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CHROMATIC NUMBER AND SIGNLESS LAPLACIAN SPECTRAL RADIUS OF GRAPHS

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Abstract. For any simple graph G, the signless Laplacian matrix of G is defined as $D(G) + A(G)$, where $D(G)$ and $A(G)$ are the diagonal matrix of vertex degrees and the adjacency matrix of G, respectively. Let $q(G)$ be the signless Laplacian spectral radius of G (the largest eigenvalue of the signless Laplacian matrix of G). In this paper we find some relations between the chromatic number and the signless Laplacian spectral radius of graphs. In particular, we characterize all graphs G of order n with odd chromatic number χ such that $q(G) = 2n\Big(1 - \frac{1}{\chi}\Big)$. Finally we show that if G is a graph of order n and with chromatic number χ, then under certain conditions, $q(G) < 2n\Big(1 - \frac{1}{\chi}\Big) - \frac{2}{n}$. This result improves some previous similar results.

1. Introduction

Throughout this paper we will consider only simple graphs (finite and undirected, without loops and multiple edges). Let $G = (V(G), E(G))$ be a simple graph. The *order* of G denotes the number of vertices of G. For two vertices u and v by $e = uv$ we mean the edge e between u and v. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $deg_G(v)$. A *pendant vertex* is a vertex with degree one. By $\Delta(G)$ we mean the maximum degree of the vertices of G. We say that G is *regular* (k–regular) if all vertices of G have the same degree (if every vertex has degree k). Let $e \in E(G)$. By $G \setminus e$ we mean the graph that obtained from G by removing e. If $e \notin E(G)$, by $G + e$ we mean the graph that obtained from G by adding the edge e. The *complement* of G, denoted by $\overline{G}$, is the simple graph with the vertex set $V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *edgeless graph* (*empty graph*) and the

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complete graph of order n are denoted by $\overline{K_n}$ and K_n respectively. Let $t \geq 2$ and $n_1, \dots, n_t$ be some positive integers. By $K_{n_1,\dots,n_t}$ we mean the *complete multipartite graph* with parts size $n_1, \dots, n_t$. In particular the *complete bipartite graph* with part sizes m and n denoted by $K_{m,n}$. The *star* of order n, denoted by S_n, it the complete bipartite graph $K_{1,n-1}$. Let x be a real number. By $\lfloor x \rfloor$ and $\lceil x \rceil$, we mean the largest integer less than or equal to x, and the smallest integer greater than or equal to x, respectively. For integers p and q, by $p|q$ (by $p \nmid q$) we mean that q is divisible by p (we mean that q is not divisible by p). For a graph G, a *clique* C of G is a subset of vertices of G such that every two distinct vertices in C are adjacent. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. An *independent* set S of G is a subset of vertices of G such that there is no edge between every two vertices of S. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent sets in G. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G so that adjacent vertices get different colors. In other words, a proper vertex coloring of a graph is a partition of the vertex set of the graph into independent sets. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors of a proper vertex coloring of G.

Let G be a graph with vertex set $\{v_1, \dots, v_n\}$. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is the $n \times n$ matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. Let $D(G)$ be the diagonal matrix $(d_1, \dots, d_n)$, where d_i is the degree of vertex v_i, for $i = 1, \dots, n$. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian* matrix of G. The matrices $A(G)$ and $Q(G)$ are symmetric, so all of the eigenvalues of $A(G)$ and $Q(G)$ are real. By the eigenvalues of G we mean those of its adjacency matrix. We denote the eigenvalues of G by $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$. By the $Spec(G)$ we mean the multiset $\{\lambda_1(G), \dots, \lambda_n(G)\}$. By the *spectral radius* of G, denoted by $\lambda(G)$, we mean the largest eigenvalue of G. In other words $\lambda(G) = \lambda_1(G)$. It is well known that $|\lambda_i(G)| \leq \lambda_1(G)$, for $i = 1, \dots, n$. The *characteristic polynomial* of G that is denoted by $P(G, \lambda)$ is $det(\lambda I - A(G))$, where I is $n \times n$ identity matrix. In fact $P(G, \lambda) = \Pi_{i=1}^{n}(\lambda - \lambda_i(G))$. Similarly, by the signless Laplacian eigenvalues of G we mean those of its signless Laplacian matrix. We denote the signless Laplacian eigenvalues of G by $q_1(G) \geq \cdots \geq q_n(G)$. It is well known that all of the signless Laplacian eigenvalues of G are non-negative (in fact $Q(G)$ is a positive semi-definite matrix). In other words, $q_n(G) \geq 0$. By the *signless Laplacian spectral radius* of G, denoted by $q(G)$, we mean the largest signless Laplacian eigenvalue of G. In other words $q(G) = q_1(G)$. The *signless Laplacian characteristic polynomial* of G that is denoted by $P_Q(G, \lambda)$ is $det(\lambda I - Q(G))$, where I is $n \times n$ identity matrix. In fact $P_Q(G, x) = \Pi_{i=1}^{n}(x - q_i(G))$. By the $Spec_Q(G)$ we mean the multiset $\{q_1(G), \dots, q_n(G)\}$. There are many papers devoted to the study of characteristic polynomial, the spectra of adjacency matrix, the spectra of signless Laplacian matrix of graphs, and in particular, studying the spectral radius and the signless Laplacian spectral radius of graphs. For instance related to the eigenvalues of graphs see [1], [2], [10], [12] and the references therein. One of the main interesting problems is the finding some relations between the graphical structures and the algebraic structures of graphs. There are some relations between the chromatic number, the clique number, the independence number and the

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spectral radius of graphs. See [4],[6],[13],[14] and the references therein. For example in [14], Wilf proved that for every graph G, $\chi(G) \leq 1 + \lambda(G)$. Also in [13] he has proved that $\lambda(G) \leq n(1 - \frac{1}{\omega(G)})$. Motivated by these results, in this paper we find some relations between the chromatic number and the signless Laplacian spectral radius of graphs. In [15] it is shown that

Theorem 1.1. *If G is a graph of order n, then*

$$q(G) \leq 2n\Big(1 - \frac{1}{\chi(G)}\Big). \tag{1.1}$$

In addition, in [15] it is proved that if $t \geq 2$ and $n_1, \ldots, n_t$ are some positive integers, then

$$q(K_{n_1,\ldots,n_t}) \leq q(T(n,t)), \tag{1.2}$$

and the equality holds if and only if $K_{n_1,\ldots,n_t}$ is isomorphic to $T(n,t)$, where $T(n,t)$ is the complete multipartite Turán graph of order n and with t parts. In [7] the author also has determined the minimum signless Laplacian spectral radius among the family of complete multipartite graphs of order n and with t parts.

In this paper first we investigate the equality in Equation (1.1) for graphs with odd chromatic number. Then we improve the upper bound of Equation (1.1) and show that under certain conditions,

$$q(G) < 2n\Big(1 - \frac{1}{\chi(G)}\Big) - \frac{2}{n}. \tag{1.3}$$

2. The equality of Equation (1.1)

In this section we find all graphs G with odd chromatic number χ such that $q(G) = 2n\Big(1 - \frac{1}{\chi}\Big)$. First we recall some results.

Theorem 2.1. [3] *Let G be a graph and e be an edge of G. Then*

$$q(G) \geq q(G \setminus e).$$

Equivalently if e' is not an edge of G, then

$$q(G + e') \geq q(G).$$

Remark 2.2. *Let G be a connected graph and H be a subgraph of G. In [3] it is mentioned that if H is proper subgraph of G, then $q(G) > q(H)$. In other words, $q(G) = q(H)$ if and only if $G = H$.*

Theorem 2.3. [15] *Let $t \geq 2$ and $n_1, \ldots, n_t$ be some positive integers, and $n = n_1 + \cdots + n_t$. Then*

$$P_Q(K_{n_1,\ldots,n_t}, x) = \prod_{i=1}^{t}(x - n + n_i)^{n_i - 1}\Big(\prod_{i=1}^{t}(x - n + 2n_i) - \sum_{i=1}^{t} n_i \prod_{j=1,\, j\neq i}^{t}(x - n + 2n_j)\Big).$$

Remark 2.4. *Using* Theorem 2.3 *one can obtain the signless Laplacian eigenvalues of complete bipartite graphs and complete multipartite regular graphs. Let r,s and t be positive integers and $t \geq 2$. In fact*

$$Spec_Q(K_{r,s}) = \Big\{r + s,\, \underbrace{r, \ldots, r}_{s-1},\, \underbrace{s, \ldots, s}_{r-1},\, 0\Big\}$$

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and

$$Spec_Q(K\underbrace{r,\dots,r}_{t}) = \Big\{(2t-2)r, \underbrace{(t-1)r,\dots,(t-1)r}_{t(r-1)}, \underbrace{(t-2)r,\dots,(t-2)r}_{t-1}\Big\}.$$

In particular, $q(K_{r,s}) = r+s$ *and* $q(K\underbrace{r,\dots,r}_{t}) = (2t-2)r$.

Theorem 2.5. [7] *Let* $t \geq 2$ *and* $n_1 \geq \cdots \geq n_t$ *be some positive integers and* $n = n_1 + \cdots + n_t$. *Let* $a = \lceil \frac{n}{t} \rceil$ *and* $b = \lfloor \frac{n}{t} \rfloor$. *Then*

$$q(K_{n_1,\dots,n_t}) \leq \frac{3n - 2(a+b) + \sqrt{n^2 - 4n(a+b) + 4(a-b)^2 + 8abt}}{2}.$$

Moreover the equality holds if and only if $t = 2$ *or* $t \geq 3$ *and* $(n_1,\dots,n_t) = (\underbrace{a,\dots,a}_{r},\underbrace{b,\dots,b}_{s})$, *where* $r = n - t\lfloor \frac{n}{t} \rfloor$ *and* $r + s = t$.

Now we can characterize all graphs with odd chromatic number that the equality of Equation (1.1) hold for them.

Theorem 2.6. *Let* G *be a graph of order* n. *Assume that* $\chi = \chi(G)$ *is odd. Then* $q(G) = 2n\Big(1 - \frac{1}{\chi}\Big)$ *if and only if* $G = \overline{K_n}$ *or* $G = K_{n_1,\dots,n_t}$ *where* $t \geq 3$ *is odd such that* $t|n$ *and* $n_1 = \cdots = n_t = \frac{n}{t}$.

Proof. Since $2n - \frac{2n}{\chi} \leq 2n - 2\Big\lfloor \frac{n}{\chi} \Big\rfloor$, by Equation (1.1) we find that

$$q(G) \leq 2n - \frac{2n}{\chi} \leq 2n - 2\Big\lfloor \frac{n}{\chi} \Big\rfloor. \tag{2.1}$$

If $G = \overline{K_n}$ or $G = K_{n_1,\dots,n_t}$ where $t \geq 3$ is odd such that $t|n$ and $n_1 = \cdots = n_t = \frac{n}{t}$, then there is nothing to prove.

Now assume that $q(G) = 2n\Big(1 - \frac{1}{\chi}\Big)$. This shows that $q(G)$ is rational. Since $q(G)$ is a root of the signless Laplacian characteristic polynomial of graph G, so $q(G)$ is *integral integer*. Hence $q(G)$ should be integer and thus $\frac{2n}{\chi}$ should be integer. Since χ is odd, we conclude that $\frac{n}{\chi}$ is integer and so $\Big\lfloor \frac{n}{\chi} \Big\rfloor = \frac{n}{\chi}$. Therefore $q(G) = 2n - 2\Big\lfloor \frac{n}{\chi} \Big\rfloor$. Now by considering the equality case of Equation (1.2) we find that $G = \overline{K_n}$ or $G = K_{n_1,\dots,n_t}$ where $t \geq 3$ is odd such that $t|n$ and $n_1 = \cdots = n_t = \frac{n}{t}$. This completes the proof. □

3. Improving the upper bound of the signless Laplacian spectral radius

In this section we improve the upper bound that is mentioned in Equation (1.1) for the signless Laplacian spectral radius of graphs. We use majorization theory to obtain the better upper bound for the signless Laplacian spectral radius of graphs. Let $X = (x_1,\dots,x_n) \in \mathbb{R}^n$. Let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$, denote the components of X in decreasing order, and let $X_\downarrow = (x_{[1]}, x_{[2]},\dots,x_{[n]})$ denote the decreasing rearrangement of X. For example if $X = (4,4,5,3,7,7,2)$, then $X_\downarrow = (7,7,5,4,4,3,2)$.

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Let $Y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $Z = (z_1, \dots, z_n) \in \mathbb{R}^n$. We let $Y \succeq_M Z$, and say Y majorizes Z, if and only if

$$\sum_{i=1}^{j} y_{[i]} \geq \sum_{i=1}^{j} z_{[i]}, \text{ for } j = 1, \dots, n-1 \text{ and } \sum_{i=1}^{n} y_{[i]} = \sum_{i=1}^{n} z_{[i]}.$$

Also we let $Y \succ_M Z$ if and only if $Y \succeq_M Z$ and $Y_\downarrow \neq Z_\downarrow$, see [5], [8], [9] and [11]. Let n and t be two positive integers where $n \geq t$. Let

$$A(n,t) = (n-t+1, \underbrace{1, \dots, 1}_{t-1}) \text{ and } B(n,t) = (\underbrace{\lceil \frac{n}{t} \rceil, \dots, \lceil \frac{n}{t} \rceil}_{r}, \underbrace{\lfloor \frac{n}{t} \rfloor, \dots, \lfloor \frac{n}{t} \rfloor}_{s}),$$

where $r = n - t\lfloor \frac{n}{t} \rfloor$ and $s = t - r$. Let

$$C_{n,t} = \Big\{ (n_1, \dots, n_t) \in \mathbb{Z}^t : n_1 \geq n_2 \geq \dots \geq n_t \geq 1, \text{ and } n_1 + \dots + n_t = n \Big\}.$$

It is easy to check that $(C_{n,t}, \succeq_M)$ is a partially ordered set. In the next theorem the maximum and the minimum elements of the poset $(C_{n,t}, \succeq_M)$ have been determined.

Theorem 3.1. [7] *Let n and t be two positive integers where $n \geq t$. Then for every $(n_1, \dots, n_t) \in C_{n,t}$,*

$$A(n,t) \succeq_M (n_1, \dots, n_t) \succeq_M B(n,t).$$

Now we find the another minimal element of the poset $(C_{n,t}, \succeq_M)$.

Theorem 3.2. *Let n and $t \geq 2$ be two positive integers where $t|n$. Then for every $(n_1, \dots, n_t) \in C_{n,t}$ except $B(n,t)$,*

$$(n_1, \dots, n_t) \succeq_M (\frac{n}{t} + 1, \underbrace{\frac{n}{t}, \dots, \frac{n}{t}}_{t-2}, \frac{n}{t} - 1).$$

Proof. Assume that $(n_1, \dots, n_t) \in C_{n,t}$ and $(n_1, \dots, n_t) \neq B(n,t)$. Since $n_1 \geq \dots \geq n_t$, $tn_1 \geq n_1 + \dots + n_t = n$. Thus $n_1 \geq \frac{n}{t}$. If $n_1 = \frac{n}{t}$, then we find that $n_1 = \dots = n_t = \frac{n}{t}$ and so $(n_1, \dots, n_t) = B(n,t)$, a contradiction. Therefore $n_1 \geq \frac{n}{t} + 1$. Hence to complete the proof it suffices to show that for every $j \in \{2, \dots, t-1\}$, $n_1 + \dots + n_j \geq \frac{jn}{t} + 1$. By contradiction, assume that for some $\alpha \in \{2, \dots, t-1\}$, $n_1 + \dots + n_\alpha \leq \frac{\alpha n}{t}$. Since by Theorem 3.1,

$$(n_1, \dots, n_t) \succeq_M (\underbrace{\frac{n}{t}, \dots, \frac{n}{t}}_{t}),$$

the latter inequality implies that

$$n_1 + \dots + n_\alpha = \frac{\alpha n}{t}. \tag{3.1}$$

Using again Theorem 3.1 we conclude that $n_1 + \dots + n_\alpha + n_{\alpha+1} \geq \frac{(\alpha+1)n}{t}$. Since $n_1 + \dots + n_\alpha = \frac{\alpha n}{t}$, we obtain that $n_{\alpha+1} \geq \frac{n}{t}$. Hence $n_2 \geq \dots \geq n_\alpha \geq n_{\alpha+1} \geq \frac{n}{t}$. On the other hand $n_1 \geq \frac{n}{t} + 1$. Therefore $n_1 + \dots + n_\alpha \geq \frac{\alpha n}{t} + 1$, a contradiction (by Equation (3.1)). The proof is complete. □

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Let n and t be two positive integers where $n > t \geq 3$ and $t|n$. By $H(n,t)$ we mean the following complete multipartite graph,

$$H(n,t) = K_{\frac{n}{t}+1,\underbrace{\frac{n}{t},\ldots,\frac{n}{t}}_{t-2},\frac{n}{t}-1}.$$

Using Theorem 2.3, one can obtain the signless Laplacian characteristic polynomial of the complete multipartite graph $H(n,t)$ as following.

Theorem 3.3. *Let n and t be two positive integers where $n > t \geq 3$ and $t|n$. Then*

$$P_Q(H(n,t),x) = (x-n+a)^{a-1}(x-n+b)^{b-1}(x-n+c)^{(t-2)(c-1)}(x-n+2c)^{t-3}\, f_{n,t}(x),$$

where $a = \frac{n}{t}+1$, $b = \frac{n}{t}-1$, $c = \frac{n}{t}$ and

$$f_{n,t}(x) = \big(x-(t-2)c\big)^3 - n\big(x-(t-2)c\big)^2 + 4n - 8c.$$

Now we investigate the signless Laplacian spectral radius of $H(n,t)$.

Theorem 3.4. *Let n and t be two positive integers where $n > t \geq 4$ and $t|n$. Then*

$$2n - \frac{2n}{t} - \frac{4}{n} < q(H(n,t)) < 2n - \frac{2n}{t} - \frac{2}{n}.$$

Proof. Using Theorem 3.3, for determining the signless Laplacian spectral radius of $H(n,t)$ it suffices to investigate the largest real root of $f(x)$, where

$$f(x) = \big(x-(t-2)\frac{n}{t}\big)^3 - n\big(x-(t-2)\frac{n}{t}\big)^2 + 4n - \frac{8n}{t}.$$

We note that the roots of $f(x)$ are some signless Laplacian eigenvalues of $H(n,t)$, and so they are real. Let $g(y) = y^3 - ny^2 + 4n - \frac{8n}{t}$. Thus $f(x) = g(x-(t-2)\frac{n}{t})$. Let $\beta(f)$ and $\beta(g)$ be the largest real root of f and g, respectively. Hence $\beta(f) = \beta(g) + (t-2)\frac{n}{t}$. The derivative g with respect to y is $g'(y) = 3y^2 - 2ny$. This implies that g is decreasing on the interval $[0, \frac{2n}{3}]$ and increasing on the interval $[\frac{2n}{3}, n]$. In addition, $g(0) = g(n) = 4n(1-\frac{2}{t}) > 0$ and $g(\frac{2n}{3}) = -4n(\frac{n^2}{27} + \frac{2}{t} - 1) < 0$. On the other hand $g(n-\frac{2}{n}) = 2n(1-\frac{4}{t}) + \frac{8}{n}(1-\frac{1}{n^2}) > 0$ and $g(n-\frac{4}{n}) = -8n(\frac{1}{t} - \frac{4}{n^2}) - \frac{64}{n^3} < 0$ (note that since $t|n$ and $n > t$, so $n \geq 2t$). Since $\frac{2n}{3} < n - \frac{4}{n} < n - \frac{2}{n} < n$, we conclude that $n - \frac{4}{n} < \beta(g) < n - \frac{2}{n}$. Thus

$$2n - \frac{2n}{t} - \frac{4}{n} < \beta(f) < 2n - \frac{2n}{t} - \frac{2}{n}.$$

By Theorem 3.3, $q(H(n,t)) = \max\{n - \frac{n}{t} + 1, \beta(f)\}$. On the other hand $2n - \frac{2n}{t} - \frac{4}{n} > n - \frac{n}{t} + 1$. Therefore $q(H(n,t)) = \beta(f)$ and so

$$2n - \frac{2n}{t} - \frac{4}{n} < q(H(n,t)) < 2n - \frac{2n}{t} - \frac{2}{n}.$$

□

Remark 3.5. *Let n and t be two positive integers where $n > t \geq 4$ and $t|n$.* Theorem 3.4 *shows that $q(H(n,t)) \simeq 2n - \frac{2n}{t}$.*

Now we can improve the upper bound (1.1). We need the following result.

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Theorem 3.6. [7] *Let $t \geq 3$ and $m_1, \dots, m_t$ and $n_1, \dots, n_t$ be some positive integers such that $m_1 \geq \cdots \geq m_t$ and $n_1 \geq \cdots \geq n_t$. If $(m_1, \dots, m_t) \succ_M (n_1, \dots, n_t)$, then*

$$q(K_{n_1,\dots,n_t}) > q(K_{m_1,\dots,m_t}).$$

Theorem 3.7. *Let G be a graph of order n and $\chi = \chi(G)$ and $\alpha = \alpha(G)$. Assume that $\chi \geq 4$ and $\chi | n$ and $\alpha\chi > n$. Then*

$$q(G) < 2n\Big(1 - \frac{1}{\chi}\Big) - \frac{2}{n}.$$

Proof. Let $t = \chi(G)$. Since $t = \chi(G)$, one can partition the vertex set of G as t non-empty independent sets $V_1, \dots, V_t$. Assume that $|V_i| = n_i$ for $i = 1, \dots, t$. Thus $n_1 + \cdots + n_t = n$. Without losing the generality suppose that $n_1 \geq n_2 \geq \cdots \geq n_t$. Therefore G is a spanning subgraph of $K_{n_1,\dots,n_t}$. Hence by applying Theorem 2.1 (many times) we obtain that

$$q(G) \leq q(K_{n_1,\dots,n_t}). \tag{3.2}$$

On the other hand, for $i = 1, \dots, t$, we have $n_i \leq \alpha$. Since $n_1 + \cdots + n_t = n$ and $t\alpha > n$, there exists $j \in \{1, \dots, t\}$ such that $n_j < \alpha$. This shows that $(n_1, \dots, n_t) \neq (\underbrace{\frac{n}{t}, \dots, \frac{n}{t}}_{t})$. Hence by Theorem 3.2,

$$(n_1, \dots, n_t) \succeq_M (\tfrac{n}{t} + 1, \underbrace{\tfrac{n}{t}, \dots, \tfrac{n}{t}}_{t-2}, \tfrac{n}{t} - 1). \tag{3.3}$$

Using Equation (3.3) and Theorem 3.6 we find that

$$q(K_{n_1,\dots,n_t}) \leq q(K_{\frac{n}{t}+1, \underbrace{\frac{n}{t},\dots,\frac{n}{t}}_{t-2}, \frac{n}{t}-1}). \tag{3.4}$$

Now by combining Equations (3.2) and (3.4), we conclude that

$$q(G) \leq q(K_{\frac{n}{t}+1, \underbrace{\frac{n}{t},\dots,\frac{n}{t}}_{t-2}, \frac{n}{t}-1}). \tag{3.5}$$

By Theorem 3.4, Equation (3.5) implies that

$$q(G) < 2n\Big(1 - \frac{1}{t}\Big) - \frac{2}{n}.$$

The proof is complete. □

Remark 3.8. *Let G be a graph of order n and $\chi = \chi(G)$ and $\alpha = \alpha(G)$. In general, $\chi\alpha \geq n$. Here we note that in* Theorem 3.7*, the assumption $\chi\alpha > n$ is critical. For example let $t \geq 2$ and $t|n$ and $G = K_{\underbrace{\frac{n}{t},\dots,\frac{n}{t}}_{t}}$. Thus $\chi(G) = t$, $\alpha(G) = \frac{n}{t}$ and so $\chi\alpha = n$. On the other hand, since G is a $(n - \frac{n}{t})$–regular graph, $q(G) = 2(n - \frac{n}{t})$. In other words, $q(G) = 2n\Big(1 - \frac{1}{\chi}\Big) \not< 2n\Big(1 - \frac{1}{\chi}\Big) - \frac{2}{n}$.*

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