}\Big[\big(2m_2-(n_2-l_2)(n_2-1)-l_2(n_2-l_2)\big)\lambda \\
&+ \big(l_2(n_2-1)-2m_2+(n_2-l_2)(n_2-1)\big)\lambda^2+l_2\Big] + \frac{l_2^2}{2}\sum_{i=1}^{n_1}\sum_{\substack{r=1\\ r\neq i}}^{n_1}\lambda - \frac{1}{2}n_1 l_2 \\
PH(G_1[G_2]) =& n_1\Big[\big(m_2-\binom{n_2}{2}+\binom{l_2}{2}\big)\lambda + \big(\binom{n_2}{2}-m_2\big)\lambda^2\Big] + l_2^2\binom{n_1}{2}\lambda
\end{aligned}
$$

Case 3: If $diam_{G_1}=2$ **and** $G_2\ncong K_{n_2}$ then,

$P(G_1[G_2]) = C_1\cup C_2$ and $d_{G_1[G_2]} = d_{G_1} = 2,$ where, $C_1 = P(G_1)\times V(G_2)$ and

$C_2 = \{(u_i,v_j)\in V(G_1)\times V(G_2) | u_i\notin P(G_1) \text{ and } deg_{G_2}(v_j) < n_2-1\}$

$$PH(G_1[G_2]) = \sum_{\{a,b\}\subset P(G_1[G_2])} \lambda^{d_{G_1[G_2]}(a,b)}$$
$$= \sum_{\{a,b\}\subset C_1} \lambda^{d_{G_1[G_2]}(a,b)} + \sum_{\{a,b\}\subset C_2} \lambda^{d_{G_1[G_2]}(a,b)} + \sum_{\{a,b\}\subset C_1\times C_2} \lambda^{d_{G_1[G_2]}(a,b)}$$

$$\sum_{\{a,b\}\subset C_1} \lambda^{d_{G_1[G_2]}(a,b)} = k_1\Big[m_2\lambda + \Big(\binom{n_2}{2} - m_2\Big)\lambda^2\Big] + n_2^2 PH(G_1)$$

$$\sum_{\{a,b\}\subset C_2} \lambda^{d_{G_1[G_2]}(a,b)} = \frac{1}{2}\sum_{a\in C_2}\sum_{b\in C_2} \lambda^{d_{G_1[G_2]}(a,b)} - \frac{1}{2}\sum_{a\in C_2} \lambda^{d_{G_1[G_2]}(a,a)}$$
$$= \frac{1}{2}\sum_{a_{ij}\in C_2}\sum_{b_{rs}\in C_2} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} - \frac{1}{2}\sum_{a\in C_2} 1,$$

where, $a_{ij} = (u_i, v_j)$ and $b_{rs} = (u_r, v_s)$

$$\sum_{\{a,b\}\subset C_2} \lambda^{d_{G_1[G_2]}(a,b)} = \frac{1}{2}\sum_{a_{ij}\in C_2}\Big[\sum_{b_{rs}\in X_{ij}^{(1)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} + \sum_{b_{rs}\in X_{ij}^{(2)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})}$$
$$+ \sum_{b_{rs}\in X_{ij}^{(3)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})} + \sum_{b_{rs}\in X_{ij}^{(4)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})}\Big] - \frac{1}{2}(n_1 - k_1)l_2,$$

where, $X_{ij}^{(1)} = \{(u_r, u_s) \in C_2 | u_r = u_i \text{ and } v_jv_s \in E(G_2)\}$,

$X_{ij}^{(2)} = \{(u_r, u_s) \in C_2 | u_r = u_i \text{ and } v_jv_s \notin E(G_2)\}$,

$X_{ij}^{(3)} = \{(u_r, u_s) \in C_3 | u_r = u_i \text{ and } v_j = v_s\}$ and

$X_{ij}^{(4)} = \{(u_r, u_s) \in C_4 | u_r \neq u_i\}$.

$$\sum_{\{a,b\}\subset C_2} \lambda^{d_{G_1[G_2]}(a,b)} = \frac{1}{2}\sum_{a_{ij}\in C_2}\Big[\sum_{b_{rs}\in X_{ij}^{(1)}} \lambda + \sum_{b_{rs}\in X_{ij}^{(2)}} \lambda^2 + 1 + \sum_{b_{rs}\in X_{ij}^{(4)}} \lambda^{d_{G_1[G_2]}(a_{ij},b_{rs})}\Big]$$
$$- \frac{1}{2}(n_1 - k_1)l_2$$
$$= \frac{1}{2}\sum_{i=k_1+1}^{n_1}\sum_{j=1}^{l_2}\Big[[deg_{G_2}(v_j) - (n_2 - l_2)]\lambda + [n_2 - 1 - deg_{G_2}]\lambda^2 + 1\Big]$$
$$+ \frac{l_2^2}{2}\sum_{i=k_1+1}^{n_1}\sum_{\substack{r=k_1+1\\ r\neq i}}^{n_1} \lambda^{d_{G_1}(a_{ij},b_{rs})} - \frac{1}{2}(n_1 - k_1)l_2$$
$$= \frac{1}{2}\sum_{i=k_1+1}^{n_1}\Big[[2m - (n_2 - l_2)(n_2 - 1) - l_2(n_2 - l_2)]\lambda + [l_2(n_2 - 1) - 2m_2$$
$$+ (n_2 - l_2)(n_2 - 1)]\lambda^2 + l_2\Big] + \frac{l_2^2}{2}\sum_{i=k_1+1}^{n_1}\sum_{\substack{r=k_1+1\\ r\neq i}}^{n_1} \lambda - \frac{1}{2}(n_1 - k_1)l_2$$

$$\sum_{\{a,b\}\subset C_2} \lambda^{d_{G_1[G_2]}(a,b)} = \Big[(n_1-k_1)\Big(m_2-\binom{n_2}{2}+\binom{l_2}{2}\Big)+l_2^2\binom{n_1-k_1}{2}\Big]\lambda + (n_1-k_1)\Big[\binom{n_2}{2}-m_2\Big]\lambda^2$$

$$\begin{aligned}\sum_{\{a,b\}\subset C_1\times C_2} \lambda^{d_{G_1[G_2]}(a,b)} &= \sum_{a\in C_1}\sum_{b\in C_2}\lambda^{d_{G_1[G_2]}(a,b)} = \sum_{(u_i,v_j)\in C_1}\sum_{(u_r,u_s)\in C_2}\lambda^{d_{G_1[G_2]}((u_i,v_j),(u_r,u_s))}\\ &= \sum_{i=1}^{k_1}\sum_{j=1}^{n_2}\sum_{r=k_1+1}^{n_1}\sum_{s=1}^{l_2}\lambda^{d_{G_1[G_2]}(u_i,u_r)} = l_2 n_2\sum_{i=1}^{k_1}\sum_{r=k_1+1}^{n_1}\lambda^{d_{G_1[G_2]}(u_i,u_r)}\\ &= l_2n_2\sum_{i=1}^{k_1}\sum_{r=k_1+1}^{n_1}\lambda, \qquad \because u_r\notin P(G_1) \text{ and } diam_{G_1}=2\\ \sum_{\{a,b\}\subset C_1\times C_2} \lambda^{d_{G_1[G_2]}(a,b)} &= l_2n_2k_1(n_1-k_1)\lambda\end{aligned}$$

$$\begin{aligned}PH(G_1[G_2]) =& k_1\Big[m_2\lambda+\Big(\binom{n_2}{2}-m_2\Big)\lambda^2\Big]+n_2^2PH(G_1)+\Big[(n_1-k_1)\Big(m_2-\binom{n_2}{2}\\ &+\binom{l_2}{2}\Big)+l_2^2\binom{n_1-k_1}{2}\Big]\lambda+(n_1-k_1)\Big[\binom{n_2}{2}-m_2\Big]\lambda^2\\ &+(n_1-k_1)k_1l_2n_2\lambda\\ PH(G_1[G_2]) =& n_2^2PH(G_1)+\Big[n_1m_2-(n_1-k_1)\Big(\binom{n_2}{2}-\binom{l_2}{2}-k_1l_2n_2\Big)\\ &+l_2^2\binom{n_1-k_1}{2}\Big]\lambda+n_1\Big[\binom{n_2}{2}-m_2\Big]\lambda^2\end{aligned}$$

Case 4: If $diam_{G_1}>2$ or $\big[diam_{G_1}=2$ and $diam_{G_2}>1\big]$ then,

By equation **??**, $P(G_1[G_2])=P(G_1)\times V(G_2)$ and $diam_{G_1[G_2]}=diam_{G_1}$.

$$\begin{aligned}PH(G_1[G_2]) =& \frac{1}{2}\sum_{a\in P_{G_1[G_2]}}\sum_{\substack{b\in P_{G_1[G_2]}\\ b\neq a}}\lambda^{d_{G_1[G_2]}(a,b)}\\ =& \frac{1}{2}\sum_{(u_i,v_j)\in P_{G_1[G_2]}}\Big[\sum_{(u_r,u_s)\in P_{G_1[G_2]}}\lambda^{d_{G_1[G_2]}((u_i,v_j),(u_r,v_s))}-1\Big]\\ PH(G_1[G_2]) =& \frac{1}{2}\sum_{(u_i,v_j)\in P_{G_1[G_2]}}\Big[\sum_{X_{ij}^{(1)}}\lambda^{d_{G_1[G_2]}((u_i,v_j),(u_r,v_s))}+\sum_{X_{ij}^{(2)}}\lambda^{d_{G_1[G_2]}((u_i,v_j),(u_r,v_s))}\\ &+\sum_{X_{ij}^{(3)}}\lambda^{d_{G_1[G_2]}((u_i,v_j),(u_r,v_s))}+\sum_{X_{ij}^{(4)}}\lambda^{d_{G_1[G_2]}((u_i,v_j),(u_r,v_s))}\Big]-\frac{1}{2}\sum_{(u_i,v_j)\in P_{G_1[G_2]}}1,\end{aligned}$$

$$\text{where } X_{ij}^{(1)} = \{(u_r,u_s)\in P_{G_1[G_2]} | u_r = u_i \text{ and } v_jv_s \in E(G_2)\}$$
$$X_{ij}^{(2)} = \{(u_r,u_s)\in P_{G_1[G_2]} | u_r = u_i \text{ and } v_jv_s \notin E(G_2)\}$$
$$X_{ij}^{(3)} = \{(u_r,u_s)\in P_{G_1[G_2]} | u_r = u_i \text{ and } v_j = v_s\}$$
$$X_{ij}^{(4)} = \{(u_r,u_s)\in P_{G_1[G_2]} | u_r \neq u_i\}.$$

$$\begin{aligned}
PH(G_1[G_2]) =& \frac{1}{2}\sum_{(u_i,v_j)\in P_{G_1[G_2]}}\Big[\sum_{X_{ij}^{(1)}}\lambda + \sum_{X_{ij}^{(2)}}\lambda^2 + 1 + \sum_{X_{ij}^{(3)}}\lambda^{d_{G_1}(u_i,u_r)}\Big] - \frac{1}{2}k_1n_2\\
=& \frac{1}{2}\sum_{i=1}^{k_1}\sum_{j=1}^{n_2}\Big[deg_{G_2}(v_j)\lambda + \Big(n_2 - 1 - deg_{G_2}(v_j)\Big)\lambda^2 + 1 + \sum_{\substack{r=1\\ r\neq i}}^{k_1}\sum_{s=1}^{n_2}\lambda^{d_{G_1}(u_i,u_r)}\Big]\\
&- \frac{1}{2}k_1n_2\\
=& \frac{1}{2}k_1\Big[2m_2\lambda + \Big(n_2(n_2-1) - 2m_2\Big)\lambda^2 + 1\Big] + \frac{1}{2}n_2^2\sum_{i=1}^{k_1}\sum_{\substack{r=1\\ r\neq i}}^{k_1}\lambda^{d_{G_1}(u_i,u_r)} - \frac{1}{2}k_1n_2\\
PH(G_1[G_2]) =& k_1\Big[m_2\lambda + \Big(\binom{n_2}{2} - m_2\Big)\lambda^2\Big] + n_2^2PH(G_1)
\end{aligned}$$

□

Theorem 2.8. *Let G_1 be a non-trivial graph with k_1 peripheral vertices and G_2 be graph of order n_2 and size m_2. Then,*

$$PH(G_1\circ G_2) = n_2^2PH(G_1)\lambda^2 + k_1\Big[m_2\lambda + \Big(\binom{n_2}{2} - m_2\Big)\lambda^2\Big]$$

Proof. Let $P(G_1) = \{u_1,u_2,\ldots,u_{k_1}\}$. Since G_1 is non-trivial graph,

$$P(G_1\circ G_2) = \bigcup_{i=1}^{k_1} V(G_2^{(i)}) \text{ and } diam_{G_1\circ G_2} = diam_{G_1} + 2.$$

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$$PH(G_1 \circ G_2) = \frac{1}{2} \sum_{a \in P_{G_1 \circ G_2}} \sum_{\substack{b \in P_{G_1 \circ G_2} \\ b \neq a}} \lambda^{d_{G_1 \circ G_2}(a,b)} = \frac{1}{2} \sum_{a \in P_{G_1 \circ G_2}} \sum_{b \in P_{G_1 \circ G_2}} \lambda^{d_{G_1 \circ G_2}(a,b)}$$
$$- \frac{1}{2} \sum_{a \in P_{G_1 \circ G_2}} \lambda^{d_{G_1 \circ G_2}(a,a)}$$
$$= \frac{1}{2} \sum_{i=1}^{k_1} \sum_{a \in V(G_2^{(i)})} \sum_{j=1}^{k_1} \sum_{b \in V(G_2^{(j)})} \lambda^{d_{G_1 \circ G_2}(a,b)} - \frac{1}{2} k_1 n_2$$
$$= \frac{1}{2} \sum_{i=1}^{k_1} \sum_{a \in V(G_2^{(i)})} \Big[\sum_{\substack{j=1 \\ j \neq i}}^{k_1} \sum_{b \in V(G_2^{(j)})} \lambda^{d_{G_1 \circ G_2}(a,b)} + \sum_{b \in V(G_2^{(i)})} \lambda^{d_{G_1 \circ G_2}(a,b)} \Big] - \frac{1}{2} k_1 n_2$$

$$= \frac{1}{2} \sum_{i=1}^{k_1} \sum_{\substack{j=1 \\ j \neq i}}^{k_1} \sum_{a \in V(G_2^{(i)})} \sum_{b \in V(G_2^{(j)})} \lambda^{d_{G_1 \circ G_2}(a,b)} + \frac{1}{2} \sum_{i=1}^{k_1} \sum_{a \in V(G_2^{(i)})} \sum_{b \in V(G_2^{(i)})} \lambda^{d_{G_1 \circ G_2}(a,b)}$$
$$- \frac{1}{2} k_1 n_2$$
$$= \frac{1}{2} \sum_{i=1}^{k_1} \sum_{\substack{j=1 \\ j \neq i}}^{k_1} \sum_{a \in V(G_2^{(i)})} \sum_{b \in V(G_2^{(j)})} \lambda^{d_{G_1}(u_i,u_j)+2} + \sum_{i=1}^{k_1} \Big[\sum_{ab \in E(G_2^{(i)})} \lambda^{d_{G_1 \circ G_2}(a,b)}$$
$$+ \sum_{ab \notin E(G_2^{(i)})} \lambda^{d_{G_1 \circ G_2}(a,b)} \Big] + \frac{1}{2} \sum_{i=1}^{k_1} \sum_{a \in V(G_2^{(i)})} \lambda^{d_{G_1 \circ G_2}(a,a)} - \frac{1}{2} k_1 n_2$$
$$= \frac{1}{2} \sum_{i=1}^{k_1} \sum_{\substack{j=1 \\ j \neq i}}^{k_1} \big(n_2^2 \lambda^{d_{G_1}(u_i,u_j)} \lambda^2 \big) + \sum_{i=1}^{k_1} \Big[m_2 \lambda + \Big(\binom{n_2}{2} - m_2 \Big) \lambda^2 \Big]$$
$$PH(G_1 \circ G_2) = n_2^2 PH(G_1) \lambda^2 + k_1 \Big[m_2 \lambda + \Big(\binom{n_2}{2} - m_2 \Big) \lambda^2 \Big]$$

□

Corollary 2.9. *Let G_1 be a non-trivial graph with k_1 peripheral vertices and G_2 be graph of order n_2 and size m_2. Then,*

$$PW(G_1 \circ G_2) = n_2^2 \big(PW(G_1) + k_1^2 \big) - k_1 (n_2 + m_2)$$

Theorem 2.10. *Let G_1 be a graph with k_1 number of peripheral vertices and G_2 be a rooted graph with root v. Then,*

$$PH(G_1\{G_2\}) = k_1 PH_0(v, G_2) + |E(v)|^2 PH(G_1) \lambda^{2e(v)}$$

where $PH_0(v, G_2) = \sum_{\{a,b\} \subset E(v)} \lambda^{d_{G_2}(a,b)}$.

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Proof. Let $P(G_1) = \{u_1, u_2, \ldots, u_{k_1}\}$ and v be the root of the rooted graph G_2. For $a, b \in V(G_1\{G_2\})$,

$$d_{G_1\{G_2\}}(a,b) = \begin{cases} d_{G_2}(a,b), & a, b \in G_2^{(i)} \\ d_{G_2}(a,v) + d_{G_1}(a,b) + d_{G_2}(v,b), & a \in G_2^{(i)} \text{ and } b \in G_2^{(j)}, i \neq j \end{cases}$$

where $G_2^{(i)}$ is a subgraph of $G_1\{G_2\}$ corresponding to the i^{th} copy of G_2.

Then,

$$P(G_1\{G_2\}) = \bigcup_{i=1}^{k_1} E^{(i)}(v) \text{ and } diam_{G_1\{G_2\}} = diam_{G_1} + 2e(v),$$

where $E^{(i)}(v)$ is a subset of $V(G_1\{G_2\})$ corresponding to $E(v)$ in the i^{th} copy of G_2.

$$\begin{aligned} PH(G_1\{G_2\}) &= \sum_{\{a,b\} \subset P_{G_1\{G_2\}}} \lambda^{d_{G_1\{G_2\}}(a,b)} \\ &= \sum_{\{a,b\} \in X_1} \lambda^{d_{G_1\{G_2\}}(a,b)} + \sum_{\{a,b\} \in X_2} \lambda^{d_{G_1\{G_2\}}(a,b)} \\ \text{where } X_1 &= \{\{a,b\} \subset P_{G_1\{G_2\}} | a, b \in E^{(i)}(v), 1 \leq i \leq k_1\} \text{ and} \\ X_2 &= \{\{a,b\} \subset P_{G_1\{G_2\}} | (a,b) \in E^{(i)}(v) \times E^{(j)}(v), 1 \leq i, j \leq k_1, i \neq j\} \\ PH(G_1\{G_2\}) &= \sum_{i=1}^{k_1} \sum_{\{a,b\} \subset E^{(i)}(v)} \lambda^{d_{G_1\{G_2\}}(a,b)} + \sum_{i=1}^{k_1} \sum_{\substack{j=1 \\ j \neq i}}^{k_1} \sum_{(a,b) \in E^{(i)}(v) \times E^{(j)}(v)} \lambda^{d_{G_1\{G_2\}}(a,b)} \\ &= \sum_{i=1}^{k_1} \sum_{\{a,b\} \subset E^{(i)}(v)} \lambda^{d_{G_2}(a,b)} + \sum_{i=1}^{k_1} \sum_{\substack{j=1 \\ j \neq i}}^{k_1} \sum_{(a,b) \in E^{(i)}(v) \times E^{(j)}(v)} \lambda^{d_{G_1}(u_i,u_j) + 2e(v)} \\ &= \sum_{i=1}^{k_1} PH_0(v, G_2) + \sum_{i=1}^{k_1} \sum_{\substack{j=1 \\ j \neq i}}^{k_1} |E(v)|^2 \lambda^{d_{G_1}(u_i,u_j) + 2e(v)} \\ &\qquad \text{where } PH_0(v, G_2) = \sum_{\{a,b\} \subset E(v)} \lambda^{d_{G_2}(a,b)} \\ PH(G_1\{G_2\}) &= k_1 PH_0(v, G_2) + |E(v)|^2 PH(G_1) \lambda^{2e(v)}. \end{aligned}$$

□

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