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A DIRECTED GRAPH ASSOCIATED WITH A T_0-QUASI-METRIC SPACE

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Abstract. Given a T_0-quasi-metric space we associate a directed graph with it and study some properties of the related directed graph. The present work complements and refines earlier work in the field in which the symmetry graph of a T_0-quasi-metric space was studied.

1. Introduction

In [10], we investigated the symmetry graph L (and its complementary graph $\overline{L}$) of a T_0-quasi-metric space (X,d). Specifically, we studied the connectedness of these two graphs in the sense of graph theory. In the present investigation, we deal with the complementary graphs of symmetry graphs of T_0-quasi-metric spaces in more detail. Indeed we need the asymmetry of our distance functions to introduce a directed graph (digraph) (compare for instance [2],[5],[8]).

Recall that the vertex set of the symmetry graph of a T_0-quasi-metric space (X,d) consists of the elements of X. The (undirected) edges $\{x,y\}$ of L are then those two element subsets of X satisfying $d(x,y)=d(y,x)$. Obviously, with this definition the symmetry graphs of (X,d) and its conjugate space (X,d^{-1}) are the same. Of course the vertex set of $\overline{L}$ is equal to X, while the edges of $\overline{L}$ are those two element subsets $\{x,y\}$ of X satisfying $d(x,y)\neq d(y,x)$, which we call an antisymmetric edge.

In this paper, we are going to direct each antisymmetric edge $\{x,y\}$ of the graph $\overline{L}$ of (X,d) by making it an arc (x,y) or (y,x) in the directed graph of (X,d) according to the following rule: We consider $\{x,y\}$ as an arc (x,y) from x to y in (X,d) if and only if $d(x,y)<d(y,x)$. (Note that if for an antisymmetric

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edge $\{x, y\}$, (x, y) is not an arc in (X, d), that is, $d(x, y) > d(y, x)$, then (y, x) is an arc in (X, d) by our definition.)

By G or G_d, we will denote the set of all arcs so obtained. The directed graph (X, G_d) is called the *directed graph* (or, digraph) of (X, d). Of course, we regain $\overline{L}$ if we neglect the orientation of the edges in (X, G_d).

It follows directly from the definitions that the directed graph $(X, G_{d^{-1}})$ of the conjugate T_0-quasi-metric space (X, d^{-1}) is $(X, (G_d)^{-1})$, where $(G_d)^{-1}$ denotes the inverse relation of

$$G_d = \{(x, y) \in X \times X : d(x, y) < d(y, x)\}$$

Hence it is just the transpose graph of the directed graph (X, G_d).

For a given T_0-quasi-metric space (X, d), it is natural to investigate the concepts related to connectedness in (X, G_d) and for instance to study directed circuits in (X, G_d). Of course, trivially, if for a T_0-quasi-metric space (X, d), the directed graph (X, G_d) is strongly connected then $\overline{L}$ is connected.

In particular we will discuss the directed graphs associated with some classes of T_0-quasi-metric spaces: Natural T_0-quasi-metrics of partial orders, weighted T_0-quasi-metrics, T_0-ultra-quasi-metrics and T_0-quasi-metrics induced by the asymmetric norms of asymmetrically normed real vector spaces.

We also show that each bounded T_0-quasi-metric space has a strongly connected T_0-quasi-metric two-point extension, while each T_0-quasi-metric space has a strongly connected T_0-quasi-metric extension with a countable remainder.

2. **Preliminaries**

Let us first fix the terminology by recalling some basic facts about distance functions.

Definition 2.1. *Let X be a set and $d : X \times X \to [0, \infty)$ a function mapping into the set of nonnegative reals. Then d is called a* quasi-pseudometric *on X if*

(a) $d(x, x) = 0$ whenever $x \in X$,

(b) $d(x, z) \le d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

*We will say that d is a T_0-*quasi-metric *provided that d also satisfies the following condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$.*

Given a T_0-quasi-metric space (X, d), we define the specialization (partial) order $\le_d$ *on X by setting for $x, y \in X$, $x \le_d y$ if $d(x, y) = 0$.*

For $x \in X$ and $\epsilon > 0$, the set $B_d(x) = \{y \in X : d(x, y) < \epsilon\}$ is the open ϵ-ball at x in (X, d).

*A T_0-quasi-metric d on a set X will be called a T_0-*ultra-quasi-metric *(see [9]) if it satisfies the* strong triangle inequality*, that is,*

$$d(x, z) \le \max\{d(x, y), d(y, z)\}$$

whenever $x, y, z \in X$.

Remark 2.2. Let the set $\mathbb{R}$ of the reals be equipped with the T_0-quasi-metric $u(x, y) = x - y$ if $x \ge y$, and $u(x, y) = 0$ if $x < y$. We will call u the *standard T_0-quasi-metric on* $\mathbb{R}$.

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For a binary relation A on a set X and $x \in X$ we set $A(x) = \{y \in X : (x, y) \in A\}$.

We will denote the diagonal $\{(x, x) : x \in X\}$ of X by Δ_X and also, $\mathbb{N}$ will denote the set of positive integers.

Now, let us repeat the next crucial definition from the introduction.

Definition 2.3. *Let (X, d) be a T_0-quasi-metric space. Set $G = \{(x, y) \in X \times X : d(x, y) < d(y, x)\}$. (We will use the notation G_d instead of G in case that there could be a confusion. Similar notations will also be used in other similar cases.) We will consider the element (x, y) in G as an arc from x to y in (X, d). As mentioned above, we will call (X, G)* the directed graph given by the T_0-quasi-metric space *(X, d).*

As usual, $G^{-1} = \{(y, x) \in X \times X : (x, y) \in G\}$ will denote the inverse relation of G. Note that the pairs $(x, y) \in G^{-1}$ are exactly the arcs of the T_0-quasi-metric space (X, d^{-1}). Sometimes (X, G^{-1}) is called the *converse graph (or the transpose graph)* of (X, G) (see e.g. [1, p. 173]). The underlying undirected graphs of (X, d) respectively (X, d^{-1}) are the same, namely $\overline{L}$ the complementary graph of symmetry graph of (X, d).

As in [10], we will set $Z = \{(x, y) \in X \times X : d(x, y) = d(y, x)\}$ and call the elements of Z the symmetric pairs in (X, d). The pairs in $G \cup G^{-1}$ will be called *antisymmetric edges (pairs).* Of course, we have that $Z = Z^{-1}$. Furthermore G determines Z, since $Z = (X \times X) \setminus (G \cup G^{-1})$: Indeed $G \cup G^{-1} \cup Z = X \times X$ and those three relations G, G^{-1} and Z on X are pairwise disjoint.

It is useful to observe that the strict order $<_d$ associated with the specialization order $\leq_d$ is a subset of G_d : Indeed for $x, y \in X$ with $x \neq y$ and $d(x, y) = 0$ we see that $0 = d(x, y) < d(y, x)$, and hence $(x, y) \in G_d$, since d is a T_0-quasi-metric on X.

Remark 2.4. *Let (X, d) be a T_0-quasi-metric space.*

Then d is a metric if and only if $G = \emptyset$, that is, (X, G) is the empty graph without any arcs.

A T_0-quasi-metric d on a set X is antisymmetric (that is, if $x, y \in X$ and $d(x, y) = d(y, x)$ then $x = y$; see [10, Lemma 9]*), if and only if the directed graph (X, G) (or (X, G^{-1})) is a tournament* [1, p. 174]*, in the sense of graph theory.*

Note that for the antisymmetric T_0-quasi-metric space $(\mathbb{R}, u)$ we have that $G =<_u$, that is, G is the usual strict linear order on $\mathbb{R}$.

Proposition 2.5. *Let (X, d) be a T_0-quasi-metric space. Then G is $\tau_{d^s} \times \tau_{d^s}$-open in $X \times X$. (Here τ_{d^s} denotes the topology of the metric d^s on X.) Furthermore G^{-1} is $\tau_{d^s} \times \tau_{d^s}$-open, too.*

Proof. Take $(a, b) \in G$, so $d(a, b) < d(b, a)$. Now, let us set $\epsilon = \frac{d(b,a)-d(a,b)}{3}$.

Then suppose that there are sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ in X such that $d^s(a, a_n) \to 0$ and $d^s(b, b_n) \to 0$ and for all $n \in \mathbb{N}$, (a_n, b_n) does not belong to G.

Recall that by the continuity of d on $(X \times X, \tau_{d^s} \times \tau_{d^s})$ by [10, Lemma 6], the sequence $(d(a_n, b_n))_{n \in \mathbb{N}}$ converges to $d(a, b)$ and the sequence $(d(b_n, a_n))_{n \in \mathbb{N}}$ converges to $d(b, a)$.

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Hence $3\epsilon = d(b,a)-d(a,b) = (d(b,a)-d(b_n,a_n))+(d(b_n,a_n)-d(a_n,b_n))+(d(a_n,b_n)-d(a,b)) < \epsilon+0+\epsilon$ for any $n \in \mathbb{N}$ sufficiently large — a contradiction.

Therefore $(X \times X) \setminus G$ is $\tau_{d^s} \times \tau_{d^s}$-closed.

For the last statement, we apply the proved result to (X, d^{-1}). □

Lemma 2.6. *Let $(X, \leq)$ be a partially ordered set equipped with its* natural T_0-quasi-metric d *defined by $d_{\leq}(x,y) = 0$ if $x \leq y$, and $d_{\leq}(x,y) = 1$ if $x \not\leq y$.*

Then $Z = \Delta_X \cup \{(x,y) \in X \times X : x$ and y are incomparable$\}$ Furthermore G is equal to the strict partial order $<$ associated with $\leq$. Hence G is transitive in this case. (Here note that $\overline{L}$ is the comparability graph of the poset $(X, \leq)$.)

Given a T_0-quasi-metric space, the relation G need not be transitive as will be seen in the following:

Example 2.7. *Let us consider the T_0-quasi-metric d on $X = \{1,2,3\}$ defined by the matrix $\mathbf{D} = (d_{ij}) = (d(i,j))$ with $i,j \in X$:*

$$\mathbf{D} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Obviously d is a T_0-quasi-metric on X. Furthermore $(1,2) \in G, (2,3) \in G,$ but $(1,3) \notin G$.

Theorem 2.8. *Let (X,d) be a T_0-ultra-quasi-metric space. Then G is transitive.*

Proof. Let $x,y,z \in X$. Suppose that $(x,y),(y,z) \in G$. Then $d(x,z) \leq d(x,y) \vee d(y,z) < d(y,x) \vee d(z,y) \leq d(y,z) \vee d(z,x) \vee d(z,x) \vee d(x,y) = d(z,x)$. Therefore $(x,z) \in G$. □

Definition 2.9. *For an arbitrary T_0-quasi-metric space (X,d), it is natural to consider the reflexive transitive hull $H = \Delta_X \cup (\bigcup_{k \in \mathbb{N}} G^k)$ of G. For $x,y \in X$, we say that y* can be reached from x *if $(x,y) \in H$. In graph-theoretic terms, this means that $x = y$ or there is a directed path $(x_0, x_1, x_2, \ldots, x_n)$ with $n \in \mathbb{N}$ such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in G$, that is, (x_i, x_{i+1}) is an arc whenever $i \in \{0,1,\ldots,n-1\}$.*

Observe also that if $(x_0, x_1, \ldots, x_n)$ is a directed path from x_0 to x_n in (X,d), then $(x_n, x_{n-1}, \ldots, x_0)$ is a directed path in (X, d^{-1}).

Trivially, H is reflexive and transitive, but H need not be antisymmetric (see Example 2.27).

Furthermore $H \cap H^{-1}$ is an equivalence relation and thus, the equivalence class $(H \cap H^{-1})(x)$, for $x \in X$, is called the *strong connectedness component* of x in X.

Of course if G is transitive, then $H = \Delta_X \cup G$.

Definition 2.10. *A T_0-quasi-metric space (X,d) will be called* acyclic *if H is a partial order on X.*

In [10], a T_0-quasi-metric space (X,d) was called symmetrically connected *if $\bigcup_{n \in \mathbb{N}} Z^n = X \times X$.*

Also in [10], a T_0-quasi-metric space (X,d) was called antisymmetrically connected *provided that $\Delta_X \cup (\bigcup_{n \in \mathbb{N}} (G \cup G^{-1})^n) = X \times X$.*

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A T_0-quasi-metric space (X,d) is called unilateral *if $H \cup H^{-1} = X \times X$.*

A T_0-quasi-metric space (X,d) will be called strongly connected *if $H = X \times X$.*

Proposition 2.11. *Let (X,d) be a T_0-quasi-metric space such that G is transitive. Then (X,d) is acyclic.*

Proof. Obviously $H = \Delta_X \cup G$ is antisymmetric. □

As the results of above proposition, by Lemma 2.6 and Theorem 2.8, respectively we have:

Corollary 2.12. *Natural T_0-quasi-metric space is acyclic.*

Corollary 2.13. *Each T_0-ultra-quasi-metric space is acyclic.*

Proposition 2.14. *Let (X,d) be a T_0-quasi-metric space, $A \subseteq X$ be τ_{d^s}-dense and $x \in A$. Then $H_A(x) = H(x) \cap A$.*

Proof. Let $y \in H_A(x)$ such that $x \neq y$. Let $P_{x,y}$ be a directed path consisting of points in A from x to y. Then $y \in H(x) \cap A$, clearly. In order to prove the converse inclusion, suppose that $y \in H(x) \cap A$ with $y \neq x$.

Then there is a directed path $Q_{x,y} = (x, x_1, \ldots, x_{n-1}, y)$ from x to y consisting of arcs in (X,d). Since G is $\tau_{d^s} \times \tau_{d^s}$-open by Proposition 2.5, with the induction on i and intersections, we can find τ_{d^s}-neighbourhoods $U(x_i)$ for $i = 0, \ldots, n$ such that $U(x_i) \times U(x_{i+1}) \subseteq G$ whenever $i \in \{0, \ldots, n-1\}$, and $x = x_0$, $y = x_n$.

Now, by the fact that A is τ_{d^s}-dense in X, choose $y_1, \ldots, y_{n-1} \in A$ with $y_i \in U(x_i)$ whenever $i = 1, \ldots, n-1$. Then the walk $(x, y_1, \ldots, y_{n-1}, y)$ (here we kept the startpoint and endpoint of the original directed path) contains a directed path from x to y that entirely lies in A. Therefore x and y are connected by a directed path in the subspace A and, we conclude that $y \in H_A(x)$.

□

Corollary 2.15. *If (X,d) is a strongly connected T_0-quasi-metric space and $S \subseteq X$ is τ_{d^s}-dense in X, then $(S, d|_{S\times S})$ is a strongly connected T_0-quasi-metric space.*

Proof. By assumption, $H(x) = X$ for $x \in X$. Thus $H_S(x) = H(x) \cap S = X \cap S = S$ whenever $x \in S$. In this case, $H_S = S \times S$, and so the subspace S of X is strongly connected. □

Proposition 2.16. *(see [6]) Let (X,d,w) be a* weighted T_0-quasi-metric space *with a so-called* weight function $w : X \to [0,\infty)$, *that is,*

$$d(x,y) + w(x) = d(y,x) + w(y)$$

whenever $x, y \in X$. Then G is transitive.

Furthermore if arc (x,y) belongs to G, its head y has a smaller weight than its tail x does. Since there are no directed cycles in (X,d),b the space (X,d) is acyclic.

Proof. Let $(x,y) \in G$ and $(y,z) \in G$. Then $0 > d(x,y) - d(y,x) = w(y) - w(x)$ and $0 > d(y,z) - d(z,y) = w(z) - w(y)$. In this case, $w(x) > w(y)$ and $w(y) > w(z)$. Also $d(x,z) - d(z,x) = w(z) - w(x) = w(z) - w(y) + w(y) - w(x) < 0$. Hence $(x,z) \in G$. □

Now, we can recall the notion of asymmetric norm:

Definition 2.17. *Let X be a real vector space equipped with a map $\| \cdot | : X \to [0,\infty)$ satisfying the conditions:*

(a) $\|x| = \| -x| = 0$ if and only if $x = \mathbf{0}$.

(b) $\|\lambda x| = \lambda \|x|$ whenever $\lambda \geq 0$ and $x \in X$.

(c) $\|x + y| \leq \|x| + \|y|$ whenever $x, y \in X$.

Then $\| \cdot |$ is called an asymmetric norm *and $(X, \| \cdot |)$ is said to be an* asymmetrically normed (real vector) space *(compare e.g. [3]). Here $\mathbf{0}$ denotes the zero vector of vector space X.*

For each $x \in X$, if we set $\|x\| = \|x| \vee \| -x|$ then $\| \cdot \|$ is a norm on X (compare [4, Proposition 2.2]*).*

Remark 2.18. *Let $(X, \| \cdot |)$ be an aymmetrically normed real vector space and $d_{\| \cdot |}$ the induced T_0-quasi-metric on X defined by the condition $d_{\| \cdot |}(x,y) = \|x - y|$, for $x, y \in X$.*

Since we have that $d_{\| \cdot |}(x,y) = d_{\| \cdot |}(x+h, y+h)$ for any $x, y, h \in X$, it is obvious that the relations G, G^{-1} and Z are determined by their values at $\mathbf{0}$, in the sense that $Z(x) = Z(\mathbf{0}) + x$, $G(x) = G(\mathbf{0}) + x$ and $G^{-1}(x) = G^{-1}(\mathbf{0}) + x$ whenever $x \in X$. (For instance, for $x, y \in X$ we have that $d(x,y) < d(y,x)$ if and only if $d(\mathbf{0}, y-x) < d(\mathbf{0}, x-y)$ if and only if $y - x \in G(\mathbf{0})$ if and only if $y \in G(\mathbf{0}) + x$.)

Proposition 2.19. *Let $(X, \| \cdot |)$ be an asymmetrically normed real vector space. Take $a, b \in X$ such that (a,b) is an arc in $(X, d_{\| \cdot |})$ and $\lambda > 0$. Then $(\lambda a, \lambda b)$ and $(-b, -a)$ are arcs in $(X, d_{\| \cdot |})$.*

If $n \in \mathbb{N}$ and $(x_1, x_2, \ldots, x_n)$ is a directed path in $(X, d_{\| \cdot |})$, then $(-x_n, -x_{n-1}, \ldots, -x_1)$ is a directed path in $(X, d_{\| \cdot |})$.

Proof. Let $(a,b) \in G$. Then $\|a - b| < \|b - a|$ and we have that $\|\lambda a - \lambda b| < \|\lambda b - \lambda a|$ by Definition 2.17(b). Furthermore $\|(-b) - (-a)| < \|(-a) - (-b)|$, which immediately implies the assertion about the directed path. Hence all the statements are established. □

Definition 2.20. *For a given asymmetrically normed real vector space $(X, \| \cdot |)$, we will call a vector $x \in X$* symmetric *in $(X, d_{\| \cdot |})$ if $x \in Z(\mathbf{0})$, and we will call x* positively biased *in $(X, d_{\| \cdot |})$ if $x \in G(\mathbf{0})$. We will say that x is* negatively biased *if $-x$ is positively biased.*

Let us note that $x \in X$ is positively biased if and only if $d_{\| \cdot |}(\mathbf{0}, x) < d_{\| \cdot |}(x, \mathbf{0})$ if and only if $\| -x| < \|x|$.

In particular, for $x, y \in X$, (x,y) is a symmetric pair if and only if the vector $y - x$ is symmetric; furthermore (x,y) is an arc in $G_{d_{\| \cdot |}}$ if and only if $y - x$ is positively biased (equivalently $x - y$ is negatively biased).

Also, if $\| -x| = 0$ then x is positively biased.

Fix $x_0 \in X$. Observe that $x_1, \ldots, x_n$ $(n \in \mathbb{N})$ are positively biased vectors in $(X, d_{\| \cdot |})$ if and only if $(x_0, x_0 + x_1, x_0 + x_1 + x_2, \ldots, x_0 + x_1 + x_2 + \cdots + x_n)$ is a directed walk from x_0 to $\sum_{i=0}^{n} x_i$.

Lemma 2.21. *Let $(X, \|\cdot|)$ be an asymmetrically normed real vector space.*

(a) The sum of two positively biased vectors is never equal to $\mathbf{0}$.

(b) If the sum of two positively biased vectors x and y is negatively biased, then $(\mathbf{0}, x, x+y, \mathbf{0})$ is a directed cycle.

Proof. (a) Suppose otherwise. Then $x + y = \mathbf{0}$ where x and y are positively biased. Therefore $x = -y$ and, x is positively biased and negatively biased, which is impossible.

(b) See the observation preceding Lemma 2.21. □

Example 2.22. *Let I be a nonempty set and consider $\ell_1(I) = \{x = (x_i)_{i\in I} : \sum_{i\in I} |x_i| < \infty\}$. For each $x \in \ell_1(I)$, we define the asymmetric norm $\|x| = \|(x_i)_{i\in I}| = \sum_{i\in I} x_i^+$. (In the following, for $x \in \mathbb{R}$ we set $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$.)*

Note that $\| - (x_i)| = \sum_{i\in I} {x_i}^-$.

We have that: $(x, y) \in G$ if and only if $\|x - y| < \|y - x|$ if and only if

$$0 < \sum_{i\in I}(y_i - x_i)^+ - \sum_{i\in I}(-(y_i - x_i))^+ = \sum_{i\in I}[(y_i - x_i)^+ - (y_i - x_i)^-] = \sum_{i\in I}(y_i - x_i)$$

if and only if $\sum_{i\in I} x_i < \sum_{i\in I} y_i$. In particular, it is easy to verify that G is transitive.

Hence for the tail x of an arc (x, y) the sum $\sum_{i\in I} x_i$ is strictly smaller than the sum $\sum_{i\in I} y_i$ of the head y, which implies that a directed path cannot return to a preceding point on the path. It immediately follows that $(\ell_1(I), d_{\|.|})$ is acyclic.

Let us note that $x \in \ell_1(I)$ is positively biased if and only if $\sum_{i\in I} x_i > 0$.

Proposition 2.23. *Let X be a set and $B(X)$ the real vector space of all bounded real-valued functions on X. Equip $B(X)$ with the asymmetric norm $\|f|_\infty = \sup_{x\in X}(f(x))^+$ whenever $f \in B(X)$. For each $f \in B(X)$ set $a_f := \frac{\sup_{x\in X} f(x) + \inf_{x\in X} f(x)}{2}$.*

Then:

(a) f is symmetric if and only if $a_f = 0$.

(b) f is positively biased if and only if $a_f > 0$.

(c) f is negatively biased if and only if $a_f < 0$.

Proof. (a) Let $a_f = 0$. Then by definition of a_f, $\sup_{x\in X} f(x) = -\inf_{x\in X} f(x)$. We consider 3 cases:

Case 1) If $\sup_{x\in X} f(x) = 0$, then $\|f|_\infty = 0 = \sup_{x\in X} f(x) = -\inf_{x\in X} f(x) = \sup_{x\in X} -f(x) = \|-f|_\infty$, thus $f = 0$ is symmetric.

Case 2) If $\sup_{x\in X} f(x) > 0$, then $0 < \|f|_\infty = \sup_{x\in X} f(x) = -\inf_{x\in X} f(x) = \sup_{x\in X} -f(x) = \|-f|_\infty$, and f is symmetric.

Case 3) If $\sup_{x\in X} f(x) < 0$, then $-\inf_{x\in X} f(x) = \sup_{x\in X} f(x) < 0 < \inf_{x\in X} f(x) \le \sup_{x\in X} f(x)$, a contradiction. Therefore this case is impossible.

Finally $a_f = 0$ implies that $\|f|_\infty = \| - f|_\infty$.

In order to establish the converse, suppose that f is symmetric. So $\|f|_\infty = \| - f|_\infty$. We consider some cases:

Case 1: Suppose that $\sup_{x\in X} f(x) \leq 0$. Then $\|f|_\infty = 0$. Thus by symmetry $\|-f|_\infty = 0$ and hence by the asymmetric norm, $f = 0$ and $\sup_{x\in X} f(x) = -\inf_{x\in X} f(x)$. In particular, it follows that $a_f = 0$.

Case 2: Suppose that $\sup_{x\in X} f(x) > 0$. Thus $\|f|_\infty = \sup_{x\in X} f(x)$.

Case 2(i): Assume also that $-\inf_{x\in X} f(x) > 0$.

Then by symmetry of f, we get $0 < -\inf_{x\in X} f(x) = \sup_{x\in X} -f(x) = \|-f|_\infty = \|f|_\infty = \sup_{x\in X} f(x)$. Therefore $a_f = 0$.

Case 2(ii): Assume also that $-\inf_{x\in X} f(x) \leq 0$.

Hence $\sup_{x\in X} -f(x) \leq 0$ and $\|-f|_\infty = 0$. By symmetry of f, $\|f|_\infty = 0$. Then, $f = 0$, and thus $a_f = 0$.

Altogether it follows that $a_f = 0$ whenever f is symmetric.

(b) Let $a_f > 0$. Then by definition of a_f, $\sup_{x\in X} f(x) > -\inf_{x\in X} f(x)$.

We consider two cases:

Case 1) If $\sup_{x\in X} f(x) \leq 0$, then $\|f|_\infty = 0 \geq \sup_{x\in X} f(x) > -\inf_{x\in X} f(x) = \sup_{x\in X} -f(x)$. Thus $\|-f|_\infty = 0$. It means that $f = 0$ and so $a_f = 0$, a contradiction, since $a_f > 0$ is assumed. Hence this case is impossible.

Case 2) If $\sup_{x\in X} f(x) > 0$, then $\|f|_\infty = \sup_{x\in X} f(x) > 0 \vee (-\inf_{x\in X} f(x)) = \|-f|_\infty$. Hence $\|f|_\infty > \|-f|_\infty$, and f is positively biased.

So $a_f > 0$ implies that $\|f|_\infty > \|-f|_\infty$, that is, f is positively biased.

In order to establish the converse, suppose that f is positively biased. So $\|f|_\infty > \|-f|_\infty$.

We have two cases:

Case 1) If $\sup_{x\in X} f(x) > 0$, then $0 < \|f|_\infty = \sup_{x\in X} f(x)$. Furthermore $\sup_{x\in X} f(x) = \|f|_\infty > \|-f|_\infty = \sup_{x\in X}(-f)(x) \vee 0 > -\inf_{x\in X} f(x)$. Therefore $\sup_{x\in X} f(x) > -\inf_{x\in X} f(x)$ and $a_f > 0$.

Case 2) If $\sup_{x\in X} f(x) \leq 0$, then $0 \geq \|f|_\infty > \|-f|_\infty$. Therefore this case is impossible, since $\|-f|_\infty \geq 0$.

That is, it follows that $a_f > 0$ whenever f is positively biased.

(c) It directly follows from the definition that $f \in B(X)$ is positively biased if and only if $-f$ is negatively biased.

Hence f is negatively biased if and only if $a_{-f} > 0$, which is equivalent to $a_f < 0$, as follows from Lemma 2.25(a) below.

□

Remark 2.24. *It was shown in* [10, Proposition 58] *that each asymmetrically normed real vector space that is not normed is antisymmetrically connected. It is also known from* [10, Example 61] *that for X with $|X| \geq 3$, the space $B(X)$ defined in Proposition 2.23 is symmetrically connected.*

Lemma 2.25. *Let X and $(B(X), \|\cdot|_\infty)$ be as above. Furthermore let $f \in B(X)$.*

(a) $a_f = -(a_{-f})$.

(b) If f is positively biased, then $\sup_{x\in X} f(x) > 0$.

(c) If f is negatively biased, then $\inf_{x\in X} f(x) < 0$.

Proof. (a) Trivially, we have $2a_f = \inf_{x\in X} f(x) + \sup_{x\in X} f(x) = -\sup_{x\in X}(-f)(x) - \inf_{x\in X}(-f)(x) = -2a_{-f}$.

(b) If f is positively biased then $0 < a_f$ by Proposition 2.23(b). Thus $0 < a_f = \frac{\sup_{x\in X} f(x)+\inf_{x\in X} f(x)}{2} \leq \sup_{x\in X} f(x)$.

(c) As similar to (b), if $a_f < 0$ then it is easy to verify that $\inf_{x\in X} f(x) < 0$. □

Proposition 2.26. *Let X be a set with at least 3 elements. Furthermore let $B(X)$ be the real vector space of all bounded real-valued functions on X equipped with the asymmetric norm $\|f|_\infty = \sup_{x\in X}(f(x))^+$ (compare Proposition 2.23). For each $f \in B(X)$, we can reach f from $\mathbf{0}$ as well as we can reach $\mathbf{0}$ from f in $(B(X), \|\cdot|_\infty)$. Hence $(B(X), \|\cdot|_\infty)$ is strongly connected.*

Proof. For $k \in \mathbb{R}$, we will denote the constant function equal to k on X by $\underline{k}$ in the following.

We first show that for each nonconstant $f \in B(X)$ there is a constant function $\underline{a}$ such that $(f, \underline{a})$ is an arc and a constant function $\underline{b}$ such that $(\underline{b}, f)$ is an arc.

Indeed, we have that $\delta := \sup_{x\in X} f(x) - \inf_{x\in X} f(x) > 0$ since f is not constant.

Now let us take
$$a = 1/2(\sup_{x\in X} f(x) + \inf_{x\in X} f(x)) + \delta/3 = a_f + \delta/3$$
and
$$b = 1/2(\sup_{x\in X} f(x) + \inf_{x\in X} f(x)) - \delta/3 = a_f - \delta/3.$$

One checks that $(f, \underline{a})$ is an arc and $(\underline{b}, f)$ is an arc.

Of course, if $\underline{a} = \mathbf{0}$, then the first part of the statement is established.

Suppose now that $a < 0$. Then it is straightforward to verify that $(f, \underline{a}, \mathbf{0})$ is a directed walk from f to $\mathbf{0}$.

So it remains to consider the case that $a > 0$.

Let $x_1, x_2, x_3 \in X$ be pairwise distinct. Here we have used that $|X| \geq 3$. Furthermore let $u_a, v_a \in B(X)$ with $u_a(x_1) = 18a, u_a(x_2) = -6a, u_a(x_3) = -6a$ and $u_a(x) = 0$ otherwise; moreover $v_a(x_1) = 9a, v_a(x_2) = 6a, v_a(x_3) = -12a$ and $v_a(x) = 0$ otherwise. Note that on the walk $(\underline{a}, u_a, v_a, \mathbf{0})$ the difference vectors $(-17a, 7a, 7a, a, a, \ldots), (9a, -12a, 6a, 0, 0, \ldots), (9a, 6a, -12a, 0, 0, \ldots)$ are negatively biased. Therefore the walk $(f, \underline{a}, u_a, v_a, \mathbf{0})$ is a directed walk from f to $\mathbf{0}$.

We now construct a directed walk from $\mathbf{0}$ to f. We already have an arc $(\underline{b}, f)$. If $\underline{b} = \mathbf{0}$, then this part of the proof is finished. If $b > 0$, then $(\mathbf{0}, \underline{b}, f)$ is a directed walk from $\mathbf{0}$ to f, as one readily verifies.

Finally if $b < 0$, then a straightforward computation analogous to the one above shows that $(\mathbf{0}, v_b, u_b, \underline{b}, f)$ is a directed walk from $\mathbf{0}$ to f.

Let us now consider arbitrary f and $g \in B(X)$ such that $f \neq g$. We can take a directed walk $(x_0, \ldots, x_n)$ from f to $\mathbf{0}$. Furthermore we can take a directed walk $(y_0, \ldots, y_n)$ from $\mathbf{0}$ to g. Then the walk $(x_0, \ldots, x_n = y_0, \ldots, y_n)$ is a directed walk from f to g. Hence $(B(X), \|\cdot|_\infty)$ is strongly connected. □

Example 2.27. *Consider $\mathbb{R}^3$ with the asymmetric supremum norm $\|(x_1, x_2, x_3)|_\infty = x_1^+ \vee x_2^+ \vee x_3^+$ whenever $x_1, x_2, x_3 \in \mathbb{R}$.*

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Of course, $(-7,-6,10)+(6,-4,-4)+(1,10,-6)=(0,0,0)$. *Observe that the non-zero vectors in the sum are positively biased in* $(\mathbb{R}^3, d_{\|\cdot|_\infty})$ *(see Proposition 2.23). Consequently the sum of any two vectors in the sum is negatively biased. Therefore* $((0,0,0),(-7,-6,10),(-1,-10,6),(0,0,0))$ *is a directed cycle in* $(\mathbb{R}^3, d_{\|\cdot|_\infty})$ *by Lemma 2.21(b). Hence* $(\mathbb{R}^3, d_{\|\cdot|_\infty})$ *is not acyclic.*

It is also worthwhile to observe that

$$((-3,2,-1),(0,0,0))\in G, ((0,0,0),(1,-2,3))\in G,$$

but $((-3,2,-1),(1,-2,3))\in Z$, *hence the sum of the two negatively biased vectors* $(-3,2,-1)$ *and* $(-1,2,-3)$ *yields the symmetric vector* $(-4,4,-4)$.

Corollary 2.28. $(\mathbb{R}^n, \|\cdot|_\infty)$ *is strongly connected whenever* $n\in\mathbb{N}$ *and* $n\geq 3$, *but not for* $n=1$ *or* $n=2$.

Proof. This follows from Proposition 2.26, by chosing X to be a subset with $n\geq 3$ elements. Let us note that the result does not hold for $\mathbb{R}$ and $\mathbb{R}^2$, as follows from Example 2.22, together with Example 3.7 below in the case of $\mathbb{R}^2$. □

Theorem 2.29. *The product of two strongly connected* T_0*-quasi-metric spaces* (X_1,d_1) *and* (X_2,d_2) *is strongly connected. Here we equip* $X_1\times X_2$ *with the* T_0*-quasi-metric* $d((x_1,y_1),(x_2,y_2))=d_1(x_1,x_2)\vee d_2(y_1,y_2)$ *whenever* $(x_1,x_2),(y_1,y_2)\in X_1\times X_2$.

Proof. Let $(x_1,x_2),(y_1,y_2)$ be two points in $X_1\times X_2$. By the strong connectedness of (X_1,d_1), y_1 can be reached from x_1 in (X_1,d_1). Similarly, y_2 can be reached from x_2 in (X_2,d_2) since (X_2,d_2) is strongly connected.

Hence in the product $X_1\times X_2$, (y_1,y_2) can be reached from (x_1,x_2), via the directed walk

$$((x_1,x_2),(,x_2),\ldots,(y_1,x_2),(y_1,),\ldots,(y_1,y_2)),$$

as one readily checks. □

We finally present an other example that illustrates some of our concepts.

Example 2.30. *Let* $(X,\|\cdot|)$ *be an asymmetrically normed real vector space of linear dimension at least 2 such that* $\|\cdot|$ *is not a norm (compare* [10, Corollary 57]*).*

For each $x\in X$, *set*

$$F(x)=d_{\|\cdot|}(\mathbf{0},x)-d_{\|\cdot|}(x,\mathbf{0})=\|-x|-\|x|$$

whenever $x\in X$.

Then $F:(X,\tau(\|\cdot\|))\to(\mathbb{R},\tau(u^s))$ *is continuous from* [10, Lemma 6]. *(Here,* $\|x\|=\|x|\vee\|-x|$ *and so* $d_{\|\cdot\|}$ *is metric, also* $\tau(\|\cdot\|)=\tau(d_{\|\cdot\|})$*.)*

Now, let us consider the set

$$T_1(\mathbf{0})=\{x\in X:\|x\|=1\}$$

and show that there is a symmetric vector in $T_1(\mathbf{0})$*: Since* $\|\cdot|$ *is not a norm, there is* $x_0\in T_1(\mathbf{0})$ *such that* x_0 *is not symmetric. Recall that the topological space* $(T_1(\mathbf{0}),\tau(\|\cdot\|))$ *is path-connected, as the image* *of the obviously path-connected topological space* $(X \setminus \{\mathbf{0}\}, \tau(\|\cdot\|))$ *under the continuous map* $x \mapsto \frac{x}{\|x\|}$. *Here the assumption on the dimension of* X *is used.*

Also, we have that $F(x_0) > 0$ *(that is,* x_0 *is negatively biased) or* $F(x_0) < 0$ *(that is,* x_0 *is positively biased). In the first case* $F(-x_0) < 0$ *and in the second case* $F(-x_0) > 0$.

Therefore, F *has a sign change on a continuous path from* x_0 *to* $-x_0$ *in the path-connected space* $(T_1(\mathbf{0}), \tau(\|\cdot\|))$. *By the intermediate value theorem, in either case there is a point* $y_0 \in T_1(\mathbf{0})$ *such that* $F(y_0) = 0$. *Therefore* y_0 *is a symmetric vector in* $T_1(\mathbf{0})$.

If the dimension of X *is one, the connectedness argument fails. Indeed the asymmetrically normed real vector space* $(\mathbb{R}, \|\cdot|)$ *equipped with the asymmetric norm* $\|x| = u(x, 0)$, *whenever* $x \in \mathbb{R}$, *is not a normed space, and also does not have any symmetric vector in* $T_1(\mathbf{0})$, *because* $\|-1| \neq \|1|$.

3. Main Results

In [10, Proposition 22], it is observed that any graph with vertex set X is the symmetry graph of some T_0-quasi-metric on X. Indeed, let Z be a reflexive and symmetric relation on a set X. Then there is a T_0-quasi-metric d on X such that $Z = Z_d$.

Similarly, we next observe that any directed graph $(X, \mathcal{A})$ is the directed graph of a T_0-quasi-metric space (X, d). Here we assume that for a directed graph the underlying undirected graph does not have any loops and no two edges have the same end points. So for two distinct vertices x and y in X not both (x, y) and (y, x) can be arcs belonging to $\mathcal{A}$.

Theorem 3.1. *Let* $(X, \mathcal{A})$ *be a directed graph and* $Z = \{(a, b) \in X \times X : (a, b) \notin \mathcal{A}$ *and* $(b, a) \notin \mathcal{A}\}$.

Define $\widetilde{d} : X \times X \to [0, \infty)$ *as follows: Let* $a, b \in X$.

(1) If $(a, b) \in \mathcal{A}$ *set* $\widetilde{d}(a, b) = 3$ *and* $\widetilde{d}(b, a) = 5$.

(2) If $(a, b) \in Z \setminus \Delta_X$ *set* $\widetilde{d}(a, b) = \widetilde{d}(b, a) = 4$.

(3) If $(a, b) \in \Delta_X$ *set* $\widetilde{d}(a, b) = 0 = \widetilde{d}(b, a)$.

Then $\widetilde{d}$ *is a* T_0*-quasi-metric on* X *with directed graph* $(X, \mathcal{A})$.

Proof. Given any $\rho > 0$, each function $d : (X \times X) \setminus \Delta_X \to [\rho, 2\rho]$ with $d(\Delta_X) = \{0\}$ is a T_0-quasi-metric on X (see [10, Remark 24]). Now, similarly the function $\widetilde{d}$ is a T_0-quasi-metric on X. Moreover, we have that $G_{\widetilde{d}} = \mathcal{A}$ and $Z_{\widetilde{d}} = \{(a, b) \in X \times X : (a, b) \notin \mathcal{A}$ and $(b, a) \notin \mathcal{A}\}$, *clearly*. That is, the directed graph of $(X, \widetilde{d})$ is indeed $(X, \mathcal{A})$. □

Remark 3.2. *Starting with a* T_0*-quasi-metric space* (X, d) *and its directed graph* (X, G_d), *the* T_0*-quasi-metric space* $(X, \widetilde{d})$ *as constructed in Theorem 3.1 may look different from* (X, d). *Observe for instance that with the method from Theorem 3.1 applied to* (X, G_d), *a nontrivial specialization order of* (X, d) *is not preserved by* $\widetilde{d}$. *The following refinement of our construction in Theorem 3.4 distinguishes between two kinds of arcs in* $\mathcal{A}$, *those that belong to a strict order relation* $<$ *and those that do not. We first prove the next lemma.*

Lemma 3.3. *Let* (X, d) *be a* T_0*-quasi-metric space and* $a, b, x, y \in X$.

(a) If $d(x,a)=0, d(b,y)=0$ and $(a,b)\in G_d$ then (x,y) is an arc in (X,d).

(b) If $d(x,a)=0, d(b,y)=0$ and $(a,b)\in Z_d$ then $(x,y)\in G_d\cup Z_d$.

Proof. (a) We have that $d(x,y)\leq d(x,a)+d(a,b)+d(b,y)<0+d(b,a)+0\leq d(b,y)+d(y,x)+d(x,a)=d(y,x)$. Hence (x,y) is an arc.

(b) The statement $d(x,y)\leq d(x,a)+d(a,b)+d(b,y)=0+d(b,a)+0\leq d(b,y)+d(y,x)+d(x,a)=d(y,x)$ is obvious since $(a,b)\in Z_d$. Thus $(x,y)\in G_d\cup Z_d$. □

Theorem 3.4. *Let (X,d) be a T_0-quasi-metric space and (X,G_d) be its directed graph. Define $\widetilde{d}: X\times X\to[0,\infty)$ as follows: Let $a,b\in X$.*

(1) If $(a,b)\in G_d\setminus\leq_d$ set $\widetilde{d}(a,b)=3$ and $\widetilde{d}(b,a)=5$.

(2) If $(a,b)\in Z_d\setminus\Delta_X$ set $\widetilde{d}(a,b)=\widetilde{d}(b,a)=4$.

(3) If $(a,b)\in\Delta_X$ set $\widetilde{d}(a,b)=0=\widetilde{d}(b,a)$.

(4) If $(a,b)\in<_d$ set $\widetilde{d}(a,b)=0$ and $\widetilde{d}(b,a)=5$.

Then $\widetilde{d}$ is a T_0-quasi-metric on X with the same specialization order as d on X. Furthermore the directed graphs of (X,d) and $(X,\widetilde{d})$ are the same.

Proof. The statements about the equality of the specialization orders and the associated directed graphs of d and $\widetilde{d}$ obviously hold.

Hence all what we need to prove is to verify that $\widetilde{d}$ satisfies the triangle inequality. For any $a,b,c\in X$ we have to show that

$$\widetilde{d}(a,c)\leq\widetilde{d}(a,b)+\widetilde{d}(b,c)$$

Here we can assume without loss of generality that $a\neq b$ and $b\neq c$. Note that the image of $\widetilde{d}$ is $\{0,3,4,5\}$.

In this case, we need only consider the following possibly problematic cases:

Case 1: $\widetilde{d}(a,b)=3$ and $\widetilde{d}(b,c)=0$. Therefore $(a,b)\in G_d\setminus\leq_d$ and $(b,c)\in\leq_d$. Then by Lemma 3.3(a), $(a,c)\in G_d$. Consequently $\widetilde{d}(a,c)=3$ or 0. Hence, in Case 1 the triangle inequality is satisfied.

Case 2: $\widetilde{d}(a,b)=0$ and $\widetilde{d}(b,c)=3$. Therefore $(b,c)\in G_d\setminus\leq_d$ and $(a,b)\in\leq_d$. In this case, by Lemma 3.3(a), $(a,c)\in G_d$. It follows that $\widetilde{d}(a,c)=3$ or 0. Hence, the triangle inequality is satisfied in Case 2.

Case 3: $\widetilde{d}(a,b)=4$ and $\widetilde{d}(b,c)=0$. Then $(a,b)\in Z_d$ and $(b,c)\in\leq_d$. In this case by Lemma 3.3(b), $(a,c)\in Z_d\cup G_d$. Consequently $\widetilde{d}(a,c)=0,3$ or 4. Hence, in Case 3 the triangle inequality is satisfied.

Case 4: $\widetilde{d}(a,b)=0$ and $\widetilde{d}(b,c)=4$. Therefore $(a,b)\in\leq_d$ and $(b,c)\in Z_d$. Thus by Lemma 3.3(b), $(a,c)\in Z_d\cup G_d$. We conclude that $\widetilde{d}(a,c)=0,3$, or 4. Hence, the triangle inequality is satisfied in Case 4. □

Definition 3.5. *Let d_1,d_2 be two T_0-quasi-metrics on a set X. Then d_1 and d_2 are called* graph-equivalent *if $G_{d_1}=G_{d_2}$.*

Note that in the preceding definition, $G_{d_1}=G_{d_2}$ implies that $Z_{d_1}=Z_{d_2}$. Of course, if both d_1 and d_2 are the induced T_0-quasi-metrics of asymmetric norms on a real vector space X, then in order to check the graph-equivalence of d_1 and d_2, it suffices to verify that $G_{d_1}(\mathbf{0})=G_{d_2}(\mathbf{0})$.

Remark 3.6. *Let* $X = \mathbb{R}^3$ *be equipped with the aymmetric norms* $\|\cdot|_1$ *and* $\|\cdot|_\infty$.

Here $\|(x_1, x_2, x_3)|_\infty = x_1^+ \vee x_2^+ \vee x_3^+$ *and* $\|(x_1, x_2, x_3)|_1 = x_1^+ + x_2^+ + x_3^+$ *whenever* $(x_1, x_2, x_3) \in \mathbb{R}^3$.

By G_∞ *we will denote the set of arcs of* $(\mathbb{R}^3, d_{\|\cdot|_\infty})$. *Similarly,* G_1 *will denote the set of arcs of* $(\mathbb{R}^3, d_{\|\cdot|_1})$. *(In similar situations below, we will use an analogous notation.)*

Then $\|(-2,-2,3)|_\infty = 3$ *and* $\|(2,2,-3)|_\infty = 2$. *On the other hand* $\|(-2,-2,3)|_1 = 3$ *and* $\|(2,2,-3)|_1 = 4$. *Hence we have that* $(-2,-2,3) \in G_1(\mathbf{0})$, *but* $(-2,-2,3) \notin G_\infty(\mathbf{0})$.

Therefore the directed graphs $(\mathbb{R}^3, G_1)$ *and* $(\mathbb{R}^3, G_\infty)$ *are distinct and so* $d_{\|\cdot|_\infty}$ *and* $d_{\|\cdot|_1}$ *are not graph-equivalent on* $\mathbb{R}^3$.

Example 3.7. *Let* $X = \mathbb{R}^2$ *and* $x = (x_1, x_2) \in \mathbb{R}^2$. *We consider the two asymmetric norms on* X *defined by* $\|(x_1, x_2)|_1 = x_1^+ + x_2^+$ *and* $\|(x_1, x_2)|_\infty = x_1^+ \vee x_2^+$ *whenever* $(x_1, x_2) \in \mathbb{R}^2$. *We are going to show that* $(\mathbb{R}^2, d_{\|\cdot|_1})$ *and* $(\mathbb{R}^2, d_{\|\cdot|_\infty})$ *have the same directed graph and thus the* T_0*-quasi-metrics* $d_{\|\cdot|_1}$ *and* $d_{\|\cdot|_\infty}$ *are graph-equivalent.*

Indeed; by Example 2.22, $x = (x_1, x_2) \in \mathbb{R}^2$ *is positively biased with respect to* $d_{\|\cdot|_1}$ *if and only if* $x_1 + x_2 > 0$.

As for $d_{\|\cdot|_\infty}$, *by Proposition 2.23 we have that* $x = (x_1, x_2) \in \mathbb{R}^2$ *is positively biased with respect to* $d_{\|\cdot|_\infty}$ *if and only if* $\frac{\max\{x_1,x_2\}+\min\{x_1,x_2\}}{2} > 0$. *But the latter inequality obviously just means that* $x_1 + x_2 > 0$.

Hence $G_1 = G_\infty$ *for* $\mathbb{R}^2$.

Remark 3.8. *(a) Consider the space* $(\mathbb{R}^2, d_{\|\cdot|_\infty})$, *where* $\|\cdot|_\infty$ *is as in Example 3.7. Then* $((1,-1),(2,-2))$ *is a symmetric pair, but there is no directed path from startpoint* $(1,-1)$ *to endpoint* $(2,-2)$, *consisting of arcs in* G, *since the sum* $z_1 + z_2$ *of coordinates strictly increases along the points* $z = (z_1, z_2)$ *on such a path and that sum is equal to* 0 *at the startpoint* $(1,-1)$ *as well as for the startpoint* $(2,-2)$. *Of course, the result still holds if startpoint and endpoint are exchanged. So the space is not unilateral. Note that on the other hand, the factor space* $(\mathbb{R}, u)$ *is unilateral.*

(b) Let us take the space $(\mathbb{R}^3, d_{\|\cdot|_\infty})$, *where* $\|\cdot|_\infty$ *is as in Remark 3.6. Then* $((0,0,0),(7,-7,7))$ *is a symmetric pair, but* $((0,0,0),(-1,-2,3),(7,-7,7))$ *is a directed path from* $(0,0,0)$ *to* $(7,-7,7)$ *consisting of two arcs in* G.

Example 3.9. *Let* $X = \mathbb{R}^2$ *be equipped with the asymmetric norm* $\|(x_1, x_2)| = |x_1| \vee x_2^+$ *whenever* $(x_1, x_2) \in \mathbb{R}^2$.

Then the set $T_1(\mathbf{0})$ *described in Example 2.30, will be :*

$$T_1(\mathbf{0}) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \vee |x_2| = 1\}$$

That is, the set $T_1(\mathbf{0})$ *will be the unit square with respect to the norm* $\|x\| = \|x| \vee \|-x|$ *on* $\mathbb{R}^2$, *whenever* $x \in \mathbb{R}^2$.

Also, by $Z_T(\mathbf{0})$ *we will denote the restriction* $Z(\mathbf{0}) \cap T_1(\mathbf{0})$ *of* $Z(\mathbf{0})$ *to the unit square* $T_1(\mathbf{0})$. *We also apply an analogous notation to* $G(\mathbf{0})$ *and* $G^{-1}(\mathbf{0})$. *By* $[-1,1]$ *resp.* $]-1,1[$, *we will denote the closed resp. open interval of the reals between* -1 *and* 1.

Then by a straightforward calculation,

$$Z_T(\mathbf{0}) = \{x \in T_1(\mathbf{0}) : \|x| = \|-x| = 1\} = \{(1,a),(-1,a) : a \in [-1,1]\},$$

$$G_T(\mathbf{0}) = \{x \in T_1(\mathbf{0}) : \|-x| < 1 = \|x|\} = \{(a,1) : a \in]-1,1[\},$$

and

$$G_T^{-1}(\mathbf{0}) = \{x \in T_1(\mathbf{0}) : \|x| < 1 = \|-x|\} = \{(a,-1) : a \in]-1,1[\}.$$

Observe that as expected the latter 3 *sets are pairwise disjoint and their union is* $T_1(\mathbf{0})$.

Note that indeed we have characterized in this way the symmetric, positively biased, and negatively biased vectors of our space, since the three properties are preserved by multiplication with a positive real scalar.

For instance a vector (x_1, x_2) *is positively biased if and only if it can be written as* (λ, a) *where* $\lambda \in \mathbb{R}$ *is positive and* $a \in]-1,1[$. *It follows immediately that* $G := G_{d_{\|\cdot|}}$ *is transitive, since the sum of two positively biased vectors is positively biased.*

Let us note that $(1,1)$ *is a symmetric vector in* $(\mathbb{R}^2, d_{\|\cdot|})$, *but the vector* $(1,1)$ *is not symmetric in* $(\mathbb{R}^2, d_{\|\cdot|_\infty})$. *Hence,* $d_{\|\cdot|}$ *and* $d_{\|\cdot|_\infty}$ *are not graph-equivalent on* $\mathbb{R}^2$.

4. Strongly connected T_0-quasi-metric extensions

In this section we obtain some results on strongly connected T_0-quasi-metric extensions of T_0-quasi-metric spaces (compare also [7]).

It is obvious that a metric space (X, m) cannot have a strongly connected T_0-quasi-metric one-point extension (Y, e) (with remainder $\{\infty\}$):

Indeed, otherwise there would be points $x, y \in X$ together with a directed path contained entirely in X from x to y and arcs (∞, x) and (y, ∞) in (Y, e). Since m is a metric, necessarily $x = y$, but the edge $\{x, \infty\}$ can only have one orientation according to our convention— a contradiction.

Now, we can modify this argument and apply it to related situations:

Proposition 4.1. *The* T_0*-quasi-metric space* $(\mathbb{R}, u)$ *does not have a strongly connected* T_0*-quasi-metric one-point extension.*

Proof. Let $Y = \mathbb{R} \cup \{\infty\}$ with $\infty \notin \mathbb{R}$ and let (Y, e) be a T_0-quasi-metric one-point extension of (X, d) that is strongly connected. Similarly as above, we find a directed circuit $(\infty, x, \ldots, y, \infty)$ with $x, y \in \mathbb{R}$. As above, we see that $x \neq y$.

By shortening the number of arcs in that circuit if necessary, we can assume without loss of generality that all points from x to y belong to a directed path in $\mathbb{R}$. Recall then that (a, b) is an arc in $(\mathbb{R}, u)$ if and only if $a < b$. Hence $x < y$ in $\mathbb{R}$. Moreover, the fact that (y, ∞) is an arc in (Y, e) implies by Lemma 3.3 that (x, ∞) is an arc in (Y, e) with $x \neq \infty$. However this again contradicts with our assumption that (∞, x) is an arc in (Y, e). Thus, we have reached a contradiction and conclude that such an extension (Y, e) of $(\mathbb{R}, u)$ does not exist. □

An analogous proof shows the following result:

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Remark 4.2. *Let $(X, \leq)$ be a linearly ordered set and $d_{\leq}$ its natural T_0-quasi-metric on X. Then (X, d) does not have a T_0-quasi-metric one-point-extension that is strongly connected.*

Theorem 4.3. *Let (X, d) be an arbitrary T_0-quasi-metric space. Then there exists a strongly connected T_0-quasi-metric extension (Y, e) of (X, d) such that the remainder $Y \setminus X$ is countable.*

Proof. Let (X, d) be a T_0-quasi-metric space and $Y = X \times \mathbb{R}^3$.

Set

$$e((x_1, r_1, r_2, r_3), (y_1, s_1, s_2, s_3)) = d(x_1, y_1) \vee (r_1 - s_1)^+ \vee (r_2 - s_2)^+ \vee (r_3 - s_3)^+$$

whenever $x_1, y_1 \in X$ and $r_1, r_2, r_3, s_1, s_2, s_3 \in \mathbb{R}$. Then e is a T_0-quasi-metric on Y.

Now let us take an arbitrary $x_0 \in X$. Then $\bigcup_{n\in\mathbb{N}} B_{d^s}(x_0, n) = X$. Moreover $x \mapsto (x, 0, 0, 0)$ yields an isometric embedding i of (X, d) into (Y, e).

Set

$$R = \{(x_0, -k, -10k, 6k), (x_0, -7k, -6k, 10k) : k \in \mathbb{N}\}$$

Furthermore, let $M = i(X) \cup R$. We want to show that $(M, e|_{M\times M})$ is a strongly connected T_0-quasi-metric extension of (X, d).

Observe that $((x_0, -7k, -6k, 10k), (x_0, -k, -10k, 6k))$ is an arc whenever $k \in \mathbb{N}$. Also, note that $((x_0, -k', -10k', 6k'), (x_0, -k, -10k, 6k))$ is an arc whenever $k, k' \in \mathbb{N}$ and $k < k'$.

Similarly $((x_0, -7k, -6k, 10k), (x_0, -7k', -6k', 10k'))$ is an arc whenever $k, k' \in \mathbb{N}$ and $k < k'$.

Let $x \in X$. In this case, there exists $k \in \mathbb{N}$ such that $d^s(x_0, x) < 6k$. Then the edges

$$((x_0, -k, -10k, 6k), (x, 0, 0, 0))$$

and

$$((x, 0, 0, 0), (x_0, -7k, -6k, 10k))$$

are arcs.

Given $k, k' \in \mathbb{N}$ with $k < k'$. Now, by setting $x = x_0$ we see that

$$((x_0, -k, -10k, 6k), (x_0, 0, 0, 0), (x_0, -7k, -6k, 10k), (x_0, -7k', -6k', 10k'), (x_0, -k', -10k', 6k')$$

$, (x_0, -k, -10k, 6k))$ is a directed cycle. Hence if $x, y \in R$, there is a directed walk from x to y. It means that all points of R belong to the same strong connectedness component in (Y, e).

Let $x, y \in X, x \neq y$. Take $k \in \mathbb{N}$ such that $d^s(x_0, x) < 6k$ and $d^s(x_0, y) < 6k$. Then

$$((x, 0, 0, 0), (x_0, -7k, -6k, 10k), (x_0, -k, -10k, 6k), (y, 0, 0, 0))$$

is a directed path in (Y, e) from x to y. Hence M is indeed strongly connected. □

Remark 4.4. *We do not know whether each T_0-quasi-metric space has a strongly connected T_0-quasi-metric two-point extension. Let us next prove some positive results under additional conditions (compare [7] for related results).*

Corollary 4.5. *Each bounded T_0-quasi-metric space (X, d) has a strongly connected T_0-quasi-metric two-point extension (Z, e).*

Proof. Since (X,d) is bounded, we can find $x_0 \in X$ and $k_0 \in \mathbb{N}$ such that $B_{d^s}(x_0, 6k_0) = X$. We consider the following subspace Z of Y as defined in the proof of Theorem 4.3 where

$$Z = i(X) \cup \{(x_0, -k_0, -10k_0, 6k_0), (x_0, -7k_0, -6k_0, 10k_0)\}.$$

Then the space (Z,e) is a T_0-quasi-metric two-point extension of (X,d). Here, e is defined as in the proof of Theorem 4.3.

Therefore, an obvious modification of the proof of Theorem 4.3 shows that the constructed T_0-quasi-metric two-point extension (Z,e) of (X,d) is strongly connected. $\square$

In fact, Corollary 4.5 can be strengthened as follows.

Theorem 4.6. *Let (X,d) be a T_0-quasi-metric space. Then (X,d) has a strongly connected T_0-quasi-metric two-point extension (Y,e) provided that there are $x_0 \in X$ and a positive real number b such that $|d(x_0,x) - d(x,x_0)| < b$ whenever $x \in X$.*

Proof. Set $Y = X \cup \{-\infty, \infty\}$ and x_0, b as described in the assertion. We extend d to Y as follows, where $x \in X$:

Let $e(\infty, x) = d(x_0, x) + 2$, and $e(x, \infty) = d(x, x_0) + b$.

Put $e(x, -\infty) = d(x, x_0) + 2b$ and $e(-\infty, x) = d(x_0, x) + b$.

Also, take $e(-\infty, \infty) = 2b$ and $e(\infty, -\infty) = b$.

Finally choose $e(\infty, \infty) = e(-\infty, -\infty) = 0$.

Now, assume that $x \in X$. Thus, $(-\infty, x, \infty, -\infty)$ is a directed cycle in (Y,e) :

Note that the verification that $(-\infty, x)$ and (x, ∞) are arcs in (Y,e) makes use of the assumption that $|d(x_0,x) - d(x,x_0)| < b$ whenever $x \in X$, which indeed implies that $e(-\infty, x) < e(x, -\infty)$ and $e(x, \infty) < e(\infty, x)$.

Hence it remains to check that e satisfies the triangle inequality:

$e(a,c) \leq e(a,b) + e(b,c)$ whenever $a,b,c \in Y$, where without loss of generality it suffices to consider the case that a,b,c are pairwise distinct.

So we have to consider the triangles with (A) all vertices in X, (B) one vertex in the remainder $Y \setminus X$ or (C) two vertices in the remainder $Y \setminus X$.

Now, we consider the various cases:

A) For $x,y,z \in X : e(x,z) \leq e(x,y) + e(y,z)$ holds, that is, $d(x,z) \leq d(x,y) + d(y,z)$.

B(1) For $x,z \in X : e(x,z) \leq e(x,\infty) + e(\infty,z)$ holds, that is, $d(x,z) \leq d(x,x_0) + b + d(x_0,z) + 2b$.

B(2) For $y,z \in X : e(\infty,z) \leq e(\infty,y) + e(y,z)$ holds, that is, $d(x_0,z) + 2b \leq d(x_0,y) + 2b + d(y,z)$.

B(3) For $x,y \in X : e(x,\infty) \leq e(x,y) + e(y,\infty)$ holds, that is, $d(x,x_0) + b \leq d(x,y) + d(y,x_0) + b$.

B(4) For $x,z \in X : e(x,z) \leq e(x,-\infty) + e(-\infty,z)$ holds, that is, $d(x,z) \leq d(x,x_0) + 2b + d(x_0,z) + b$.

B(5) For $y,z \in X : e(-\infty,z) \leq e(-\infty,y) + e(y,z)$ holds, that is, $d(x_0,z) + b \leq d(x_0,y) + b + d(y,z)$.

B(6) For $x,y \in X : e(x,-\infty) \leq e(x,y) + e(y,-\infty)$ holds, that is, $d(x,x_0) + 2b \leq d(x,y) + d(y,x_0) + 2b$.

C(1) For $z \in X : e(\infty,z) \leq e(\infty,-\infty) + e(-\infty,z)$ holds, that is, $d(x_0,z) + 2b \leq b + d(x_0,z) + b$.

C(2) For $z \in X : e(-\infty,z) \leq e(-\infty,\infty) + e(\infty,z)$ holds, that is, $d(x_0,z) + b \leq 2b + d(x_0,z) + 2b$.

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C(3) For $x \in X : e(x,\infty) \leq e(x,-\infty) + e(-\infty,\infty)$ holds, that is, $d(x,x_0) + b \leq d(x,x_0) + 2b + 2b$.

C(4) For $x \in X : e(x,-\infty) \leq e(x,\infty) + e(\infty,-\infty)$ holds, that is, $d(x,x_0) + 2b \leq d(x,x_0) + b + b$.

C(5) For $y \in X : e(-\infty,\infty) \leq e(-\infty,y) + e(y,\infty)$ holds, that is, $2b \leq d(x_0,y) + b + d(y,x_0) + b$.

C(6) For $y \in X : e(\infty,-\infty) \leq e(\infty,y) + e(y,-\infty)$ holds, that is, $b \leq d(x_0,y) + 2b + d(y,x_0) + 2b$.

It follows that the triangle inequality is satisfied in all these cases. Hence e satisfies the triangle inequality on Y and the statement of the theorem is established. □

Corollary 4.7. *Let (X,d) be a T_0-quasi-metric space having a symmetric point $x_0 \in X$, that is $d(x_0,x) = d(x,x_0)$ whenever $x \in X$ (see* [10, Lemma 30]*). Then (X,d) has a strongly connected T_0-quasi-metric two-point extension.*

Therefore, we have also the next result.

Corollary 4.8. *Let (X,d) be a metric space. Then (X,d) has a strongly connected T_0-quasi-metric two-point extension.*

Let us finish the paper with an open question:

Remark 4.9. *Which asymmetrically normed real vector spaces $(X, \| \cdot |)$ induce a T_0-quasi-metric $d_{\|\cdot|}$ such that $(X, d_{\|\cdot|})$ is strongly connected?*

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