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ON EIGENSPACES OF SOME COMPOUND COMPLEX UNIT GAIN GRAPHS

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ABSTRACT. Let $\mathbb{T}$ be the multiplicative group of complex units, and let $L(\Phi)$ denote the Laplacian matrix of a nonempty $\mathbb{T}$-gain graph $\Phi = (\Gamma, \mathbb{T}, \gamma)$. The gain line graph $\mathcal{L}(\Phi)$ and the gain subdivision graph $\mathcal{S}(\Phi)$ are defined up to switching equivalence. We discuss how the eigenspaces determined by the adjacency eigenvalues of $\mathcal{L}(\Phi)$ and $\mathcal{S}(\Phi)$ are related with those of $L(\Phi)$.

1. Introduction

Let Γ be a nonempty simple graph with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$, and let $\vec{E}(\Gamma)$ be the set of oriented edges. Such set contains two copies of each edge of Γ with opposite directions. We write e_{ij} for the oriented edge from v_i to v_j. Given any group $\mathfrak{G}$, a ($\mathfrak{G}$-)*gain graph* is a triple $\Phi = (\Gamma, \mathfrak{G}, \gamma)$ consisting of an *underlying graph* Γ, the *gain group* $\mathfrak{G}$ and a map $\gamma : \vec{E}(\Gamma) \to \mathfrak{G}$ such that $\gamma(e_{ij}) = \gamma(e_{ji})^{-1}$ called the *gain function*. Let 1 denote the identity element of $\mathfrak{G}$. The gain graph Φ is said to be balanced if, for every directed cycle $\vec{C} = e_{i_1i_2} \ldots e_{i_ki_1}$ in Γ (if any), we have $\gamma(e_{i_1i_2})\gamma(e_{i_2i_3}) \ldots \gamma(e_{i_ki_1}) = 1$. Most of the concepts defined for simple graphs directly extend to gain graphs. For instance, we say that a gain graph $(\Gamma, \mathfrak{G}, \gamma)$ is k-cyclic if the underlying graph Γ is connected and $k = m - n + 1$. As usual, the words *unicyclic* and *bicyclic* stand as synonyms for 1-cyclic and 2-cyclic respectively. Gain graphs (also known in the literature as *voltage graphs*) are studied in many research areas (see [27] and the annotated bibliography [28]).

In particular, a *complex unit* gain graph is a $\mathfrak{G}$-gain graph with $\mathfrak{G} = \mathbb{T}$, the multiplicative group of all complex numbers with norm 1. The theory of complex unit gain graphs embodies those of signed

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graphs and mixed graphs (as defined in [16]). In fact, a signed graph (resp. mixed graph) can be seen as a particular $\mathbb{T}$-gain graph with gains in the subset $\{\pm 1\}$ (resp. $\{1, \pm i\}$) of $\mathbb{T}$. Clearly, every $\mathbb{T}_n$-gain graph, where $n \in \mathbb{N}$ and $\mathbb{T}_n$ denotes the group of n-th roots of unity, can be regarded as a complex unit gain graph. Empty graphs can be thought as $\mathbb{T}$-gain graph equipped with the empty gain function $\varnothing \to \mathbb{T}$ and are obviously balanced.

Over the last few years, there has been a growing interest for the study of matrices and eigenvalues associated to $\mathbb{T}$-gain graphs (see, for instance [3, 5, 9, 12, 17, 18, 20, 21, 22, 25]).

In [23], Reff introduced a notion of orientation for gain graphs in order to provide a suitable setting to build up line graphs of gain graphs. In the wake of this inspiring paper, some work has been done to develop a spectral theory for line and subdivision graphs of $\mathbb{T}$-gain graphs [1, 2, 10, 11]. A fundamental result in this context is the mutual interrelationship between the Laplacian spectrum of a $\mathbb{T}$-gain graph Φ and the adjacency spectra of a spectral line graph $\mathcal{L}(\Phi)$ and a subdivision graph $\mathcal{S}(\Phi)$ determined by Φ. The relationship is expressed by Theorem 1.1 proved in [1] for $\mathbb{T}_4$-gain graphs, but the proof for complex unit gain graphs is formally identical. In the statement of Theorem 1.1 and throughout the paper, we denote by $A(\Phi)$ and $L(\Phi)$ the adjacency and the Laplacian matrix associate to a $\mathbb{T}$-gain graph Φ, and by

$$\phi(\Phi, x) = \det(xI - A(\Phi)) \qquad (\text{resp.} \quad \psi(\Phi, x) = \det(xI - L(\Phi)))$$

its adjacency (resp. Laplacian) characteristic polynomial.

Theorem 1.1. *Let Γ be the underlying graph of a nonempty $\mathbb{T}$-gain graph Φ of order n and size m. The following equalities of polynomials hold.*

(1) $\phi(\mathcal{L}(\Phi), x) = (x+2)^{m-n}\psi(\Phi, x+2)$;

(2) $\phi(\mathcal{S}(\Phi), x) = x^{m-n}\psi(\Phi, x^2)$.

Since the roots of the Laplacian characteristics polynomial are all real and nonnegative, the minimum possible (adjacency) eigenvalue of $\mathcal{L}(\Phi)$ is -2. We have recently proved in [2] that such minimum is attained whenever Φ has a connected component which is neither a tree or a balanced unicyclic gain graph. In these cases, we detected a basis for the -2-eigenspace by using the star complement technique and generalizing the routine successfully applied in the past to simple graphs (see [13, 14, 15]) and to signed graphs (see [6, 7]).

In Section 2, we recall how to associate a line graph $\mathcal{L}(\Phi)$ and a subdivision $\mathcal{S}(\Phi)$ to a $\mathbb{T}$-gain graph Φ. These two compound graphs are well-defined once we choose a fixed incidence matrix for $\Phi = (\Gamma, \mathbb{T}, \gamma)$ (or, adopting the terminology of [11], a suitable *represented* $\mathbb{T}$*-phase matrix* of Γ). In this paper, we show how the eigenspaces of $L(\Phi)$ are related with those of $A(\mathcal{L}(\Phi))$ and $A(\mathcal{S}(\Phi))$. The main results are summarized in Theorems 4.5, 4.8, 4.9 and 4.10.

It turns out that, apart from at most the minimal eigenvalue, the eigenspaces of any pair of matrices in the set $\mathcal{T} = \{L(\Phi), A(\mathcal{L}(\Phi)), A(\mathcal{S}(\Phi))\}$ can be quickly deduced from the matrix H chosen to define

$\mathcal{L}(\Phi)$ and $\mathcal{S}(\Phi)$, and from the eigenspaces of the remaining one in $\mathcal{T}$. Our way of arguing generalizes to complex unit gain graphs the corresponding results in [24] involving unsigned graphs and those in [7] holding for signed graphs. The remainder of the paper is organized as follows. Section 2 contains preliminaries on complex unit graphs, on the associated line and subdivision graphs built from a fixed incidence matrix, and on the procedure investigated in [2] to determine a basis for the (-2)-eigenspace of $A(L(\Phi))$ when -2 belongs to its spectrum. Section 3 is devoted to a comparison between the eigenspaces of $L(\Phi)$ and those of $A(\mathcal{L}(\Phi))$. Section 4 focuses on the eigenspaces of the subdivision graph. In the short Section 5 it is studied how the eigenbases of the compound graphs behave when Φ is replaced by a switching equivalent graph. Finally, Section 6 provides the explicit computation of some eigenspaces.

2. Preliminaries

2.1. Complex unit gain graphs.

From now on, a $\mathbb{T}$-gain graph will be simply denoted by $\Phi = (\Gamma, \gamma)$. Given a nonempty $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ of order n and size $m > 0$, we adopt the notation

$$V(\Gamma) = \{ v_1, \dots, v_n \} \qquad \text{and} \qquad \mathrm{E}(\Gamma) = \{ e_1, \dots, e_m \}$$

for the set of vertices and the set of (unoriented) edges of Γ respectively.

Let $M_{m,n}(\mathbb{C})$ be the set of $m \times n$ complex matrices. For a matrix $A = (a_{ij}) \in M_{m,n}(\mathbb{C})$, we denote by $A^* = (a^*_{ij}) \in M_{n,m}(\mathbb{C})$ its *conjugate* (or *Hermitian*) *transpose*; i.e. $a^*_{ij} = \overline{a_{ji}}$.

The *adjacency matrix* $A(\Phi) = (a_{ij}) \in M_{n,n}(\mathbb{C})$ of a $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ is defined by

$$a_{ij} = \begin{cases} \gamma(e_{ij}) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If v_i is adjacent to v_j, then $a_{ij} = \gamma(e_{ij}) = \gamma(e_{ji})^{-1} = \overline{\gamma(e_{ji})} = \overline{a_{ji}}$. Consequently, $A(\Phi)$ is Hermitian and its eigenvalues $\lambda_1(\Phi) \geq \cdots \geq \lambda_n(\Phi)$ are real. The *Laplacian matrix* $L(\Phi)$, defined as $D(\Gamma) - A(\Phi)$, where $D(\Gamma) = \mathrm{diag}(d(v_1), \dots, d(v_n))$ stands for the diagonal matrix of vertex degrees of Γ, is Hermitian as well, and all its eigenvalues $\mu_1(\Phi) \geq \cdots \geq \mu_n(\Phi)$ are nonnegative [22]. By definition, the spectrum $\mathrm{Spec}(M(\Phi))$ is the multiset of eigenvalues of $M(\Phi)$, where $M \in \{A, L\}$. For each $\lambda \in \mathbb{R}$, we set

$$\mathcal{E}_M(\lambda, \Phi) = \{\mathbf{x} \in \mathbb{C}^{|V(\Gamma)|} \mid M(\Phi)\,\mathbf{x} = \lambda \mathbf{x}\},$$

and denote by $m_{M(\Phi)}(\lambda)$ its dimension as a $\mathbb{C}$-vector space. Clearly, $m_{M(\Phi)}(\lambda) > 0$ if and only if $\lambda \in \mathrm{Spec}(M(\Phi))$.

A *switching function* for a gain graph Φ is a map $\zeta : V(\Gamma) \to \mathbb{T}$. Switching a nonempty $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ means replacing γ by γ^ζ, where $\gamma^\zeta(e_{ij}) = \zeta(v_i)^{-1}\gamma(e_{ij})\zeta(v_j)$, and obtaining in this way the new $\mathbb{T}$-gain graph $\Phi^\zeta = (\Gamma, \gamma^\zeta)$. We say that $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ (and their corresponding gain functions) are *switching equivalent* if there exists a switching function ζ such that $\Phi_2 = \Phi_1^\zeta$. By writing $\Phi_1 \sim \Phi_2$ we mean that Φ_1 and Φ_2 are switching equivalent.

To each switching function ζ we associate a diagonal matrix $D(\zeta) = \mathrm{diag}(\zeta(v_1), \ldots, \zeta(v_n))$. Note that

$$M(\Phi_2) = D(\zeta)^* M(\Phi_1) D(\zeta) \qquad \text{for } M \in \{A, L\}; \tag{2.1}$$

therefore,

$$\mathrm{Spec}(M(\Phi_1)) = \mathrm{Spec}(M(\Phi_2)), \qquad \text{whenever } \Phi_1 \sim \Phi_2.$$

One of the key notions in the theory of gain graphs (and of the more general theory of biased graphs as well) is balance (see [27]). An oriented edge $e_{i_h i_k} \in \vec{E}(\Gamma)$ is said to be *neutral* for $\Phi = (\Gamma, \gamma)$ if $\gamma(e_{i_h i_k}) = 1$. Similarly, the walk $W = e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_{l-1} i_l}$ is said to be *neutral* if its *gain*

$$\gamma(W) := \gamma(e_{i_1 i_2})\gamma(e_{i_2 i_3}) \cdots \gamma(e_{i_{l-1} i_l})$$

is equal to 1. We write $(\Gamma, 1)$ for the $\mathbb{T}$-gain graph with all neutral edges.

An edge set $S \subseteq E$ is said to be *balanced* if no nonneutral directed cycles with edges in S exist. A subgraph is *balanced* if its edge set is balanced (see [1, 5, 22] for further details).

By [25, Theorem 2.8] we get the following proposition.

Proposition 2.1. *A connected $\mathbb{T}$-gain graph Φ of order n is balanced if and only if its least Laplacian eigenvalue $\mu_n(\Phi)$ is 0.*

The next proposition specializes [23, Lemma 2.2] to $\mathbb{T}$-gain graphs.

Proposition 2.2. *Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be $\mathbb{T}$-gain graphs with the same underlying graph Γ. Φ_1 and Φ_2 are switching equivalent if and only if, for every cycle C in Γ, there exists a directed cycle with base vertex v such that $\gamma_1(\vec{C}_v) = \gamma_2(\vec{C}_v)$.*

By Proposition 2.2 it follows that a gain graph Φ is balanced if and only if all its directed cycles are neutral. Furthermore, the following corollary holds.

Corollary 2.3. *A nonempty $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$ is balanced if and only if it is switching equivalent to $(\Gamma, 1)$.*

The last result of this section is a consequence of Proposition 2.1 and Corollary 2.3.

Corollary 2.4. *Let Φ be any complex unit gain graph. The number of its balanced components is equal to $m_{L(\Phi)}(0)$, the multiplicity of 0 as a Laplacian eigenvalue of Φ.*

2.2. Line and subdivision graphs associated to $\mathbb{T}$-gain graphs.

Unless otherwise specified, $\Phi = (\Gamma, \gamma)$ will always denote a $\mathbb{T}$-gain graph of order n and size $m > 0$. We say that the $n \times m$ complex matrix $\mathrm{H}(\Phi) = (\eta_{ve})$ with entries in $\mathbb{T} \cup \{0\}$ is an *incidence matrix* of Φ if

$$\eta_{v_i e_h} = \begin{cases} -\eta_{v_j e_h}\gamma(e_{ij}) & \text{if the endpoints of } e_h \text{ are precisely } v_i \text{ and } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

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In the case when e_h joins v_i and v_j, we also require that $\eta_{v_i e_h}$ is nonzero. We say 'an' incidence matrix, because by multiplying each column by any element in $\mathbb{T}$, we still obtain an incidence matrix of the same gain graph. Indeed, Proposition 2.5 says that all the other possible incidence matrices can be obtained from a fixed $\mathrm{H}(\Phi)$ in that way.

Proposition 2.5. [2, Proposition 2.10] *Let* $\mathrm{H}(\Phi) = (\eta_{ve})$ *and* $\mathrm{H}(\Phi)' = (\eta'_{ve})$ *be two incidence matrices both related to the* $\mathbb{T}$*-gain graph* $\Phi = (\Gamma, \gamma)$*. There exists an* $m \times m$ *diagonal matrix* S *with entries in* $\mathbb{T} \cup \{0\}$ *such that* $\mathrm{H}(\Phi)' = \mathrm{H}(\Phi)S$ *and* $S^*S = I$*.*

By definition (or equivalently by Proposition 2.5), for a fixed edge $e_h \in E(\Gamma)$ with endpoints v_i and v_j, the nonzero elements on the corresponding column of $\mathrm{H}(\Phi)$, i.e. $\eta_{v_i e_h}$ and $\eta_{v_j e_h}$, satisfy

$$(\eta_{v_i e_h}, \eta_{v_j e_h}) = (\mathrm{e}^{i\theta}\gamma(e_{ij}), \mathrm{e}^{i(\theta+\pi)})$$

for a suitable $\mathrm{e}^{i\theta} \in \mathbb{T}$ (depending on h).

In what follows, we denote by H a specific incidence matrix related to the $\mathbb{T}$-gain graph $\Phi = (\Gamma, \gamma)$. We next explain how H and an *involution* $\mathfrak{s} \in \{-1, 1\} \subset \mathbb{C}$ determine a $\mathbb{T}$-gain structure on the line graph $\mathcal{L}(\Gamma)$. It is well-known that $V(\mathcal{L}(\Gamma)) = E(\Gamma)$, and $ef \in E(\mathcal{L}(\Gamma))$ whenever e and f share an endpoint. We denote by $\mathcal{L}^{\mathfrak{s}}_{\mathrm{H}}(\Phi)$ the $\mathbb{T}$-gain graph $(\mathcal{L}(\Gamma), \gamma^{\mathcal{L}^{\mathfrak{s}}}_{\mathrm{H}})$, where

$$\gamma^{\mathcal{L}^{\mathfrak{s}}}_{\mathrm{H}} : ef \in \vec{E}(\mathcal{L}(\Gamma)) \longrightarrow \mathfrak{s}\,\overline{\eta}_{we}\eta_{wf} \in \mathbb{T},$$

and w is the endpoint shared by the edges e and f. It is easy to verify that $\gamma^{\mathcal{L}^{\mathfrak{s}}}_{\mathrm{H}}$ is a gain function. In fact,

$$\gamma^{\mathcal{L}^{\mathfrak{s}}}_{\mathrm{H}}(fe) = \overline{\gamma^{\mathcal{L}^{\mathfrak{s}}}_{\mathrm{H}}(ef)}.$$

As recalled in [2, Section 2.2] and [8, Remark 2.1], there is not a scientific consensus on which compound graph between $\mathcal{L}^{1}_{\mathrm{H}}(\Phi)$ and $\mathcal{L}^{-1}_{\mathrm{H}}(\Phi)$ enjoys nicer properties. In [8] they are respectively named *spectral line graph* and *combinatorial line graph* (it is also instructive the comprehensive discussion in [10, Section 1] on the several definitions of gain line graphs existing in literature). In this paper, we only deal with $\mathcal{L}_{\mathrm{H}}(\Phi) := \mathcal{L}^{1}_{\mathrm{H}}(\Phi)$, and simply write $\gamma^{\mathcal{L}}_{\mathrm{H}}$ to denote $\gamma^{\mathcal{L}^1}_{\mathrm{H}}$. Note, in any case, that

$$\lambda \in \mathrm{Spec}(A(\mathcal{L}_{\mathrm{H}}(\Phi))) \Longleftrightarrow -\lambda \in \mathrm{Spec}(A(\mathcal{L}^{-1}_{\mathrm{H}}(\Phi)));$$

moreover,

$$\mathcal{E}_A(\lambda, \mathcal{L}_{\mathrm{H}}(\Phi)) = \mathcal{E}_A(-\lambda, \mathcal{L}^{-1}_{\mathrm{H}}(\Phi)).$$

When $\gamma(\vec{E}(\Gamma)) \subseteq \{-1, 1\}$, i.e. when the $\mathbb{T}$-gain graph Φ is actually a signed graph, the map $\gamma^{\mathcal{L}}_{\mathrm{H}}$ assigns to $\mathcal{L}(\Gamma)$ the same signature prescribed in [6, Section 1] and [7, Section 2].

Theorem 2.6. [23, Theorem 5.1] *Let* H *be one of the incidence matrices related to the* $\mathbb{T}$*-gain graph* $\Phi = (\Gamma, \gamma)$*. Then,*

$$\mathrm{H}(\Phi)^*\mathrm{H}(\Phi) = 2I_m + A(\mathcal{L}_{\mathrm{H}}(\Phi)).$$

We end this section with two results essentially encapsulating [1, Proposition 5] and [2, Propositions 2.12 and 2.13].

Proposition 2.7. *Let* H *and* H′ *be two of incidence matrices both associated to the same* $\mathbb{T}$*-gain graph* $\Phi = (\Gamma, \gamma)$.

(1) $\mathcal{L}_{\mathrm{H}}(\Phi)$ *and* $\mathcal{L}_{\mathrm{H}'}(\Phi)$ *share the same adjacency spectrum. Moreover, if* S *is the diagonal matrix such that* $\mathrm{H}(\Phi)' = \mathrm{H}(\Phi)S$*, then*

$$A(\mathcal{L}_{\mathrm{H}'}(\Phi)) = S^* A(\mathcal{L}_{\mathrm{H}}(\Phi))S.$$

(2) Every $\mathbb{T}$*-gain graph being switching equivalent to* $\mathcal{L}_{\mathrm{H}}(\Phi)$ *is a line graph associated to* Φ.

Proposition 2.8. *Line graphs of switching equivalent* $\mathbb{T}$*-gain graphs* $\Phi_1 = (\Gamma, \gamma_1)$ *and* $\Phi_2 = (\Gamma, \gamma_2)$ *are switching equivalent. Moreover, if* $\zeta : V(\Gamma) \to \mathbb{T}$ *is the switching function such that* $\Phi_2 = \Phi_1^{\zeta}$*, and* H_1 *is an incidence matrix for* Φ_1*, then* $D(\zeta)^*\mathrm{H}_1$ *is an incidence matrix for* Φ_2*, and*

$$\mathcal{L}_{\mathrm{H}_1}(\Phi_1) = \mathcal{L}_{D(\zeta)^*\mathrm{H}_1}(\Phi_2).$$

From Propositions 2.7 and 2.8, it follows that all line graphs built from any representative of a switching equivalence class of $\mathbb{T}$-gain graphs share the same A-spectrum and the same L-spectrum. That is why it has been possible to drop the incidence matrix H out of notation in Theorem 1.1.

For any nonempty graph Γ, the subdivision graph $\mathcal{S}(\Gamma)$ is obtained from Γ by replacing each of its edges by a path of length 2, or, equivalently, by inserting an additional vertex into each edge e of Γ. As it is usual in this context, we denote by e the additional vertex inserted on the homonymous edge. For the set $V(\mathcal{S}(\Gamma))$ we choose the ordering $\{v_1, \dots, v_n, e_1, \dots, e_m\}$.

Any incidence matrix $\mathrm{H} = (\eta_{ve})$ of Φ induces a gain structure on $\mathcal{S}(\Gamma)$ through the map $\gamma_{\mathrm{H}}^{\mathcal{S}} : \vec{E}(\mathcal{S}(\Gamma)) \to \mathbb{T}$ defined in the following way:

$$\gamma_{\mathrm{H}}^{\mathcal{S}}(ve) = \overline{\gamma_{\mathrm{H}}^{\mathcal{S}}(ev)} = \eta_{ve}$$

for any $v \in V(\Gamma)$ and for any $e \in E(\Gamma)$.

According to the chosen vertex ordering the adjacency matrix of the gain graph $\mathcal{S}_{\mathrm{H}}(\Phi) = (\mathcal{S}(\Gamma), \gamma_{\mathrm{H}}^{\mathcal{S}})$ is

$$A(\mathcal{S}_{\mathrm{H}}(\Phi)) = \begin{pmatrix} O_n & \mathrm{H} \\ \mathrm{H}^* & O_m \end{pmatrix}. \tag{2.2}$$

The following proposition can be regarded as the subdivision counterpart of Proposition 2.7.

Proposition 2.9. *Let* H *and* $\mathrm{H}' = \mathrm{H}S$ *be two incidence matrices of a* $\mathbb{T}$*-gain graph* $\Phi = (\Gamma, \gamma)$*, with* $S = \mathrm{diag}(s_1, \dots, s_m)$*. Then,*

(1) $A(\mathcal{S}_{\mathrm{H}}(\Phi))$ *and* $A(\mathcal{S}_{\mathrm{H}'}(\Phi))$ *are similar and share the same adjacency spectrum. In fact,* $\mathcal{S}_{\mathrm{H}'}(\Phi) = \mathcal{S}_{\mathrm{H}}(\Phi)^{\zeta_{\mathrm{H}}^{\mathrm{H}'}}$*, the switching function* $\zeta_{\mathrm{H}}^{\mathrm{H}'} : V(\mathcal{S}_{\mathrm{H}}(\Phi)) \to \mathbb{T}$ *being*

$$\zeta_{\mathrm{H}}^{\mathrm{H}'}(v_h) = 1 \quad \text{for } 1 \le h \le n, \qquad \text{and} \qquad \zeta_{\mathrm{H}}^{\mathrm{H}'}(e_k) = s_k \qquad \text{for } 1 \le k \le m.$$

(2) Subdivision graphs of two nonempty switching equivalent $\mathbb{T}$*-gain graphs* $\Phi_1 = (\Gamma, \gamma_1)$ *and* $\Phi_2 = (\Gamma, \gamma_2)$ *are switching equivalent. More explicitly, if* H_2 *is a fixed incidence matrix of* Φ_2 *and* $\gamma_2 = \gamma_1^{\zeta}$*, then* $\mathcal{S}_{\mathrm{H}_2}(\Phi_2) = \mathcal{S}_{\mathrm{H}_1}(\Phi_1)^Z$*, where* $\mathrm{H}_1 := D(\zeta)\mathrm{H}_2$ *and the switching map* $Z : V(\mathcal{S}_{\mathrm{H}_1}(\Phi_1)) \to \mathbb{T}$ *is defined as follows:*

$$Z(v_h) = \zeta(v_h), \quad \textit{for } 1 \le h \le n, \qquad \textit{and} \qquad Z(e_k) = 1, \qquad \textit{for } 1 \le k \le m.$$

Proof. The proofs of [1, Propositions 7 and 8] can be followed verbatim, once $\mathbb{T}_4$ is replaced with $\mathbb{T}$ whenever it occurs. □

Remark 2.10. *As it happens for ordinary graphs, a complex unit gain graph* Φ *is connected if and only if* $\mathcal{L}(\Phi)$ *and* $\mathcal{S}(\Phi)$ *are both connected. Moreover, since for a (disjoint) union* $\Phi_1 \dot\cup \Phi_2$ *of complex unit graphs, we have* $\mathcal{L}(\Phi_1 \dot\cup \Phi_2) = \mathcal{L}(\Phi_1) \dot\cup \mathcal{L}(\Phi_2)$ *and* $\mathcal{S}(\Phi_1 \dot\cup \Phi_2) = \mathcal{S}(\Phi_1) \dot\cup \mathcal{S}(\Phi_2)$*, in a comparison of eigenspaces it would not be really restrictive to assume that* Φ*, and* a fortiori $\mathcal{L}(\Phi)$ *and* $\mathcal{S}(\Phi)$*, are connected; in fact, the spectral results can be extended component-wise to disconnected* $\mathbb{T}$*-gain graphs.*

2.3. The eigenspace $\mathcal{E}_A(-2, \mathcal{L}(\Phi))$.

Let $\Phi = (\Gamma, \gamma)$ be a nonempty k-cyclic complex unit gain graph, and let $\mathcal{L}(\Phi) = (\mathcal{L}(\Gamma), \gamma^{\mathcal{L}})$ be the associated line graph arising from a fixed incidence matrix H of Φ. We denote by $\widehat{\Phi}$ the unique minimal k-cyclic subgraph of Φ. The graph $\widehat{\Phi}$ is known as the *base* of Φ, and can be also characterized as the only k-cyclic subgraph of Φ with no pendant vertices.

In this section we assume that -2 belongs to $\mathrm{Spec}(A(\mathcal{L}(\Phi))$. As a consequence of Theorem 1.1(1), it is not hard to show that this is equivalent to require that Φ is neither a tree nor unbalanced unicyclic. The same fact can also be directly deduced from [2, Theorem 3.1].

The eigenspace $\mathcal{E}_A(-2, \mathcal{L}(\Phi))$ is described in every detail in [2, Section 3], and the reader is referred to it for all further information. Here, we extract just what is needed to make this paper reasonably self-contained.

It can be proved that there exists a spanning subgraph $\Psi = (\Lambda, \gamma_{|\vec{E}(\Lambda)})$ of Φ satisfying the following properties:

(i) $\mathcal{L}(\Psi)$ is connected;
(ii) $-2 \notin \mathrm{Spec}(A(\mathcal{L}(\Psi)))$;
(iii) $-2 \in \mathrm{Spec}(A(\mathcal{L}(\Psi_e)))$ for each $e \in E(\Gamma) \setminus E(\Lambda)$, where Ψ_e is a one-edge extension of Ψ, i.e. $\Psi_e = (\Lambda_e, \gamma_{|\vec{E}(\Lambda_e)})$, with $V(\Lambda_e) = V(\Lambda)$ and $E(\Lambda_e) = E(\Lambda) \cup \{e\}$

A spanning subgraph Ψ satisfying (i)-(iii) is known as a *complex unit foundation*. Complex unit foundations are either trees (if Φ is balanced) or unbalanced unicyclic graphs. Moreover, $\widehat{\Psi}_e$ can either be a balanced cycle, or an ∞-graph with two unbalanced cycles, or a dumbbell with two unbalanced cycles, (see Fig. 1). Recall that a dumbbell is a graph consisting of two disjoint cycles joined by a nontrivial path; whereas an ∞-graph consists of two cycles with just one vertex in common.

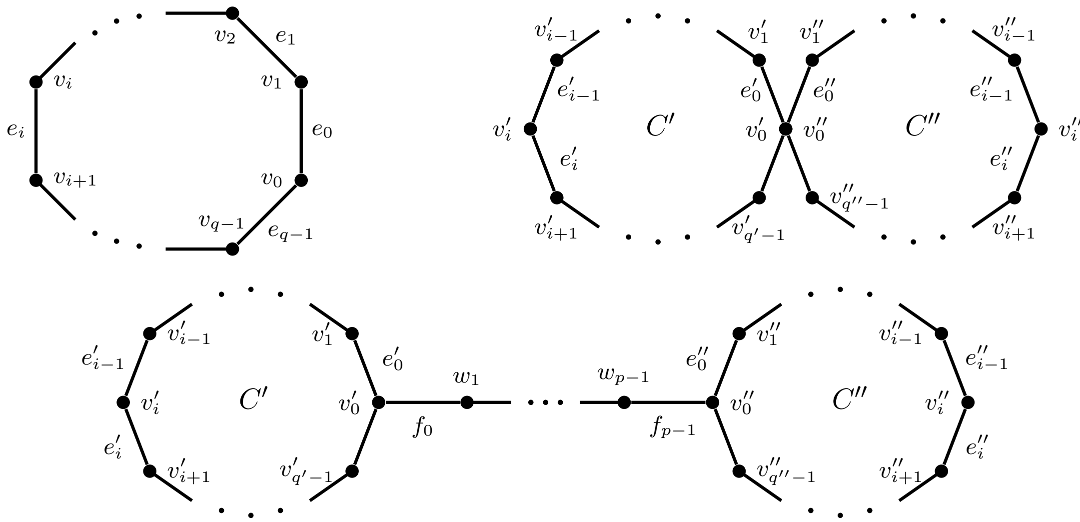

FIGURE 1. Vertex and edge labeling for the three types of cores Θ_e.

Let now $\mathbf{x_e}$ be a (-2)-eigenvector of $\mathcal{L}(\Psi_e)$. Clearly, each of its coordinates is labelled by a suitable edge in Ψ_e. Since the multiplicity of -2 in $\mathrm{Spec}(A(\mathcal{L}(\Psi_e)))$ is 1, every (-2)-eigenvector of $\mathcal{L}(\Psi_e)$ is proportional to $\mathbf{x_e}$; in particular, it shares with $\mathbf{x_e}$ the same nonzero versus zero pattern.

In view of the latter observation, we can distinguish two types of edges in Ψ_e. We say that an edge is *heavy* (resp. *light*) if the corresponding entry in $\mathbf{x_e}$ is nonzero (resp. zero). The unique subgraph Θ_e of Ψ_e induced by its heavy edges will be called the *core* of Ψ_e. It turns out that

(i) the core Θ_e is precisely the base $\widehat{\Psi}_e$ of Ψ_e; therefore, it can either be a balanced cycle, or an ∞-graph with two unbalanced cycles, or a dumbbell with two unbalanced cycles;
(ii) the edge e belongs to some cycle of $\Theta_e = \widehat{\Psi}_e$; hence, it is heavy for $\mathcal{L}(\Psi_e)$;
(iii) the vector $\mathbf{x_e}$ can be extended to a (-2)-eigenvector $\mathbf{y_e}$ of $A(\mathcal{L}(\Phi))$ by inserting zeros at the entries corresponding to edges in $E(\Gamma) \setminus E(\Lambda_e)$;
(iv) the set $\{\mathbf{y_e} \mid e \in E(\Gamma) \setminus E(\Lambda)\}$ forms a basis for $\mathcal{E}_A(-2, \mathcal{L}(\Phi))$.

The components of any nonzero generator of $\mathcal{E}_A(-2, \mathcal{L}(\Theta_e))$ are described by the following three theorems.

Theorem 2.11. [2, Theorem 3.5] *Let the core $\Theta_e = (C, \gamma_{|\vec{E}(C)})$ be a balanced cycle. After labeling the $q \geq 3$ vertices of C and its edges as in Fig. 1, a generator $\mathbf{a} = (a_0, a_1, \dots, a_{q-1})^\top$ of the -2-eigenspace of $A(\mathcal{L}(\Theta_e))$ is given by the formulæ*

$$a_i = (-1)^i \Big[\prod_{s=1}^{i} \overline{\nu(s)} \Big] a_0 \qquad \text{for } 1 \leq i \leq q-1 \quad \text{and} \quad a_0 \neq 0.$$

where the component a_i corresponds to the edge e_i, and

$$\nu(i) = \gamma^{\mathcal{L}}(e_{i-1}e_i) = \overline{\eta}_{ie_{i-1}} \eta_{ie_i} \in \mathbb{T} \qquad \text{for } 1 \leq i \leq q-1.$$

We now fix some notation to investigate the cases when the underlying graph of Θ_e consists of two cycles C' and C'' (of length q' and q'' respectively) joined by a path P of length $p \geq 0$. In literature, this bicyclic graph is often denoted by $B(q', p, q'')$ (see, for instance, [4, 13]). We label vertices and edges of Θ_e as follows:

$$V(C') = \{v'_0, \dots, v'_{q'-1}\}, \quad V(C'') = \{v''_0, \dots, v''_{q''-1}\},$$

$$E(C') = \{e'_i = v'_i v'_{i+1} \,|0 \leq i \leq q'-1\} \cup \{e'_{q'-1} = v'_{q'-1} v'_0\},$$

$$E(C'') = \{e''_i = v''_i v''_{i+1} \,|0 \leq i \leq q''-1\} \cup \{e''_{q''-1} = v''_{q''-1} v''_0\}.$$

If P is nontrivial, i.e. its length is $p > 0$, we assume that

$$V(P) = \{w_0, \dots, w_p\}, \qquad E(P) = \{f_i = w_i w_{i+1} \,|\, 0 \leq i \leq p-1\},$$

and its end-vertices w_0 and w_p are respectively identified with vertices $v'_0 \in V(C')$ and $v''_0 \in V(C'')$ (see Fig. 1).

Let $\mathbf{x}$ be a -2-eigenvector for $A(\mathcal{L}(\Theta_e))$. For convenience, we split its ordered set of components into two (resp. three) parts if $p = 0$ (resp. $p > 0$), each corresponding to its constituents C', P (if nontrivial) and C''. Namely, we write $\mathbf{x} = \mathbf{a}' \dotplus \mathbf{b} \dotplus \mathbf{a}''$ where $\mathbf{a}' = (a'_0, a'_1, \dots, a'_{q'-1})^\top$, $\mathbf{b} = (b_0, b_1, \dots, b_{p-1})^\top$ and $\mathbf{a}'' = (a''_0, a''_1, \dots, a''_{q''-1})^\top$, and the components a'_i, b_i and a''_i respectively correspond to the edges e'_i, f_i and e''_i. In the statements of Theorems 2.12 and 2.13, the following two directed cycles

$$\vec{C}' = e'_0 e'_1 \cdots e'_{q'-1} \qquad \text{and} \qquad \vec{C}'' = e''_0 e''_1 \cdots e''_{q''-1} \tag{2.3}$$

play an important role.

Theorem 2.12. [2, Theorem 3.7] *Let the core $\Theta_e = (B(q', 0, q''), \gamma_{|\vec{E}(B(q',0,q''))})$ be a complex unit ∞-graph with two unbalanced cycles. Under the above notation (see also Fig. 1), for each nonzero complex number a'_0, a generator $\mathbf{a}' \dotplus \mathbf{a}''$ of the -2-eigenspace of $A(\mathcal{L}(\Theta_e))$ is given by the formulæ*

$$a''_0 = -\left(1 - \overline{\gamma(\vec{C}'')}\right)^{-1} \left(1 - \overline{\gamma(\vec{C}')}\right) \gamma^{\mathcal{L}}(e''_0 e'_0) a'_0, \tag{2.4}$$

and

$$\begin{aligned} a'_i &= (-1)^i \Big[\prod_{s=1}^{i} \overline{\nu'(s)}\Big] a'_0 \qquad \textit{for } 1 \leq i \leq q'-1, \\ a''_i &= (-1)^i \Big[\prod_{s=1}^{i} \overline{\nu''(s)}\Big] a''_0 \qquad \textit{for } 1 \leq i \leq q''-1, \end{aligned}$$

where

$$\nu'(i) = \gamma^{\mathcal{L}}(e'_{i-1} e'_i) = \overline{\eta}_{ie'_{i-1}} \eta_{ie'_i} \in \mathbb{T} \qquad \textit{for } 1 \leq i \leq q'-1, \tag{2.5}$$

and

$$\nu''(i) = \gamma^{\mathcal{L}}(e''_{i-1} e''_i) = \overline{\eta}_{ie''_{i-1}} \eta_{ie''_i} \in \mathbb{T} \qquad \textit{for } 1 \leq i \leq q''-1. \tag{2.6}$$

Theorem 2.13. [2, Theorem 3.6] *Let the core $\Theta_e = (B(q',p,q''), \gamma_{|\vec{E}(B(q',p,q''))})$ be a complex unit dumbbell with two unbalanced cycles (hence, $p>0$). Under the above notation (see also Fig. 1), for each nonzero complex number b_0, a generator $\mathbf{a}'\dot{+}\mathbf{b}\dot{+}\mathbf{a}''$ of the -2-eigenspace of $A(\mathcal{L}(\Theta_e))$ is given by the formulæ*

$$a'_0 = -\left(1-\overline{\gamma(\vec{C}')}\right)^{-1}\gamma^{\mathcal{L}}(e'_0f_0)b_0, \qquad a''_0 = -\left(1-\overline{\gamma(\vec{C}'')}\right)^{-1}\gamma^{\mathcal{L}}(e''_0f_{p-1})b_{p-1}, \tag{2.7}$$

and

$$a'_i = (-1)^i\Big[\prod_{s=1}^{i}\overline{\nu'(s)}\Big]a'_0 \qquad \text{for } 1\le i\le q'-1, \tag{2.8}$$

$$b_i = (-1)^i\Big[\prod_{s=1}^{i}\overline{\nu(s)}\Big]b_0 \qquad \text{for } 1\le i\le p-1 \quad \text{and} \quad b_0\neq 0, \tag{2.9}$$

$$a''_i = (-1)^i\Big[\prod_{s=1}^{i}\overline{\nu''(s)}\Big]a''_0 \qquad \text{for } 1\le i\le q''-1, \tag{2.10}$$

where the $\nu'(i)$'s and the $\nu''(i)$'s satisfy (2.5) and (2.6), and

$$\nu(i) = \gamma^{\mathcal{L}}(f_{i-1}f_i) = \overline{\eta}_{if_{i-1}}\eta_{if_i} \in \mathbb{T} \qquad \text{for } 1\le i\le p-1.$$

The geometric procedure described in [7, Remark 3.6] to quickly identify ‘heavy’ subgraphs of a signed graph can be improved and employed in a complex unit gain context. Suppose that Φ is k-cyclic and $-2\in \operatorname{Spec}(A(\mathcal{L}(\Phi))$. Theorem 1.1 and Corollary 2.4 yield

$$m_{A(\mathcal{L}(\Phi))}(-2) = \begin{cases} k & \text{if } \Phi \text{ is balanced;}\\ k-1 & \text{if } \Phi \text{ is unbalanced.}\end{cases}$$

We first pick k independent cycles $\Theta_i = (\Lambda_i, \gamma_{|\vec{E}(\Lambda_i)})$ for $1\le i\le k$, choosing them in order to get as many balanced cycles as possible. This implies that each pair of chosen unbalanced cycles (if any) is edge-disjoint.

If Φ is balanced, then all Θ_i's are neutral. A foundation Ψ can be obtained from Φ by removing k pairwise distinct edges $e_1,\dots,e_k$ with $e_i\in\Lambda_i$. For $1\le i\le k$, the core of Ψ_{e_i} is precisely Θ_i. Thus, we form an eigenbasis for $\mathcal{E}_A(-2,\mathcal{L}(\Phi))$ by considering the vectors $\mathbf{y}_{\mathbf{e_i}}$ $(1\le i\le k)$ built as explained above from a generator $\mathbf{x}_{\mathbf{e_i}}$ of $\mathcal{E}_A(-2,\mathcal{L}(\Psi_{e_i}))$. The components of $\mathbf{x}_{\mathbf{e_i}}$ are given in Theorem 2.11.

If Φ is unbalanced, it not restrictive to assume that the unbalanced graphs among the Θ_i's are the first $\ell>0$. If $\ell\ge 2$, for $2\le i\le \ell$ we denote by Ξ_i the minimal connected subgraph of Φ containing Θ_1 and Θ_i. Each Ξ_i is either an ∞- or a dumbbell complex unit gain graph. A set of $k-1$ heavy subgraphs giving rise to an eigenbasis for $\mathcal{E}_A(-2,\mathcal{L}(\Phi))$ is

$$\begin{array}{ll} \{\Theta_i \mid 2\le i\le k\} & \text{if } \ell=1;\\ \{\Xi_2,\dots,\Xi_\ell,\Theta_{\ell+1},\dots,\Theta_k\} & \text{if } 1<\ell<k;\\ \{\Xi_i \mid 2\le i\le k\} & \text{if } \ell=k.\end{array}$$

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and the nonzero components of the vector in the eigenbasis corresponding to each Ξ_i can be computed, as done in [2], by Theorems 2.12 and 2.13.

3. Eigenspaces of $A(\Phi)$ and $A(\mathcal{L}(\Phi))$: a comparison

In this section, we fix once for all an incidence matrix H of the $\mathbb{T}$-gain graph Φ, and consider the graph $\mathcal{L}(\Phi) := \mathcal{L}^1_{\mathrm{H}}(\Phi)$, i.e. the complex unit line graph determined by H (see Section 2.2).

By definition, the following two equalities hold (see the proofs of [1, Proposition 4 and Theorem 1] if needed):

$$\mathrm{HH}^* = L(\Phi) \qquad \text{and} \qquad \mathrm{H}^*\mathrm{H} = 2I + A(\mathcal{L}(\Phi)). \tag{3.1}$$

By Theorem 1.1, it is immediately seen that the map

$$\mu \in \mathrm{Spec}(L(\Phi)) \setminus \{0\} \longmapsto \mu - 2 \in \mathrm{Spec}(A(\mathcal{L}_{\mathrm{H}}(\Phi))) \setminus \{-2\}$$

is bijective. Moreover, for $\mu > 0$, the eigenspaces $\mathcal{E}_L(\mu, \Phi)$ and $\mathcal{E}_A(\mu - 2, \mathcal{L}(\Phi))$ are isomorphic. The following lemma shows how the matrix H allows to define isomorphisms in both directions.

Lemma 3.1. *Let μ a nonzero element in* $\mathrm{Spec}(L(\Phi))$*. The maps*

$$\alpha_{\mathrm{H}^*} : \mathbf{x} \in \mathcal{E}_L(\mu, \Phi) \longmapsto \mathrm{H}^*\mathbf{x} \in \mathcal{E}_A(\mu - 2, \mathcal{L}(\Phi)) \tag{3.2}$$

and

$$\alpha_{\mathrm{H}} : \mathbf{y} \in \mathcal{E}_A(\mu - 2, \mathcal{L}(\Phi)) \longmapsto \mathrm{H}\mathbf{y} \in \mathcal{E}_L(\mu, \Phi) \tag{3.3}$$

are both isomorphisms.

Proof. Let $\mathbf{x} \in \mathcal{E}_L(\mu, \Phi)$. We first prove that $\mathbf{y} = \mathrm{H}^*\mathbf{x}$ actually belongs to $\mathcal{E}_A(\mu - 2, \mathcal{L}(\Phi))$. By the two equalities of (3.1), we respectively obtain

$$\mathrm{H}\mathbf{y} = \mathrm{HH}^*\mathbf{x} = L(\Phi)\mathbf{x} = \mu\mathbf{x} \tag{3.4}$$

and

$$A(\mathcal{L}(\Phi))\mathbf{y} = \mathrm{H}^*\mathrm{H}\mathbf{y} - 2\mathbf{y} = \mathrm{H}^*\mathrm{H}(\mathrm{H}^*\mathbf{x}) - 2\mathbf{y} = \mathrm{H}^*(\mu\mathbf{x}) - 2\mathbf{y} = (\mu - 2)\mathbf{y}.$$

Since $\mu \neq 0$, we also deduce from (3.4) that $\mathbf{x} \neq \mathbf{0}$ implies $\mathbf{y} \neq \mathbf{0}$. In other words, the map α_{H^*} is a monomorphism between two vector spaces of same dimension, and thence an isomorphism.

Knowing that α_{H^*} is an isomorhism, the fact that α_{H} bijectively maps $\mathcal{E}_A(\mu-2, \mathcal{L}(\Phi))$ onto $\mathcal{E}_L(\mu, \Phi)$ comes from the equality $\alpha_{\mathrm{H}} \circ \alpha_{\mathrm{H}^*} = \mu \mathbf{1}_{L(\Phi)}$ (recall that $\mu \neq 0$). ☐

With the aid Lemma 3.1, the proof of the following theorem becomes straightforward.

Theorem 3.2. *Let μ a nonzero element in* $\mathrm{Spec}(L(\Phi))$*.*

(i) $\{\mathbf{x_1}, \ldots, \mathbf{x_s}\}$ *is a basis for the eigenspace $\mathcal{E}_L(\mu, \Phi)$ if and only if* $\{\mathrm{H}^*\mathbf{x_1}, \ldots, \mathrm{H}^*\mathbf{x_s}\}$ *is a basis for the eigenspace $\mathcal{E}_A(\mu - 2, \mathcal{L}(\Phi))$.*

(ii) $\{\mathbf{y_1},\dots,\mathbf{y_s}\}$ *is a basis for the eigenspace* $\mathcal{E}_A(\mu-2,\mathcal{L}(\Phi))$ *if and only if* $\{\mathrm{H}\mathbf{y_1},\dots,\mathrm{H}\mathbf{y_s}\}$ *is a basis for the eigenspace* $\mathcal{E}_L(\mu,\Phi)$.

The next proposition suggests how to obtain an eigenbasis for $\mathcal{E}_L(0,\Phi)$ when $0\in\mathrm{Spec}(L(\Phi))$.

Proposition 3.3. *Let the* $\mathbb{T}$*-gain graph* Φ *be balanced and connected, and let* $\xi: V(\Gamma)\longrightarrow\mathbb{T}$ *be the switching function such that* $\Phi^\xi=(\Gamma,1)$*. Then, the* 1*-dimensional eigenspace* $\mathcal{E}_L(0,\Phi)$ *is generated by* $\mathbf{w}^\top=(\xi(v_1),\xi(v_2),\dots,\xi(v_n))$.

Proof. Corollary 2.4 implies that $m_{L(\Phi)}(0)=1$, and by definition of ξ we see that $\xi(v_i)=\gamma(e_{ij})\xi(v_j)$. Now, the i-th element of the column vector $L(\Phi)\mathbf{w}$ is given by

$$d(v_i)\xi(v_i)-\sum_{v_h\sim v_i}\gamma(e_{ih})\xi(v_h)=d(v_i)\xi(v_i)-d(v_i)\xi(v_i)=0.$$

Hence, $L(\Phi)\mathbf{w}=0\mathbf{w}$ as claimed. □

The proof of the following corollary easily comes from Proposition 3.3.

Corollary 3.4. *Let* $\Phi=(\Gamma,\gamma)$ *be a complex unit gain graph with at least one balanced component. Denoted by* $\Lambda_1,\dots,\Lambda_r$ *the balanced connected components of* Γ*, and by* $\xi: V(\Gamma)\longrightarrow\mathbb{T}$ *a switching function such that all edges of* Φ^ξ *in* $\overrightarrow{E}(\cup_{i=1}^r\Lambda_i)$ *are neutral, an eigenbasis of* $\mathcal{E}_L(0,\Phi)$ *is given by* $\{\mathbf{w_1},\dots,\mathbf{w_r}\}$*, where*

$$\mathbf{w_i}=(w_{iv})_{v\in V(\Gamma)},\qquad \textit{with}\qquad w_{iv}=\begin{cases}\xi(v) & \textit{if } v\in\Lambda_i;\\ 0 & \textit{otherwise.}\end{cases}$$

When Φ is balanced, it makes sense to ask whether, for $\mu=0$, the maps α_{H^*} and α_{H} defined in (3.2) and (3.3) map eigenvectors onto eigenvectors. The answer is trivially negative when Φ is a forest, since in this case $-2\notin\mathrm{Spec}_A(\mathcal{L}(\Phi))$. Proposition 3.5 shows that the answer is negative in all cases.

Proposition 3.5. *Let the* $\mathbb{T}$*-gain graph* Φ *be balanced. Then, the maps*

$$\alpha_{\mathrm{H}^*}:\mathbf{x}\in\mathcal{E}_L(0,\Phi)\longmapsto \mathrm{H}^*\mathbf{x}\in\mathcal{E}_A(-2,\mathcal{L}(\Phi))$$

and

$$\alpha_{\mathrm{H}}:\mathbf{y}\in\mathcal{E}_A(-2,\mathcal{L}(\Phi))\longmapsto \mathrm{H}\mathbf{y}\in\mathcal{E}_L(0,\Phi)$$

are both null.

Proof. Let $\mathbf{z}$ be any element in $\mathcal{E}_L(0,\Phi)$. By multiplying the extremal sides of

$$\mathrm{H}\mathrm{H}^*\mathbf{z}=L(\Phi)\mathbf{z}=\mathbf{0}$$

by $\mathbf{z}^*$, it turns out that the norm of $\mathrm{H}^*\mathbf{z}$ is null, and this is possible only if $\mathrm{H}^*\mathbf{z}=\mathbf{0}$. Hence, α_{H^*} is null.

The proof of the nullity of α_{H} in nontrivial cases, i.e. when $-2\in\mathrm{Spec}_A(\mathcal{L}(\Phi))$, is slightly more complicated. It relies on Theorems 2.11-2.13. Indeed, we have to distinguish three cases. Using notation and terminology of Section 2.3, let $\Psi = (\Lambda, \gamma_{|\vec{E}(\Lambda)})$ be a complex unit foundation of any connected component $(\Omega, \gamma_{|\vec{E}(\Omega)})$ of $\Phi = (\Gamma, \gamma)$, and, for each $e \in E(\Omega) \setminus E(\Lambda)$, let Θ_e denote the core of Ψ_e. We need to show that $\mathrm{H}\mathbf{y_e} = \mathbf{0}$, where $\mathbf{y_e} \in \mathcal{E}_A(-2, \mathcal{L}(\Phi))$ is obtained from the generator $\mathbf{x_e}$ of $\mathcal{E}_A(-2, \mathcal{L}(\Psi_e))$ by inserting zeroes at the entries corresponding to edges in $E(\Gamma) \setminus E(\Lambda_e)$.

Case 1: Θ_e is a balanced cycle. We label the edges of Θ_e as done for the graph at the top-left corner of Fig. 1, and consider the directed cycle $\vec{C} = e_0 e_1 \cdots e_{q-1}$. The nonzero components of $\mathbf{y_e}$ are computed in Theorem 2.11. The only possible nontrivial components of $\mathrm{H}\mathbf{y_e}$ are in correspondence of vertices in $\{v_0, \ldots, v_{q-1}\}$. The v_0-entry of $\mathrm{H}\mathbf{y_e}$ is

$$\begin{aligned}\eta_{v_0 e_0} a_0 + \eta_{v_0 e_{q-1}} a_{q-1} &= \eta_{v_0 e_0} a_0 + \eta_{v_0 e_{q-1}} (-1)^{q-1} \Big[\textstyle\prod_{s=1}^{q-1} \overline{\nu(s)} \Big] a_0 \\ &= \eta_{v_0 e_0} a_0 + \eta_{v_0 e_{q-1}} (-1)^{q-1} (-1)^q \nu(0) a_0 \\ &= \eta_{v_0 e_0} a_0 - \eta_{v_0 e_{q-1}} \overline{\eta_{v_0 e_{q-1}}} \eta_{v_0 e_0} a_0 \\ &= 0.\end{aligned}$$

For the second equality we have used the identity

$$\prod_{s=1}^{q-1} \overline{\nu(s)} = (-1)^q \nu(0) \overline{\gamma(\vec{C})} \tag{3.5}$$

obtained from $\prod_{s=0}^{q-1} \overline{\nu(s)} = (-1)^q \overline{\gamma(\vec{C})}$ by multiplying both sides by $\nu(0) = \overline{\eta}_{v_0 e_{q-1}} \eta_{v_0 e_0}$ (recall that, in this case, $\gamma(\vec{C}) = 1$, the graph Θ_e being balanced).

The v_i-entry of $\mathrm{H}\mathbf{y_e}$ for $i > 0$ is instead given by

$$\begin{aligned}\eta_{v_i e_{i-1}} a_{i-1} + \eta_{v_i e_i} a_i &= \eta_{v_i e_{i-1}} (-1)^{i-1} \Big[\textstyle\prod_{s=1}^{i-1} \overline{\nu(s)} \Big] a_0 + \eta_{v_i e_i} (-1)^i \Big[\textstyle\prod_{s=1}^{i} \overline{\nu(s)} \Big] a_0 \\ &= (-1)^{i-1} \Big[\textstyle\prod_{s=1}^{i-1} \overline{\nu(s)} \Big] \Big[\eta_{v_i e_{i-1}} - \overline{\nu(i)} \eta_{v_i e_i} \Big] a_0 \\ &= (-1)^{i-1} \Big[\textstyle\prod_{s=1}^{i-1} \overline{\nu(s)} \Big] \Big[\eta_{v_i e_{i-1}} - \eta_{v_i e_{i-1}} \overline{\eta}_{v_i e_i} \eta_{v_i e_i} \Big] a_0 \\ &= 0.\end{aligned}$$

It turns out that $\mathrm{H}\mathbf{y_e} = \mathbf{0}$ as claimed.

Case 2: Θ_e is a complex unit ∞-graph. We label the edges of Θ_e as done for the graph at the top-right corner of Fig. 1, and consider the directed cycles $\vec{C}'$ and $\vec{C}''$ as in (2.3). We only need to prove that the entry c of $\mathrm{H}\mathbf{y_e}$ corresponding to $v_0 := v_0' = v_0''$ vanishes (the argument for the nullity of the entries corresponding to the other vertices of Θ_e is given in Case 1). By definition of H and by Theorem 2.12, we have

$$\begin{aligned}c &= \eta_{v_0 e_0'} a_0' + \eta_{v_0 e_{q'-1}'} a_{q'-1}' + \eta_{v_0 e_0''} a_0'' + \eta_{v_0 e_{q''-1}''} a_{q''-1}'' \\ &= \Big[\eta_{v_0 e_0'} + \eta_{v_0 e_{q'-1}'} (-1)^{q'-1} \textstyle\prod_{s=1}^{q'-1} \overline{\nu'(s)} \Big] a_0' + \Big[\eta_{v_0 e_0''} + \eta_{v_0 e_{q''-1}''} (-1)^{q''-1} \textstyle\prod_{s=1}^{q''-1} \overline{\nu''(s)} \Big] a_0''\end{aligned}$$

Once we plug (2.4) in such expression, together with $\gamma^{\mathcal{L}}(e_0'' e_0') = \overline{\eta}_{v_0 e_0''} \eta_{v_0 e_0'}$ and

$$\prod_{s=1}^{q'-1} \overline{\nu'(s)} = (-1)^{q'} \overline{\eta}_{v_0' e_{q'-1}'} \eta_{v_0' e_0'} \overline{\gamma(\vec{C}')} \quad \text{and} \quad \prod_{s=1}^{q''-1} \overline{\nu''(s)} = (-1)^{q''} \overline{\eta}_{v_0'' e_{q''-1}''} \eta_{v_0'' e_0''} \overline{\gamma(\vec{C}'')} \tag{3.6}$$

obtained by specializing (3.5) to $\vec{C}'$ and $\vec{C}''$ respectively, an elementary algebraic manipulation shows that $c = 0$.

Case 3: Θ_e is a complex unit dumbbell. We label the edges of Θ_e as done for the graph at the bottom of Fig. 1, and once again consider the directed cycles $\vec{C}'$ and $\vec{C}''$ as in (2.3). Arguing as in Case 1, we deduce that it is zero each entry of $\mathrm{H}\mathbf{y_e}$ corresponding to vertices of degree 2 in $C' \cup C''$. Therefore, we only need to prove that the entries

$$c_{v_0'} = \eta_{v_0'e_0'}a_0' + \eta_{v_0'e_{q'-1}'}a_{q'-1}' + \eta_{v_0'f_0}b_0,$$
$$c_{w_i} = \eta_{w_if_{i-1}}b_{i-1} + \eta_{w_if_i}b_i \qquad (1 \le i \le p-1)$$

and

$$c_{v_0''} = \eta_{v_0''f_{p-1}}b_{p-1} + \eta_{v_0''e_0''}a_0'' + \eta_{v_0''e_{q''-1}''}a_{q''-1}''$$

of $\mathrm{H}\mathbf{y_e}$ respectively corresponding to v_0', w_i $(0 \le i \le p-1)$ and v_0'' vanish. This can be done by suitably plugging in the three equalities above (2.7)-(2.10) together with (3.6). □

4. Eigenspaces of $A(\mathcal{S}(\Phi))$

As in Section 3, we fix a particular incidence matrix H of the $\mathbb{T}$-gain graph Φ, and consider the graphs $\mathcal{L}(\Phi) := \mathcal{L}^1_{\mathrm{H}}(\Phi)$ and $\mathcal{S}(\Phi) = \mathcal{S}_{\mathrm{H}}(\Phi)$. The latter is the complex unit subdivision graph determined by H defined in Section 2.2. Once again, we choose for $V(\mathcal{S}(\Gamma))$ the ordering $\{v_1, \dots, v_n, e_1, \dots, e_m\}$. Moreover, for every $n+m$ vector $\mathbf{z}$, we denote by $\mathbf{x} = \mathbf{z}(V_1)$ (resp. $\mathbf{y} = \mathbf{z}(V_2)$) the projection of $\mathbf{z}$ onto its first n-components (resp. its last m components). Thus, we can write $\mathbf{z} = \mathbf{x}\dot{+}\mathbf{y}$.

The first of the following list of lemmas immediately comes from the definition of $A(\mathcal{S}(\Phi))$.

Lemma 4.1. *Let $\mathbf{z} := \mathbf{x}\dot{+}\mathbf{y}$, where $\mathbf{x} = \mathbf{z}(V_1)$ and $\mathbf{y} = \mathbf{z}(V_2)$. The following conditions are equivalent.*

(1) $\mathrm{H}\mathbf{y} = \hat{\lambda}\mathbf{x}$ and $\mathrm{H}^*\mathbf{x} = \hat{\lambda}\mathbf{y}$.
(2) The vector $\mathbf{z}$ belongs to $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$.
(3) The vector $\mathbf{x}\dot{+}(-\mathbf{y})$ belongs to $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$.

Lemma 4.2. *Let $\hat{\lambda}$ be any real number, and let $\mathbf{z} = \mathbf{x}\dot{+}\mathbf{y} \in \mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$, with $\mathbf{x} = \mathbf{z}(V_1)$ and $\mathbf{y} = \mathbf{z}(V_2)$. Then,*

$$L(\Phi)\mathbf{x} = \hat{\lambda}^2\mathbf{x} \qquad \text{and} \qquad A(\mathcal{L}(\Phi))\mathbf{y} = (\hat{\lambda}^2 - 2)\mathbf{y}.$$

Proof. Clearly, $A^2(\mathcal{S}(\Gamma))\mathbf{z} = \hat{\lambda}^2\mathbf{z}$. From (2.2), we easily compute

$$A^2(\mathcal{S}(\Gamma)) = \begin{pmatrix} \mathrm{H}\mathrm{H}^* & O_{n\times m} \\ O_{m\times n} & \mathrm{H}^*\mathrm{H} \end{pmatrix} = \begin{pmatrix} L(\Phi) & O_{n\times m} \\ O_{m\times n} & A(\mathcal{L}(\Phi)) + 2I_m \end{pmatrix} := L(\Phi)\dot{+}(A(\mathcal{L}(\Phi)) + 2I_m).$$

It is now clear that $L(\Phi)\mathbf{x} = \hat{\lambda}^2\mathbf{x}$ and $(A(\mathcal{L}(\Phi)) + 2I_m)\mathbf{y} = \hat{\lambda}^2\mathbf{y}$. □

Lemma 4.3. *Let $S = \{\mathbf{z_i} := \mathbf{x_i}\dot{+}\mathbf{y_i} \mid 1 \le i \le k\}$ and $S' = \{\mathbf{z_i}' := \mathbf{x_i}\dot{+}(-\mathbf{y_i}) \mid 1 \le i \le k\}$ be two subsets of $\mathbb{C}^{n+m}$, where $\mathbf{x_i} = \mathbf{z_i}(V_1)$ and $\mathbf{y} = \mathbf{z_i}(V_2)$. If $\hat{\lambda}$ is a nonzero real number, then S spans (resp. is a basis of) $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$ if and only if S' spans (resp. is a basis of) $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$.*

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Proof. By Theorem 1.1(2), $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$ and $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$ are isomorphic, and by elementary matrix theory $\mathrm{Span}(S)$ and $\mathrm{Span}(S')$ are isomorphic as well. The statement now comes from Lemma 4.1. □

Lemma 4.4. *Let $\mathbf{z} := \mathbf{x} \dot{+} \mathbf{y} \in \mathbb{C}^{n+m}$, where $\mathbf{x} = \mathbf{z}(V_1)$ and $\mathbf{y} = \mathbf{z}(V_2)$. If $\hat{\lambda} \neq 0$ and $\mathbf{z}$ is a (nonzero) eigenvector of $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$, then $\mathbf{x}$ and $\mathbf{y}$ are both nonzero vectors.*

Proof. Let us write $\mathbf{x} = (x_v)_{v \in V(\Gamma)}$ and $\mathbf{y} = (y_e)_{e \in E(\Gamma)}$. Given any pair $(u, f) \in V(\Gamma) \times E(\Gamma)$, the eigenvalue equations at the u-row and the f-row of $A(\mathcal{S}(\Phi))$ yield

$$x_u = \hat{\lambda}^{-1} \left(\sum_{e \in E(\Gamma)} \eta_{ue} y_e \right) \qquad \text{and} \qquad y_f = \hat{\lambda}^{-1} \left(\sum_{v \in V(\Gamma)} \overline{\eta}_{vf} x_v \right),$$

implying that $\mathbf{x} = \mathbf{0} \iff \mathbf{y} = \mathbf{0}$. But $\mathbf{x} \dot{+} \mathbf{y}$ is by hypothesis an eigenvector; hence, $\mathbf{x}$ and $\mathbf{y}$ are both nonzero as claimed. □

Given a set $S = \{\mathbf{z_1} = \mathbf{x_1} \dot{+} \mathbf{y_1}, \dots, \mathbf{z_k} = \mathbf{x_k} \dot{+} \mathbf{y_k}\} \subset \mathbb{C}^{n+m}$, where $\mathbf{x_i} = \mathbf{z_i}(V_1)$ and $\mathbf{y_i} = \mathbf{z_i}(V_2)$ for each $i \in \{1, \dots, k\}$, we introduce the following notation:

$$S(V_1) := \{\mathbf{x_1}, \dots, \mathbf{x_k}\} \subset \mathbb{C}^n \qquad \text{and} \qquad S(V_2) := \{\mathbf{y_1}, \dots, \mathbf{y_k}\} \subset \mathbb{C}^m.$$

The next theorem explains how to quickly determine the L-eigenspaces of Φ and the A-eigenspaces of $\mathcal{L}(\Phi)$ from the A-eigenspaces of $\mathcal{S}(\Phi)$.

Theorem 4.5. *Let $\hat{\lambda} \in \mathrm{Spec}(A(\mathcal{S}(\Phi))$, and let $B = \{\mathbf{z_1} = \mathbf{x_1} \dot{+} \mathbf{y_1}, \dots, \mathbf{z_k} = \mathbf{x_k} \dot{+} \mathbf{y_k}\}$ be an eigenbasis for $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$, where $\mathbf{x_i} = \mathbf{z_i}(V_1)$ and $\mathbf{y_i} = \mathbf{z_i}(V_2)$ for each $i \in \{1, \dots, k\}$.*

(1) If $\hat{\lambda} \neq 0$, then $B(V_1)$ is an eigenbasis for $\mathcal{E}_L(\hat{\lambda}^2, \Phi)$, and $B(V_2)$ is an eigenbasis for $\mathcal{E}_A(\hat{\lambda}^2 - 2, \mathcal{L}(\Phi))$.

(2) If $\hat{\lambda} = 0$, then $B(V_1)$ spans $\mathcal{E}_L(0, \Phi)$, and $B(V_2)$ spans $\mathcal{E}_A(-2, \mathcal{L}(\Phi))$.

Proof. By Lemma 4.2, surely $B(V_1) \subset \mathcal{E}_L(\hat{\lambda}^2, \Phi)$ and $B(V_2) \subset \mathcal{E}_A(\hat{\lambda}^2 - 2, \mathcal{L}(\Phi))$.

In order to prove (1) it suffices to show the $B(V_1)$ and $B(V_2)$ are linearly independent when $\hat{\lambda} \neq 0$; in fact, if this is the case, the eigenspaces $\mathcal{E}_L(\hat{\lambda}^2, \Phi)$ and $\mathcal{E}_A(\hat{\lambda}^2 - 2, \mathcal{L}(\Phi))$ have both dimension k by Theorem 1.1(2).

Thus, suppose $\sum_{i=1}^k \alpha_i \mathbf{x_i} = \mathbf{0}$ for some $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$. Since $\sum_{i=1}^k \alpha_i \mathbf{z_i}$ belongs to $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$, we infer from Lemma 4.4 that $\sum_{i=1}^k \alpha_i \mathbf{z_i} = \mathbf{0}$, hence all the α_i's are null, the set B being independent. The linear independence of $B(V_2)$ is proved similarly.

We now prove (2). Let $\hat{\lambda} = 0$. We already observed at the beginning of the proof that

$$\mathrm{Span}(B(V_1)) \subseteq \mathcal{E}_L(0, \Phi) \qquad \text{and} \qquad \mathrm{Span}(B(V_2)) \subseteq \mathcal{E}_A(-2, \mathcal{L}(\Phi)). \tag{4.1}$$

We now show that in (4.1) the equalities hold. Pick any pair $(\mathbf{x}, \mathbf{y}) \in \mathcal{E}_L(0, \Phi) \times \mathcal{E}_A(-2, \mathcal{L}(\Phi))$. By Proposition 3.5, the vectors $\mathrm{H}^*\mathbf{x}$ and $\mathrm{H}\mathbf{y}$ are both null. Therefore, Condition (1) of Lemma 4.1 holds for $\mathbf{z} = \mathbf{x} \dot{+} \mathbf{y}$ and $\hat{\lambda} = 0$. Now, Lemma 4.1 ensures that its Condition (2) holds as well for $\mathbf{z}$; in other words $\mathbf{x} \dot{+} \mathbf{y}$ belongs to $\mathcal{E}_A(0, \mathcal{S}(\Phi))$. Thus, $\mathbf{x} \dot{+} \mathbf{y} = \sum_{i=1}^k \beta_i (\mathbf{x_i} \dot{+} \mathbf{y_i})$ for a suitable $(\beta_1, \dots, \beta_k) \in \mathbb{C}^k$. This proves that $\mathbf{x} \in \mathrm{Span}(B(V_1))$ and $\mathbf{y} \in \mathrm{Span}(B(V_2))$, and the proof is complete. □

Remark 4.6. *If the gains of Φ are in $\{\pm 1\} \subset \mathbb{T}$, the complex unit gain graph Φ can be regarded as a signed graph; hence,* [7, Theorem 3.7] *becomes a consequence of our Theorem 4.5. Yet, our line of attack does not need the orthogonality assumption for the basis B, which is instead decisive in* [7, Section 3].

The next result could be obtained from Theorem 1.1, but the required case analysis is quite annoying. As we see in a moment, Theorem 4.5 provides instead a much more direct argument.

Corollary 4.7. *If 0 belongs to $\mathrm{Spec}(A(\mathcal{S}(\Phi)))$ then either 0 belongs to $\mathrm{Spec}(L(\Phi))$ or -2 belongs to $\mathrm{Spec}(A(\mathcal{L}(\Phi)))$.*

Proof. Let $B = \{\mathbf{z_1} = \mathbf{x_1} \dot{+} \mathbf{y_1}, \ldots, \mathbf{z_k} = \mathbf{x_k} \dot{+} \mathbf{y_k}\}$ be an eigenbasis for $\mathcal{E}_A(0, \mathcal{S}(\Phi))$, where $\mathbf{x_i} = \mathbf{z_i}(V_1)$ and $\mathbf{y_i} = \mathbf{z_i}(V_2)$ for each $i \in \{1, \ldots, k\}$. Clearly, at least one set between $B(V_1)$ and $B(V_2)$ contains a nonzero vector. Our statement now comes from Theorem 4.5(2). □

Compared to Theorem 4.5, the next three results go the other way around. In fact, eigenbases of $A(\mathcal{S}(\Phi))$ are obtained from those of $L(\Phi)$ and $A(\mathcal{L}(\Phi))$.

Theorem 4.8. *Let $B = \{\mathbf{x_1}, \ldots, \mathbf{x_k}\}$ be a basis of the eigenspace $\mathcal{E}_L(\mu, \Phi)$ for $\mu \neq 0$, and let $\hat{\lambda} = \sqrt{\mu}$. Then, the sets*

$$\hat{\lambda} B \dot{+} \mathrm{H}^*(B) := \{\hat{\lambda}\, \mathbf{x_1} \dot{+} \mathrm{H}^* \mathbf{x_1}, \ldots, \hat{\lambda}\, \mathbf{x_k} \dot{+} \mathrm{H}^* \mathbf{x_k}\}$$

and

$$-\hat{\lambda} B \dot{+} \mathrm{H}^*(B) := \{(-\hat{\lambda}\, \mathbf{x_1}) \dot{+} \mathrm{H}^* \mathbf{x_1}, \ldots, (-\hat{\lambda}\, \mathbf{x_k}) \dot{+} \mathrm{H}^* \mathbf{x_k}\}$$

are a basis for $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$ and $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$ respectively.

Proof. First of all, set $\mathbf{X_i} = \hat{\lambda}\, \mathbf{x_i}$ and $\mathbf{Y_i} = \mathrm{H}^* \mathbf{x_i}$ for $1 \leq i \leq k$. We see that

$$\mathrm{H}\, \mathbf{Y_i} = \mathrm{H}\mathrm{H}^*\, \mathbf{x_i} = L(\Phi)\, \mathbf{x_i} = \hat{\lambda}^2\, \mathbf{x_i} = \hat{\lambda}\, \mathbf{X_i} \qquad \text{and} \qquad \mathrm{H}^*\, \mathbf{X_i} = \hat{\lambda}\, \mathbf{Y_i}.$$

By Lemma 4.1, this implies that $\hat{\lambda} B \dot{+} \mathrm{H}^*(B)$ is a subset of $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$. Since, by elementary matrix theory, $\hat{\lambda} B \dot{+} \mathrm{H}^*(B)$ is linearly independent and $\dim_{\mathbb{C}}(\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))) = k$ by Theorem 1.1, the set $\hat{\lambda} B \dot{+} \mathrm{H}^*(B)$ is actually a basis for $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$.

The argument to prove that $-\hat{\lambda} B \dot{+} \mathrm{H}^*(B)$ is a basis of $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$ is analogous. □

Theorem 4.9. *Let $B' = \{\mathbf{y_1}, \ldots, \mathbf{y_k}\}$ be a basis of the eigenspace $\mathcal{E}_A(\lambda, \mathcal{L}(\Phi))$ for $\lambda \neq -2$, and let $\hat{\lambda} = \sqrt{\lambda + 2}$. Then, the sets*

$$\mathrm{H}(B') \dot{+} \hat{\lambda} B' := \{\mathrm{H}\, \mathbf{y_1} \dot{+} \hat{\lambda}\, \mathbf{y_1}, \ldots, \mathrm{H}\, \mathbf{y_k} \dot{+} \hat{\lambda}\, \mathbf{y_k}\}$$

and

$$-B' \dot{+} \mathrm{H}^*(B') := \{(-\mathrm{H}\, \mathbf{y_1}) \dot{+} \hat{\lambda}\, \mathbf{y_1}, \ldots, (-\mathrm{H}\, \mathbf{y_k}) \dot{+} \hat{\lambda}\, \mathbf{y_k}\}$$

are a basis for $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$ and $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$ respectively.

Proof. We set $\mathbf{X_i} = \mathrm{H}\,\mathbf{y_i}$ and $\mathbf{Y_i} = \hat{\lambda}\,\mathbf{y_i}$ for $1 \leq i \leq k$. Now,

$$\mathrm{H}\,\mathbf{Y_i} = \mathrm{H}\hat{\lambda}\mathbf{y_i} = \hat{\lambda}\mathbf{X_i} \qquad \text{and} \qquad \mathrm{H}^*\,\mathbf{X_i} = \mathrm{H}^*\mathrm{H}\,\mathbf{y_i} = A(\mathcal{L}(\Phi) + 2I)\mathbf{y_i} = (\lambda + 2)\,\mathbf{y_i} = \hat{\lambda}\mathbf{Y_i}.$$

As in the previous proof, by Lemma 4.1 and elementary matrix theory, $\mathrm{H}(B')\dot{+}\hat{\lambda}B'$ is an independent set in $\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi))$. Since its cardinality is $\dim_{\mathbb{C}}(\mathcal{E}_A(\hat{\lambda}, \mathcal{S}(\Phi)))$, the set $\mathrm{H}(B')\dot{+}\hat{\lambda}B'$ is a basis of such eigenspace, as claimed. The analogous argument proving that $-B'\dot{+}\mathrm{H}^*(B')$ is a basis for $\mathcal{E}_A(-\hat{\lambda}, \mathcal{S}(\Phi))$ is left to the reader. □

Let r be the number of balanced connected components of a complex unit gain graph Φ, and let $s = m_{A(\mathcal{L}(\Phi))}(-2)$. In order to state the last theorem of this section, we need some extra notation. Let C_1 and C_2 denote the following subsets of $\mathbb{C}^{n+m}$.

$$(4.2) \qquad C_1 = \begin{cases} \varnothing & \text{if } r = 0, \\ \{\mathbf{w_1}\dot{+}\mathbf{0}, \dots, \mathbf{w_r}\dot{+}\mathbf{0}\} & \text{if } r > 0, \end{cases} \quad \text{and} \quad C_2 = \begin{cases} \varnothing & \text{if } s = 0, \\ \{\mathbf{0}\dot{+}\mathbf{y_1}, \dots, \mathbf{0}\dot{+}\mathbf{y_s}\} & \text{if } s > 0, \end{cases}$$

where $\{\mathbf{w_1}, \dots, \mathbf{w_r}\}$ is a basis for $\mathcal{L}(0, \Phi)$ (for instance the one described in Corollary 3.4), and $\{\mathbf{y_1}, \dots, \mathbf{y_s}\}$ is a basis of $\mathcal{E}_A(-2, \mathcal{L}(\Phi))$ (for instance constructed according to the procedure explained in Section 2.2).

Theorem 4.10. *Let $\Phi = (\Gamma, \gamma)$ be a $\mathbb{T}$-gain graphs such that $m_{A(\mathcal{S}(\Phi))}(0) > 0$, and let C_1 and C_2 the sets defined in (4.2). An eigenbasis of $\mathcal{E}_A(0, S(\Phi))$ is given by $C_1 \cup C_2$.*

Proof. By Corollary 4.7, the sets C_1 and C_2 cannot be both empty. Furthermore, $C_1 \cup C_2$ is linearly independent. Proposition 3.5 ensures that $(C_1 \cup C_2)(V_1)$ is in the kernel of H^* and $(C_1 \cup C_2)(V_2)$ is in the kernel of H. Now, Lemma 4.1 guarantees that $C_1 \cup C_2$ is a subset of $\mathcal{E}_A(0, S(\Phi))$. By Part (1) (resp. Part (2)) of Theorem 1.1, it turns out that $s = |C_2| = m - n + r$ (resp. $m_{A(\mathcal{S}(\Phi))}(0) = m - n + 2r$). Thus, $|C_1 \cup C_2| = |C_1| + |C_2| = r + (m - n + r) = m_{A(\mathcal{S}(\Phi))}(0)$. This proves that $C_1 \cup C_2$ is a basis of $\mathcal{E}_A(0, S(\Phi))$, and the proof is complete. □

5. Switching and eigenvector components

Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be two complex unit gain graphs of order n and size $m > 0$ such that $\Phi_2 = \Phi_1^{\zeta}$ for a suitable switching function $\zeta : V(\Gamma) \longrightarrow \mathbb{T}$, and let H_1 (resp. H_2) be an incidence matrix of the complex unit gain graph Φ_1 (resp. Φ_2). The relationship between the eigenspaces of $\mathcal{L}_{\mathrm{H}_1}\Phi_1$ and $\mathcal{L}_{\mathrm{H}_2}\Phi_2$ has been already investigated in [1, 2]. In this short section, we recap the outcomes of that discussion, by adding a comparison between the A-eigenspaces of $\mathcal{S}_{\mathrm{H}_1}\Phi_1$ and those of $\mathcal{S}_{\mathrm{H}_2}\Phi_2$. Specifically, Parts (1) and (2) of Theorem 5.1 easily come from [2, Propositions 2.12 and 2.13], whereas Part (3) is new.

Theorem 5.1. *Let $\Phi_1 = (\Gamma, \gamma_1)$ and $\Phi_2 = (\Gamma, \gamma_2)$ be $\mathbb{T}$-gain graphs such that $\Phi_2 = \Phi_1^{\zeta}$ for a suitable switching function $\zeta : V(\Gamma) \longrightarrow \mathbb{T}$, let H_i be an incidence matrix of Φ_i for $i \in \{1, 2\}$, and let S be a diagonal matrix such that $\mathrm{H}_2 = D(\zeta)^*\mathrm{H}_1 S$. For every $\lambda \in \mathbb{R}$, the following three statements hold.*

(1) For $M \in \{A, L\}$, a vector $\mathbf{x}$ belongs to $\mathcal{E}_M(\lambda, M(\Phi_1))$ if and only if $D(\zeta)^*\mathbf{x}$ belongs to $\mathcal{E}_M(\lambda, M(\Phi_2))$.

(2) A vector $\mathbf{y}$ belongs to $\mathcal{E}_A(\lambda, \mathcal{L}_{\mathrm{H}_1}(\Phi_1))$ if and only if $S^*\mathbf{y}$ belongs to $\mathcal{E}_A(\lambda, \mathcal{L}_{\mathrm{H}_2}(\Phi_2))$.

(3) Let $\mathbf{z} = \mathbf{x}\dot{+}\mathbf{y}$ be a vector in $\mathbb{C}^{n+m}$ with $\mathbf{x} = \mathbf{z}(V_1)$ and $\mathbf{y} = \mathbf{z}(V_2)$. Then, $\mathbf{z}$ belongs to $\mathcal{E}_A(\lambda, \mathcal{S}_{\mathrm{H}_1}(\Phi_1))$ if and only if $\mathbf{z}' = D(\zeta)^*\mathbf{x}\dot{+}S^*\mathbf{y}$ belongs to $\mathcal{E}_A(\lambda, \mathcal{S}_{\mathrm{H}_2}(\Phi_2))$.

Proof. By Proposition 2.8, $D(\zeta)^*\mathrm{H}_1$ is an incidence matrix for Φ_2 such that $\mathcal{L}_{\mathrm{H}_1}(\Phi_1) = \mathcal{L}_{D(\zeta)^*\mathrm{H}_1}(\Phi_2)$. Hence, an $m \times m$ diagonal matrix S such that $\mathrm{H}_2 = D(\zeta)^*\mathrm{H}_1 S$ exists by Proposition 2.5. Now, Statement (1) is a direct consequence of (2.1), whereas Statement (2) comes from Proposition 2.7. In order to prove that Statement (3), we recall that

$$\mathbf{z} = \mathbf{x}\dot{+}\mathbf{y} \in \mathcal{E}_A(\lambda, \mathcal{S}_{\mathrm{H}_1}(\Phi_1)) \Longleftrightarrow \mathrm{H}_1\mathbf{y} = \lambda\mathbf{x} \quad \text{and} \quad \mathrm{H}_1^*\mathbf{x} = \lambda\mathbf{y} \qquad \text{(by Lemma 4.1)},$$

and the two conditions on the right hold if and only if $\mathbf{z}' \in \mathcal{E}_A(\lambda, \mathcal{S}_{\mathrm{H}_2}(\Phi_2))$. In fact,

$$A(\mathcal{S}_{\mathrm{H}_2}(\Phi_2))\mathbf{z}' = \begin{pmatrix} O_n & D(\zeta)^*\mathrm{H}_1 S \\ S^*\mathrm{H}_1^* D(\zeta) & O_m \end{pmatrix} \begin{pmatrix} D(\zeta)^*\mathbf{x} \\ S^*\mathbf{y} \end{pmatrix} = \begin{pmatrix} D(\zeta)^*\mathrm{H}_1\mathbf{y} \\ S^*\mathrm{H}_1^*\mathbf{x} \end{pmatrix}.$$

□

Along the proof of Theorem 5.1 we have used the following equality of matrices

$$A(\mathcal{S}_{\mathrm{H}_2}(\Phi_2)) = (D(\zeta) \oplus S)^* A(\mathcal{S}_{\mathrm{H}_1}(\Phi))(D(\zeta) \oplus S), \tag{5.1}$$

where the symbol $\oplus$ denote the block diagonal sum of two matrices. If we want to compare the adjacency matrices of gain subdivision graphs arising from a single nonempty complex unit gain graph Φ, once we fix two different incidence matrices H and $\mathrm{H}' = \mathrm{H}_1 S$, from (5.1) we deduce

$$A(\mathcal{S}_{\mathrm{H}'}(\Phi)) = (I_n \oplus S)^* A(\mathcal{S}_{\mathrm{H}}(\Phi))(I_n \oplus S),$$

consistently with Proposition 2.9(1).

6. Examples

In order to depict $\mathbb{T}$-gain graphs in Fig. 2, each continuous (resp. dashed) thick undirected line represents two opposite oriented edges with gain 1 (resp. -1); whereas the arrows detect the oriented edges uv's with an imaginary gain. The value $\gamma(uv)$ is specified near the correspondent arrow.

Let $\Phi = (\Gamma, \gamma)$ be the complex unit gain graph depicted in Fig. 2. The graph Φ could be regarded as a $\mathbb{T}_6$-gain graph or a signed digraph in the sense of [26] (see also [19]).

The vertex and the edge labeling is consistent with the one used in Fig. 1. Namely $e_i' = v_1'v_{i+1}'$ and $e_i'' = v_1'v_{i+1}'$ for $i \in \{0, 1\}$; $e_2' = v_2'v_0'$, $e_2'' = v_2''v_0''$ and $f_0 = v_0'v_0''$.

In order to write down the Laplacian matrix $L(\Phi)$, an incidence matrix H for Φ and the adjacency matrix of the corresponding line graph $\mathcal{L}(\Phi)$, we choose the ordering v_0', v_1', v_2', v_0'', v_1'', v_2'' for the

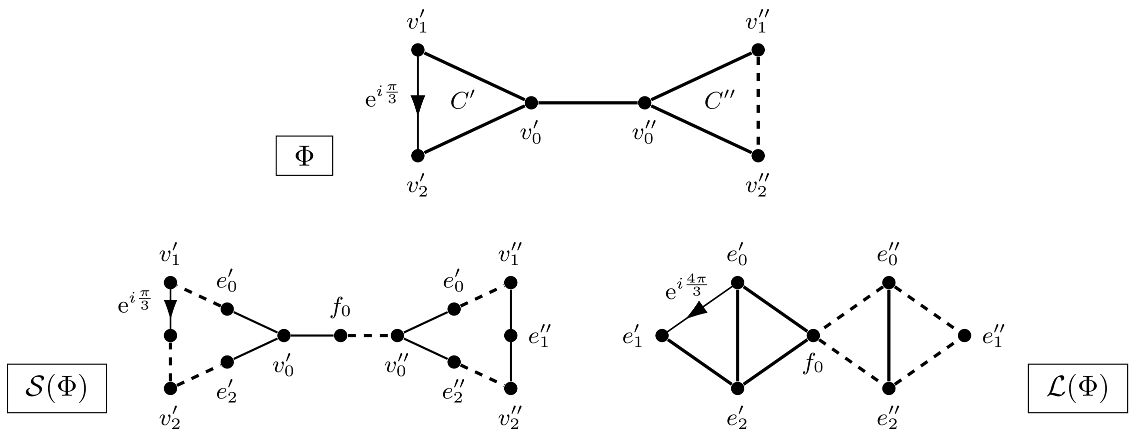

FIGURE 2. A complex unit dumbbell Φ, one of its subdivision graphs $\mathcal{S}(\Phi)$, and one of its line graphs $\mathcal{L}(\Phi)$.

elements in $V(\Gamma)$, and the ordering $e_0',e_1',e_2',f_0,e_0'',e_1'',e_2''$ for those in $E(\Gamma)$. The gains of the directed cycles $C_0' := e_0'e_1'e_2'$ and $C_0'' := e_0''e_1''e_2''$ are

$$\gamma(C_0') = \mathrm{e}^{i\frac{\pi}{3}} \qquad \text{and} \qquad \gamma(C_0'') = -1.$$

The Laplacian matrix $L(\Phi)$ and an incidence matrix H for Φ are respectively given by

$$L(\Phi) = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & -\mathrm{e}^{i\frac{\pi}{3}} & 0 & 0 & 0 \\ -1 & -\mathrm{e}^{-i\frac{\pi}{3}} & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathrm{H} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & \mathrm{e}^{i\frac{\pi}{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

According to the rules explained in Section 3, the adjacency matrix of $\mathcal{L}_{\mathrm{H}}(\Phi)$ is

$$A(\mathcal{L}_{\mathrm{H}}(\Phi)) = \begin{pmatrix} 0 & \mathrm{e}^{i\frac{4\pi}{3}} & 1 & 1 & 0 & 0 & 0 \\ \mathrm{e}^{i\frac{2\pi}{3}} & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}.$$

For instance, $\gamma^{\mathcal{L}}(e_0'e_1') = \overline{\eta}_{v_1'e_0'}\eta_{v_1'e_1'} = -\mathrm{e}^{i\frac{\pi}{3}} = \mathrm{e}^{i\frac{4\pi}{3}}$. The graph $\mathcal{L}_{\mathrm{H}}(\Phi)$ is depicted in Fig. 2.

A MATLAB computation shows that

$$\mathrm{Spec}(L(\Phi)) = \left\{2-\sqrt{3},\, 1,\, 3-2\cos\left(\frac{2\pi}{9}\right),\, 3-2\sin\left(\frac{\pi}{18}\right),\, 2+\sqrt{3},\, 3+2\cos\left(\frac{\pi}{9}\right)\right\}$$

and

$$\mathrm{Spec}(A(\mathcal{L}_{\mathrm{H}}(\Phi))) = \left\{-2,\, -\sqrt{3},\, -1,\, 1-2\cos\left(\frac{2\pi}{9}\right),\, 1-2\sin\left(\frac{\pi}{18}\right),\, \sqrt{3},\, 1+2\cos\left(\frac{\pi}{9}\right)\right\},$$

http://dx.doi.org/10.22108/TOC.2021.130013.1888

confirming that the map

$$\mu \in \mathrm{Spec}(L(\Phi)) \setminus \{0\} \longmapsto \mu - 2 \in \mathrm{Spec}(A(\mathcal{L}_{\mathrm{H}}(\Phi))) \setminus \{-2\}$$

is bijective (since Φ is unbalanced, in this case $\mathrm{Spec}(L(\Phi)) \setminus \{0\} = \mathrm{Spec}(L(\Phi))$.

Row-column products can be performed in order to verify that $\mathcal{E}_L(2-\sqrt{3},\Phi) = \mathrm{Span}(\mathbf{x})$, where

$$\mathbf{x}^\top = \left(2,\ \sqrt{3}+\mathrm{e}^{i\frac{\pi}{3}},\ \sqrt{3}+\mathrm{e}^{-i\frac{\pi}{3}},\ 1,\ \frac{\sqrt{3}-1}{2},\ \frac{\sqrt{3}-1}{2}\right).$$

and $\mathcal{E}_A(-\sqrt{3},\mathcal{L}(\Phi))$, as predicted by Theorem 3.2, is equal to $\mathrm{Span}(\mathrm{H}^*\mathbf{x})$, where

$$(\mathrm{H}^*\mathbf{x})^\top = \left(2-\sqrt{3}-\mathrm{e}^{i\frac{\pi}{3}},\ (\sqrt{3}-1)(\mathrm{e}^{-i\frac{\pi}{3}}-1),\ 2-\sqrt{3}-\mathrm{e}^{-i\frac{\pi}{3}},\ 1,\ \frac{3-\sqrt{3}}{2}, \sqrt{3}-1, \frac{3-\sqrt{3}}{2}\right).$$

Theorem 4.5 now ensures that the 1-dimensional eigenspaces

$$\mathcal{E}_A(\sqrt{2-\sqrt{3}}, A(\mathcal{S}(\Phi))) \qquad \text{and} \qquad \mathcal{E}_A(-\sqrt{2-\sqrt{3}}, A(\mathcal{S}(\Phi)))$$

are respectively spanned by

$$\left(\sqrt{2-\sqrt{3}}\cdot\mathbf{x}\right)\dot{+}\mathrm{H}^*\mathbf{x} \qquad \text{and} \qquad \left(-\sqrt{2-\sqrt{3}}\cdot\mathbf{x}\right)\dot{+}\mathrm{H}^*\mathbf{x}.$$

As already observed in [2, Section 4], the 1-dimensional eigenspace $\mathcal{E}_A(-2,\mathcal{L}(\Phi))$ is equal to $\mathrm{Span}(\mathbf{y})$, where

$$\mathbf{y}^\top = (2\mathrm{e}^{i\frac{2\pi}{3}}, 2\mathrm{e}^{i\frac{\pi}{3}}, 2\mathrm{e}^{i\frac{4\pi}{3}}, 2, 1, 1, 1)^\top.$$

The components of $\mathbf{y}$ satisfy Theorem 2.13. Since $m_{L(\Phi)}(0) = 0$, consistently with Theorem 4.10, the eigenspace $\mathcal{E}_A(0,\mathcal{S}(\Phi))$ is spanned by $\mathbf{0}\dot{+}\mathbf{y}$, where $\mathbf{0} \in \mathbb{C}^6$.

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