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THE IDENTIFYING CODE NUMBER AND MYCIELSKI'S CONSTRUCTION OF GRAPHS

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Abstract. Let $G = (V, E)$ be a simple graph. A set C of vertices G is an identifying code of G if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. Given a graph G, the smallest size of an identifying code of G is called the identifying code number of G and denoted by $\gamma^{ID}(G)$. Two vertices x and y are twins when $N_G[x] = N_G[y]$. Graphs with at least two twin vertices are not an identifiable graph. In this paper, we deal with the identifying code number of Mycielski's construction of graph G. We prove that the Mycielski's construction of every graph G of order $n \geq 2$, is an identifiable graph. Also, we present two upper bounds for the identifying code number of Mycielski's construction G, such that these two bounds are sharp. Finally, we show that Foucaud et al.'s conjecture is holding for Mycielski's construction of some graphs.

1. Introduction

All graphs $G = (V, E)$ throughout this paper considered simple, finite, and undirected. If two vertices x and y are *adjacent*, then it is denoted by $x \sim y$, otherwise, $x \nsim y$. For every vertex $x \in V(G)$, the *open neighborhood* of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) |\ x \sim y\}$. Also, the *close neighborhood* of vertex $x \in V(G)$, $N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. The *degree* of a vertex $x \in V(G)$ is $\deg_G(x) =\mid N_G(x) \mid$. The *maximum degree* and *minimum degree* of graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex is called universal if it is adjacent to all of the vertices of the graph.

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $G_1 \bowtie G_2$ is the join graph of G_1 and G_2. Its vertex set is $V_1 \cup V_2$ and its edge set is $E(G_1) \cup E(G_2) \cup \{x_1x_2 \mid x_1 \in V_1, x_2 \in V_2\}$.

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For graph $G=(V,E)$, let $V^{'}(G)=\{v_1^{'},v_2^{'},\dots,v_n^{'}\}$ be a copy of $V(G)=\{v_1,v_2,\dots,v_n\}$ and w be a new vertex. The *Mycielski's construction* of graph G, is denoted by $\mu(G)$ and it is a graph with vertex set $V(\mu(G))=V(G)\cup V^{'}(G)\cup\{w\}$ and edge set $E(\mu(G))=E(G)\cup\{v_i\sim v_j^{'}\mid v_i\sim v_j \textit{ and } 1\leq i,j\leq n\}\cup\{w\sim v_i^{'}\mid 1\leq i\leq n\}$. The Mycielski's construction of graph G, was introduced by J. Mycielski for the purpose to construct triangle-free graphs with an arbitrary large chromatic number [13]. In recent years, there have been results reported on Mycielski's graph related to several domination parameters [7, 11, 12].

A set of vertices G such as C is a *dominating set* of graph G if for every vertex x of $V(G)$, is either in C or is adjacent to a vertex in C. It is clear that every isolated vertex is in every dominating set of G. Also, a set C is called a *separating set* of G if for each pair $u,\ v$ of vertices of G, $N_G[u]\cap C\neq N_G[v]\cap C$ (equivalently, $(N_G[u]\bigtriangleup N_G[v])\cap C\neq\emptyset$). If a dominating set C in graph G is a separating set of G then we said that C is an identifying code of graph G and if G has an identifying code, then we say that G is an *identifiable graph*. Given a graph G, the smallest size of an identifying code of G is called the *identifying code number* of G and denoted by $\gamma^{ID}(G)$. If for two distinct vertices x and y, $N_G[x]=N_G[y]$, then they are called *twins*. It is noteworthy that a graph G is identifiable if and only if G is twin free.

The identifying code concept was introduced by Karpovsky et al. [10] in 1998. Later, several various families of graphs have been studied; cycles and paths [2, 8], trees [1], triangular and square grids, [9] and triangle-free graphs [5]. Also, identifying codes have found applications in various fields. For example, sensor network monitoring [3], identifying codes in random networks [6] and Communication Networks[14]. Identifying codes are used to model fault diagnosis in multiprocessor systems. In these systems, it may happen that some of the processors become faulty, in a way that depends on the purpose of the system. Then such processors are detected and replaced, so that the system can work properly.

This paper deals with the study of Mycielski's construction of some graphs. In section 2, we show that the Mycielski's construction of a graph of order $n\geq 2$ is an identifiable graph and we give two sharp bounds for $\gamma^{ID}(\mu(G))$. The identifying code number of the Mycielski's construction of some graphs such as complete bipartite and $(n-2)$-regular graph of order n is considered. In section 3, we discuss the identifying code number of the Mycielski's construction of some graphs, which are not identifiable.

2. $\gamma^{ID}(\mu(G))$, where G is an identifiable graph

In this section, we prove that the Mycielski's construction of every graph is identifiable. Also, we determine an upper bound for the identifying code number of Mycielski's construction if G is an identifiable graph of order $n\geq 2$ and another bound for it if $\delta(G)\geq 2$. In the following, we show that these two bounds can be achieved. At the of this section, we obtain the identifying code number of Mycielski's construction of a complete bipartite graph and $(n-2)$-regular graph of order n.

Lemma 2.1. *Let G be a graph of order n. Then $\mu(G)$ is an identifiable graph if and only if $n\geq 2$.*

Proof. Let $\mu(G)$ is not an identifiable graph. Then there are two vertices x and y in $V(\mu(G))$, such that $N_{\mu(G)}[x]=N_{\mu(G)}[y]$. It is clear that $\{x,y\}\not\subseteq V(G)$ and $\{x,y\}\not\subseteq V^{'}(G)$. So, $x\in V^{'}(G)$ and $y=w$.

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Since $N_{\mu(G)}(w) = V'(G)$ and induced subgraph on $V'(G)$ is empty. Hence, we have $n = 1$. Conversely, if $n = 1$, then $\mu(G) \cong P_2 \cup K_1$ and so $\mu(G)$ is not an identifiable graph. □

Observation 2.2. *Let G be a graph of order n. Then $\gamma^{ID}(G) = 1$ if and only if $n = 1$.*

Proof. By the formula $\gamma^{ID}(G) \geq \lceil log_2^{n+1} \rceil$ ([10]), the proof is straightforward. □

Lemma 2.3. *Let G be a graph and C be an identifying code of G. If $N_G[x] \bigtriangleup N_G[y] = \{y_1, y_2\}$, then $y_1 \in C$ or $y_2 \in C$.*

Proof. Let $\{y_1, y_2\} \cap C = \emptyset$. Then $N_G[x] \cap C = N_G[y] \cap C$, which is not true. □

Theorem 2.4. *Let G be a graph of order n such that for every two distinct vertices u and v of G, $N_G(u) \neq N_G(v)$. If $\delta(G) \geq 2$, then $\gamma^{ID}(\mu(G)) \leq n$.*
Furthermore, this bound is sharp.

Proof. By Lemma 2.1, $\mu(G)$ is an identifiable graph. Let $C = V'(G)$. Then for each $y \in V'(G)$, $N_{\mu(G)}[y] \cap C = \{y\}$ and $N_{\mu(G)}[w] \cap C = C$. If x and y are two vertices in $V(G)$, such that $N_{\mu(G)}[x] \cap C = N_{\mu(G)}[y] \cap C$, then $N_G(x) = N_G(y)$. That is not true. If x and y are two vertices in $V'(G)$, then $N_{\mu(G)}[x] \cap C \neq N_{\mu(G)}[y] \cap C$. If $x \in V(G)$ and $y \in V'(G)$, such that $N_G[x] \cap C = N_G[y] \cap C$, then $N_{\mu(G)}[x] \cap C = \{y\}$. So, $\deg_G(x) = 1$. Which is a conradiction. It is clear that $N_{\mu(G)}[w] \cap C \neq N_{\mu(G)}[x] \cap C$, for every $x \in V(G) \cup V'(G)$. Hence, C is an identifying code of $\mu(G)$ and so $\gamma^{ID}(\mu(G)) \leq |C| = n$.

Now let $n \geq 3$ and C be an identifying code of $\mu(K_n)$ with minimum cardinality. Also, let $|C| \leq n-1$. Since C is a dominating set of $\mu(K_n)$, $C \nsubseteq V'(K_n)$. Hence, there are two vertices x' and y' of $V'(G)$ which are not in C. So, we have $N_{\mu(K_n)}[x] \cap C = N_{\mu(K_n)}[y] \cap C$, where x and y are the correspondings vertices of x' and y', respectively. That is a contradiction. So $|C| > n - 1$. Therefore, $\gamma^{ID}(\mu(K_n)) = n$. This shows that this bound is sharp. □

Theorem 2.5. *Let G be an identifiable graph of order $n \geq 2$. Then $\gamma^{ID}(\mu(G)) \leq 2\gamma^{ID}(G)$.*

Proof. By Lemma 2.1, $\mu(G)$ is an identifiable graph. Let C_1 be an identifying code of G and $C = C_1 \cup C_1'$, where $C_1' \subseteq V'(G)$ is the corresponding to C_1. Then for each pair x, y of vertices in $V(\mu(G))$, we have the following cases.

Case 1: Let x and y are two vertices in $V(G)$. Then $N_G[x] \cap C_1 \neq N_G[y] \cap C_1$ and so $N_{\mu(G)}[x] \cap C \neq N_{\mu(G)}[y] \cap C$.

Case 2: Let x and y be two vertices in $V'(G)$. For simplicity, let $x = a'$ and $y = b'$. If $\{a', b'\} \cap C = \{a', b'\}$ or $\{a', b'\} \cap C = \{a'\}$, then $a' \in N_{\mu(G)}[a'] \cap C$ and $a' \notin N_{\mu(G)}[b'] \cap C$. If $\{a', b'\} \cap C = \emptyset$, then since $N_G[a] \cap C_1 \neq N_G[b] \cap C_1$, then there exists $z \in C_1$, such that $z \sim a$ and $z \nsim b$. So $z \in N_{\mu(G)}[a'] \cap C$ and $z \notin N_{\mu(G)}[b'] \cap C$.

Case 3: Suppose that $x \in V(G)$ and $y \in V'(G)$. If $b \in V(G)$, $b \neq x$ and $y = b'$, then since C_1 is an identifying code of G, so $N_G[x] \cap C_1 \neq N_G[b] \cap C_1$. Hence, there exists $z \in C_1$, such that $z \sim x$ and $z \nsim b$. Thus $z \in N_{\mu(G)}[x] \cap C$ and $z \notin N_{\mu(G)}[b'] \cap C$. If $y = x'$ and $z \in N_G[x] \cap C_1$, then $z' \in N_{\mu(G)}[x] \cap C$ and $z' \notin N_{\mu(G)}[y] \cap C$.

Case 4: Assume that $x \in V(G)$ and $y = w$. Then since $N_G[x] \cap C_1 \neq \emptyset$, there exists $z \in N_G[x] \cap C$. So, $z \in N_{\mu(G)}[x] \cap C$ and $z \notin N_{\mu(G)}[w] \cap C$.
Case 5: Let $x \in V^{'}(G)$ and $y = w$. By Observation 2.2, $|C_1| \geq 2$. So $|C_1^{'}| \geq 2$. Hence, $N_{\mu(G)}[w] \cap C = C_1^{'}$ and $N_{\mu(G)}[x] \cap C = \{x\}$. Hence, $N_{\mu(G)}[x] \cap C \neq N_{\mu(G)}[w] \cap C$.
However, $N_{\mu(G)}[x] \cap C \neq N_{\mu(G)}[y] \cap C$. Therefore, C is an identifying code of $\mu(G)$ and so $\gamma^{ID}(\mu(G)) \leq |C| = 2\gamma^{ID}(G)$. □

For an integer $k \geq 1$, let $A_k = (V_k, E_k)$ be the graph with vertex set $V_k = \{x_1, ..., x_{2k}\}$ and edge set $E_k = \{x_i \sim x_j \mid |i-j| \leq k-1\}$. Also, let $\mathscr{A}$ be the closure of $\{A_i \mid i = 1, 2, \ldots\}$ with respect to operation $\bowtie$. In the next theorem, Foucaud et al. showed that for any twin free graph $G \notin \{K_{1,n-1}\} \cup (\mathscr{A}, \bowtie) \cup (\mathscr{A}, \bowtie) \bowtie K_1$, $\gamma^{ID}(G) \leq |V(G)| - 2$.

Theorem 2.6. [4] *Let G be an identifiable graph of order n. Then $\gamma^{ID}(G) = n-1$ if and only if $G \ncong \overline{K_2}$ and $G \in \{K_{1,n-1}\} \cup (\mathscr{A}, \bowtie) \cup (\mathscr{A}, \bowtie) \bowtie K_1$.*

Lemma 2.7. *If $G \cong K_{m,n}$ and $n \geq 2$, then*

$$\gamma^{ID}(G) = \begin{cases} n, & m = 1 \\ 3, & m = n = 2 \\ m + n - 2 & o.w. \end{cases}$$

Proof. Let the bipartition of $K_{m,n}$ be X and Y with $|X| = n$ and $|Y| = m$. If $m = 1$, then by Theorem 2.6, $\gamma^{ID}(G) = n$. If $m = n = 2$, then $G \cong C_4$ and so $\gamma^{ID}(G) = 3$.

Otherwise, let C be an identifying code of $K_{m,n}$. By Lemma 2.3, $|X \cap C| \geq n-1$ and $|Y \cap C| \geq m-1$. So $|C| \geq m + n - 2$. By Theorem 2.6, $\gamma^{ID}(G) = m + n - 2$. □

Theorem 2.8. *Let $G \cong K_{m,n}$ and $n \geq 2$. Then*

$$\gamma^{ID}(\mu(G)) = \begin{cases} 2n, & m = 1 \\ 2(m+n) - 4, & m \neq 1 \end{cases}$$

Proof. Let $m = 1$. By Theorem 2.5 and Lemma 2.7, $\gamma^{ID}(\mu(G)) \leq 2n$.

Now let $X = V(G) \setminus \{a\} = \{x_1, x_2, \ldots, x_n\}$, where a is the universal vertex of G. Also, let C be a minimum identifying code of $\mu(G)$ and $|C| \leq 2n - 1$. By Lemma 2.3, $|X \cap C| \geq n - 1$ and $|X^{'} \cap C| \geq n - 1$. If $X \subseteq C$, then $|X^{'} \cap C| = n - 1$ and $\{a, a^{'}, w\} \cap C = \emptyset$. Without loss of generality, let $x_1^{'} \notin C$. Then $x_1^{'}$ is not dominated by C, that is not true. So $X \nsubseteq C$. Similarly, $X^{'} \nsubseteq C$. Hence, $|X \cap C| = |X^{'} \cap C| = n - 1$. Since C is a dominating set of $\mu(G)$, $a \in C$ or $a^{'} \in C$. If $a \in C$ and $C = \{x_k, x_k^{'}, a \mid 1 \leq k \leq n\} \setminus \{x_i, x_j^{'}\}$, then $N_{\mu(G)}[x_i] \cap C = \{a\} = N_{\mu(G)}[x_j^{'}] \cap C$, that is a contradiction. If $a^{'} \in C$ and $C = \{x_k, x_k^{'}, a^{'} \mid 1 \leq k \leq n\} \setminus \{x_i, x_j^{'}\}$, then $x_j^{'}$ is not dominated by C. That is not true. Therefore, $|C| \geq 2n$ and so $\gamma^{ID}(\mu(G)) = 2n$.

Let $m \geq 2$. By Theorem 2.5 and Lemma 2.7, $\gamma^{ID}(\mu(G)) \leq 2(m + n) - 4$.

Now, let C be a minimum identifying code of $\mu(G)$. Also, let the bipartition of $K_{m,n}$ be $X = \{x_k \mid 1 \leq k \leq n\}$ and $Y = \{y_k \mid 1 \leq k \leq m\}$. Then by Lemma 2.3, we have $|X \cap C| \geq n - 1$, $|X^{'} \cap C| \geq n - 1$, $|Y \cap C| \geq m-1$ and $|Y^{'} \cap C| \geq m-1$. Hence, $|C| \geq 2(n+m)-4$. Therefore, $\gamma^{ID}(\mu(G)) = 2(n+m)-4$. □

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Corollary 2.9. *If $n \geq 2$, then $\gamma^{ID}(\mu(K_{1,n})) = 2\gamma^{ID}(K_{1,n})$.*

Proof. By Lemma 2.7 and Theorem 2.8 the proof is straightforward. □

This show that the upper bound of Theorem 2.5 is sharp.

Theorem 2.10. *Let G be a $(n-2)$-regular graph of order n. Then $\gamma^{ID}(\mu(G)) = n$.*

Proof. By Lemma 2.1, $\mu(G)$ is an identifiable graph. It is clear that n is even. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, such that $v_i \nsim v_{i+\frac{n}{2}}$ for every $1 \leq i \leq \frac{n}{2}$. Also, let $C_1 = \{v_i | 1 \leq i \leq \frac{n}{2}\}$ and $C = C_1 \cup C_1^{'}$, where $C_1^{'} = \{v_i^{'} \in V^{'}(G) | 1 \leq i \leq \frac{n}{2}\}$. Then for every $1 \leq i \leq \frac{n}{2}$, we have $N_{\mu(G)}[v_i] \cap C = C \setminus \{v_i^{'}\}$, $N_{\mu(G)}[v_i^{'}] \cap C = C_1 \setminus \{v_i\} \cup \{v_i^{'}\}$, $N_{\mu(G)}[v_{i+\frac{n}{2}}] \cap C = C \setminus \{v_i, v_i^{'}\}$, $N_{\mu(G)}[v_{i+\frac{n}{2}}^{'}] \cap C = C_1 \setminus \{v_i\}$ and $N_{\mu(G)}[w] \cap C = C_1^{'}$. So for each pair u, v of the vertices in $V(\mu(G))$, $N_{\mu(G)}[v] \cap C \neq N_{\mu(G)}[u] \cap C$ and so C is an identifying code of $\mu(G)$. Therefore, $\gamma^{ID}(\mu(G)) \leq |C| = n$.

Now let D be an identifying code of $\mu(G)$ such that $\gamma^{ID}(\mu(G)) = |D|$. By Lemma 2.3, there is a $X \subseteq V(G)$, such that $|X \cap D| \geq \dfrac{n}{2}$. Similarly, ther exists $Y^{'} \subseteq V^{'}(G)$, such that $|Y^{'} \cap D| \geq \dfrac{n}{2}$. Hence, $|D| \geq n$. Therefore, $\gamma^{ID}(\mu(G)) = n$. □

3. $\gamma^{ID}(\mu(G))$, where G is not an identifiable graph

In this section, we discuss identifying code number of the Mycielski's construction of some graphs, which are not identifiable. We introduce two families of graphs, which are not identifiable graphs but their Mycielski's construction are identifiable graphs. By computing the identifying code number of Mycielski's construction of these families, we show that Foucaud et al.'s conjecture is true.

Theorem 3.1. *Let G_1 be a empty graph of order s and G_2 be a complete graph of order r, such that $(s, r) \notin \{(0, 2), (1, 1)\}$. Then*

$$\gamma^{ID}(\mu(G_1 \bowtie G_2)) = \begin{cases} r, & s = 0 \\ r+1, & s = 1 \\ 2s, & r = 0, 1 \\ 2s + r - 2, & s \geq 2 \end{cases}$$

Proof. Let $G = G_1 \bowtie G_2$. By Lemma 2.1, $\mu(G)$ is an identifiable graph. If $s \in \{0, 1\}$, then by the proof of Theorem 2.4, the proof is straightforward. If $r \in \{0, 1\}$, then by Theorem 2.8, $\gamma^{ID}(\mu(G)) = 2s$.

Now let $s \geq 2$, $V(G_1) = \{v_1, v_2, \ldots, v_s\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_r\}$. Also, let C be an identifying code of $\mu(G)$. By Lemma 2.3, there are $A \subseteq \{v_1, v_2, \ldots, v_s\}$ and $B^{'} \subseteq \{v_1^{'}, v_2^{'}, \ldots, v_s^{'}\}$, such that $|A \cap C| = |B^{'} \cap C| = s - 1$. By Lemma 2.3, there are $F^{'} \subseteq \{u_1^{'}, u_2^{'}, \ldots, u_r^{'}\}$, such that $|F^{'} \cap C| = r - 1$. So $|C| \geq 2s + r - 3$. If $|C| = 2s + r - 3$, then $|C \cap \{v_1, \ldots, v_s\}| = s - 1$ and $|C \cap \{v_1^{'}, \ldots, v_s^{'}\}| = s - 1$. Without loss of generality, assume that $v_1^{'} \notin C$. Then $v_1^{'}$ is not dominated by C, which is a contradiction. So $|C| \geq 2s + r - 2$.

Now let $X = \{v_i \mid 2 \leq i \leq s\} \cup V^{'}(G) \setminus \{u_1^{'}\}$. Then $N_{\mu(G)}[v_1] \cap X = \{u_k^{'} \mid 2 \leq k \leq r\}$, for $2 \leq i \leq s$, $N_{\mu(G)}[v_i] \cap X = \{u_k^{'} \mid 2 \leq k \leq r\} \cup \{v_i\}$ and for $1 \leq i \leq r$, $N_{\mu(G)}[u_i] \cap X = X \setminus \{u_i^{'}\}$.

Also, we have for $1 \leq i \leq s$, $N_{\mu(G)}[v_i^{'}] \cap X = \{v_i^{'}\}$, $N_{\mu(G)}[u_1^{'}] \cap X = \{v_k \mid 2 \leq k \leq s\}$, for $2 \leq i \leq r$, $N_{\mu(G)}[u_i^{'}] \cap X = \{v_k \mid 2 \leq k \leq s\} \cup \{u_i^{'}\}$ and $N_{\mu(G)}[w] \cap X = V^{'}(G) \setminus \{u_1^{'}\}$. Then X is an identifying code of $\mu(G)$ and so $\gamma^{ID}(\mu(G)) \leq 2s + r - 2$. Therefore, $\gamma^{ID}(\mu(G)) = 2s + r - 2$. This complete the proof. □

Theorem 3.2. *Let G be a graph of order n and a be a universal vertex of G. Also, let $G \setminus \{a\} = \bigcup_{i=1}^{s} K_1 \bigcup_{i=1}^{r} K_{n_i}$, such that $2 \leq n_1 \leq n_2 \leq \cdots \leq n_r$ (see Figure 1). Then*

$$\gamma^{ID}(\mu(G)) = \begin{cases} n-1, & s = 0,\ n_r \geq 3 \\ n, & s = 0,\ n_r = 2 \\ n+s-1, & s \geq 1 \end{cases}$$

Proof. By Lemma 2.1, $\mu(G)$ is an identifiable graph. Let $V(K_{n_i}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ and $V(G \setminus \{a\}) = V(\bigcup_{i=1}^{r} K_{n_i}) \cup \{v_j | 1 \leq j \leq s\}$. Let $s = 0$, $n_1 \geq 3$ and $X_1 = V^{'}(G) \setminus \{v_{i1}^{'}, a^{'} \mid 1 \leq i \leq r\} \cup \{v_{i2}, a \mid 1 \leq i \leq r-1\}$. Then for $j \geq 3$ and $1 \leq i \leq r-1$, $N_{\mu(G)}[v_{ij}^{'}] \cap X_1 = \{v_{ij}^{'}, v_{i2}, a\}$, $N_{\mu(G)}[v_{ij}] \cap X_1 = \{a, v_{i2}\} \cup \{v_{i\ell}^{'} \mid 2 \leq \ell \leq n_i, \ell \neq j\}$, $N_{\mu(G)}[v_{rj}^{'}] \cap X_1 = \{v_{rj}^{'}, a\}$, $N_{\mu(G)}[v_{rj}] \cap X_1 = \{a\} \cup \{v_{r\ell}^{'} \mid 2 \leq \ell \leq n_r, \ell \neq j\}$, $N_{\mu(G)}[v_{i1}^{'}] \cap X_1 = \{a, v_{i2}\}$, $N_{\mu(G)}[v_{i1}] \cap X_1 = \{a, v_{i2}\} \cup \{v_{i\ell}^{'} \mid 2 \leq \ell \leq n_i\}$ and $N_{\mu(G)}[v_{i2}] \cap X_1 = \{a, v_{i2}\} \cup \{v_{i\ell}^{'} \mid 3 \leq \ell \leq n_i\}$. Also, $N_{\mu(G)}[a^{'}] \cap X_1 = \{v_{i2} \mid 1 \leq i \leq r-1\}$, $N_{\mu(G)}[a] \cap X_1 = X_1$, $N_{\mu(G)}[w] \cap X_1 = V^{'}(G) \setminus \{v_{i1}^{'}, a^{'} \mid 1 \leq i \leq r\}$, $N_{\mu(G)}[v_{r1}^{'}] \cap X_1 = \{a\}$, $N_{\mu(G)}[v_{r1}] \cap X_1 = \{a\} \cup \{v_{r\ell}^{'} \mid 2 \leq \ell \leq n_r\}$, $N_{\mu(G)}[v_{r2}] \cap X_1 = \{a, v_{r\ell}^{'} \mid 3 \leq \ell \leq n_r\}$ and $N_{\mu(G)}[v_{r2}^{'}] \cap X_1 = \{a, v_{r2}^{'}\}$. So, X_1 is an identifying code of $\mu(G)$. Similarly, we see that if $s = 0$, $n_r \geq 3$ and $n_1 = 2$, then $X_2 = V^{'}(G) \setminus \{v_{i1}^{'}, a^{'} \mid 1 \leq i \leq r\} \cup \{v_{i2}, w \mid 1 \leq i \leq r-1\}$ is an identifying code of $\mu(G)$. Hence, $\gamma^{ID}(\mu(G)) \leq n-1$. Let $s = 0$ and $n_r = 2$. Then $X_3 = V^{'}(G) \setminus \{v_{i1}^{'}, a^{'} \mid 1 \leq i \leq r\} \cup \{v_{i2}, a, w \mid 1 \leq i \leq r-1\}$ is an identifying code of $\mu(G)$. So $\gamma^{ID}(\mu(G)) \leq n$.

Now let $s \geq 1$ and $X_4 = V^{'}(G) \setminus \{a^{'}, v_1^{'}, v_{i1}^{'} \mid 1 \leq i \leq r-1\} \cup \{v_i \mid 2 \leq i \leq s\} \cup \{v_{i2} \mid 1 \leq i \leq r-1\} \cup \{a, w\}$. Then X_4 is an identifying code of $\mu(G)$. So $\gamma^{ID}(\mu(G)) \leq n+s-1$. We have two following cases.

Case 1: Let $s = 0$ and C be an identifying code of $\mu(G)$ with $\gamma^{ID}(\mu(G)) = |C|$. For $2 \leq j \leq n_i$, we have $N_{\mu(G)}[v_{i1}] \triangle N_{\mu(G)}[v_{ij}] = \{v_{i1}^{'}, v_{ij}^{'}\}$. By Lemma 2.3, $v_{i1}^{'} \in C$ or $v_{ij}^{'} \in C$. Let $v_{i1}^{'} \notin C$, for $1 \leq i \leq r$. Then there exists $A^{'} \subseteq \bigcup_{i=1}^{r} V^{'}(K_{n_i})$, such that $|A^{'} \cap C| = \sum_{i=1}^{r}(n_i - 1)$. Since $N_{\mu(G)}(v_{11}^{'}) = \{a, w\} \cup V(K_{n_1}) \setminus \{v_{11}\}$ and $N_{\mu(G)}(v_{21}^{'}) = \{a, w\} \cup V(K_{n_2}) \setminus \{v_{21}\}$, $|(V(K_{n_1}) \setminus \{v_{11}\} \cup V(K_{n_2}) \setminus \{v_{21}\}) \cap C| \geq 1$. Without loss of generality assume that $v_{12} \in C$. Similarly, we may assume that for $2 \leq i \leq r-1$, $v_{i2} \in C$. Hence, $\gamma^{ID}(\mu(G)) = |C| \geq n-2$. Therefore, if $s = 0$ and $n_r \geq 3$, then $\gamma^{ID}(\mu(G)) \in \{n-1, n-2\}$ and if $(s, n_r) = (0, 2)$, then $\gamma^{ID}(\mu(G)) \in \{n, n-1, n-2\}$.

Subcase 1: Let $n_r \geq 3$, $\gamma^{ID}(\mu(G)) = n-2$ and C be an identifying code of $\mu(G)$ with minimum cardinality. Then $C = V^{'}(G) \setminus \{v_{i1}^{'}, a^{'} \mid 1 \leq i \leq r\} \cup \{v_{i2} \mid 1 \leq i \leq r-1\}$. We see that $v_{r1}^{'}$ is not dominated by C, which is a contradiction. Therefore, $\gamma^{ID}(\mu(G)) = n-1$.

Subcase 2: Let $n_r = 2$, $\gamma^{ID}(\mu(G)) \in \{n-1, n-2\}$ and C be an identifying code of $\mu(G)$ with minimum cardinality. If $\gamma^{ID}(\mu(G)) = n-2$, then $C = V^{'}(G) \setminus \{v_{i1}^{'}, a^{'} \mid 1 \leq i \leq r\} \cup \{v_{i2} \mid 1 \leq i \leq r-1\}$ and so $v_{r1}^{'}$ is not dominated by C, which is a contradiction. Therefore, $\gamma^{ID}(\mu(G)) = n-1$. If $|\{a, a^{'}\} \cap C| = \emptyset$, then v_{r2} is not dominated by C, which is a contradiction. So $a \in C$ or $a^{'} \in C$. Let $a \in C$. Then

$C = V'(G) \setminus \{v'_{i1}, a' \mid 1 \leq i \leq r\} \cup \{v_{i2}, a \mid 1 \leq i \leq r-1\}$. Hence, $N_{\mu(G)}[v_{r2}] \cap C = \{a\} = N_{\mu(G)}[v'_{r1}] \cap C$. That is not true. Thus $a' \in C$ and so $C = V'(G) \setminus \{v'_{i1} \mid 1 \leq i \leq r\} \cup \{v_{i2} \mid 1 \leq i \leq r-1\}$. It is easy to see that v'_{r1} is not dominated by C. That is impossible. Therefore, $\gamma^{ID}(\mu(G)) = n$.

Case 2: Let $s \geq 1$, $\gamma^{ID}(\mu(G)) \leq n+s-2$ and C be an identifying code of $\mu(G)$ with $\gamma^{ID}(\mu(G)) = |C|$. By Lemma 2.3, there are $B \subseteq \{v_j \mid 1 \leq j \leq s\}$ and $F' \subseteq \{v'_j \mid 1 \leq j \leq s\}$, such that $|B \cap C| = |F' \cap C| = s-1$. Without loss of generality assume that $v_1 \notin B$ and $v'_1 \notin F'$. For $2 \leq j \leq n_i$, we have $N_{\mu(G)}[v_{i1}] \triangle N_{\mu(G)}[v_{ij}] = \{v'_{i1}, v'_{ij}\}$. By Lemma 2.3, $v'_{i1} \in C$ or $v'_{ij} \in C$. Let $v'_{i1} \notin C$, for $1 \leq i \leq r$. So, there exists $A' \subseteq \bigcup_{i=1}^{r} V'(K_{n_i})$, such that $|A' \cap C| = \sum_{i=1}^{r}(n_i - 1)$. It is easy to see that $|\{v_1, v'_1, a, a', w\} \cap C| \geq 2$. So $|C| \geq n+s-r-1$. On the other hand, since $N_{\mu(G)}(v'_{11}) = \{a, w\} \cup V(K_{n_1}) \setminus \{v_{11}\}$ and $N_{\mu(G)}(v'_{21}) = \{a, w\} \cup V(K_{n_2}) \setminus \{v_{21}\}$, $|(V(K_{n_1}) \setminus \{v_{11}\} \cup V(K_{n_2}) \setminus \{v_{21}\}) \cap C| \geq 1$. Without loos of generality assume that $v_{12} \in C$. Similarly, we may assume that for $2 \leq i \leq r-1$, $v_{i2} \in C$. Hence, $\gamma^{ID}(\mu(G)) \geq n+s-2$ and so $\gamma^{ID}(\mu(G)) = |C| = n+s-2$. Thus $|\{v_1, v'_1, a, a', w\} \cap C| = 2$. If $v'_1 \notin C$, then $N_{\mu(G)}(v'_1) \cap C = N_{\mu(G)}(v'_{r_1}) \cap C$. That is not true. So $v'_1 \in C$. If $a \notin C$, then since $N_{\mu(G)}(v'_{r_1}) = \{a, w\} \cup V(K_{n_r}) \setminus \{v_{r_1}\}$, $w \in C$. Hence, $\{a', v_1\} \cap C = \emptyset$ and so v_1 is not dominated by C. That is impossible. So, $a \in C$. Hence, $\{a', v_1, w\} \cap C = \emptyset$. We have $N_{\mu(G)}(v_1) \cap C = \{a\} = N_{\mu(G)}(v'_{r_1}) \cap C$, which is a contradiction. Therefore, $\gamma^{ID}(\mu(G)) = n+s-1$.

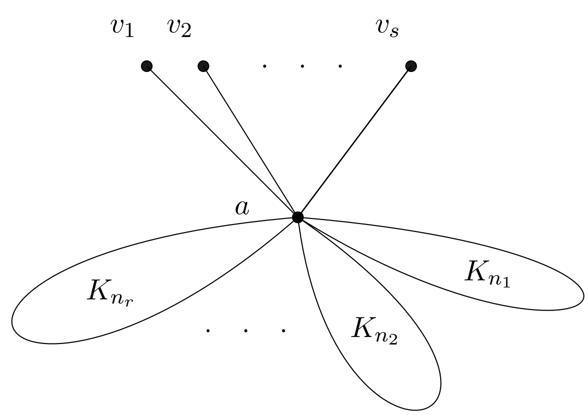

Figure 1

□

The friendship graph F_3^m is a collection of m triangles with a common vertex.

Corollary 3.3. *Let $G \cong F_3^m$. Then $\gamma^{ID}(\mu(G)) = 2m+1$.*

Proof. By Theorem 3.2, the proof is straightforward. □

Conjecture 3.4. [5] *There exists a constant c such that for any nontrivial connected twin-free graph G of maximum degree $\Delta(G)$, $\gamma^{ID}(G) \leq n - \dfrac{n}{\Delta(G)} + c$.*

Note: *The conjecture 3.4, holds for $\mu(K_n)$, $\mu(K_{m,n})$, $\mu(K_3^m)$, $\mu(K_n \setminus \{e\})$ and $\mu(G)$, where G is $(n-2)$-regular graph of order n, with $c = 0$.*

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