

## UNICYCLIC GRAPHS WITH NON-ISOLATED RESOLVING NUMBER 2

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ABSTRACT. Let  $G$  be a connected graph and  $W = \{w_1, w_2, \dots, w_k\}$  be an ordered subset of vertices of  $G$ . For any vertex  $v$  of  $G$ , the ordered  $k$ -vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \dots, d(v, w_k))$$

is called the metric representation of  $v$  with respect to  $W$ , where  $d(x, y)$  is the distance between the vertices  $x$  and  $y$ . A set  $W$  is called a resolving set for  $G$  if distinct vertices of  $G$  have distinct metric representations with respect to  $W$ . The minimum cardinality of a resolving set for  $G$  is its metric dimension denoted by  $\dim(G)$ . A resolving set  $W$  is called a non-isolated resolving set for  $G$  if the induced subgraph  $\langle W \rangle$  of  $G$  has no isolated vertices. The minimum cardinality of a non-isolated resolving set for  $G$  is called the non-isolated resolving number of  $G$  and denoted by  $nr(G)$ . The aim of this paper is to find properties of unicyclic graphs that have non-isolated resolving number 2 and then to characterize all these graphs.

### 1. Introduction

In this section, we present some definitions and notations which are necessary to prove main results. Throughout this paper,  $G$  is a simple connected graph with vertex set  $V(G)$  and edge set  $E(G)$ . The distance between two vertices  $u$  and  $v$ , denoted by  $d(u, v)$ , is the length of a shortest path between  $u$  and  $v$ . The number of all neighbors of a vertex  $v$  is  $\deg(v)$ . We use symbols  $(v_1, v_2, \dots, v_n)$  and

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$(v_1, v_2, \dots, v_n, v_1)$  for a path of order  $n$ ,  $P_n$ , and a cycle of order  $n$ ,  $C_n$ , respectively. A unicyclic graph is a graph containing exactly one cycle.

For an ordered subset  $W = \{w_1, \dots, w_k\}$  of  $V(G)$  and a vertex  $v$  of  $G$ , the *metric representation* of  $v$  with respect to  $W$  is

$$r(v|W) = (d(v, w_1), \dots, d(v, w_k)).$$

The set  $W$  is a *resolving set* for  $G$  if the distinct vertices of  $G$  have different metric representations, with respect to  $W$ . A resolving set  $W$  for  $G$  with minimum cardinality is a *metric basis* of  $G$ , and its cardinality is the *metric dimension* of  $G$ , denoted by  $\dim(G)$ . A resolving set  $W$  is called a *non-isolated resolving set* for  $G$  if the induced subgraph  $\langle W \rangle$  of  $G$  has no isolated vertices. A non-isolated resolving set for  $G$  with minimum cardinality is called a *non-isolated basis* for  $G$  and its cardinality is called the *non-isolated resolving number* of  $G$  and denoted by  $nr(G)$ .

The concepts of resolving sets and metric dimension of a graph were introduced independently by Slater [16] and by Harary and Melter [10]. Resolving sets have several applications in diverse areas such as coin weighing problems [15], network discovery and verification [5], robot navigation [13], mastermind game [7], problems of pattern recognition and image processing [14], and combinatorial search and optimization [15]. For more results about resolving sets and metric dimension see [4, 6, 7, 8, 11]. Several variations of resolving sets were introduced by imposing conditions on the subgraph induced by a resolving set. One of these variations is the concept of non-isolated resolving sets. The concepts of non-isolated resolving sets and non-isolated resolving number were introduced in [12]. The non-isolated resolving number of some families of graphs such as paths, complete graphs, bipartite graphs and some friendship graphs are obtained in [12]. For more results about non-isolated resolving sets see [1, 2, 3, 12].

It is easy to see that if  $G$  is a graph of order  $n$ , then  $2 \leq nr(G) \leq n - 1$ . In [12] all graphs of order  $n$  and resolving number  $n - 1$  are characterized. Also in [12] the problem of characterization all graphs with non-isolated resolving number 2 is proposed as an open problem. An important family of graphs is the family of unicyclic graphs. Unicyclic graphs with metric dimension 2 are studied in [9]. In this paper we investigate unicyclic graphs with non-isolated resolving number 2. The aim of the paper is to find properties of these graphs and then to characterize all of them.

## 2. Main Results

The main goal of this section is to characterize all unicyclic graphs with non-isolated resolving number 2. To do this, we need to consider some properties of these graphs. By the following lemma, in a unicyclic graph with non-isolated resolving number 2 the vertices not in the cycle can not be in a minimum non-isolated resolving set.

**Lemma 2.1.** *Let  $G$  be a unicyclic graph with  $nr(G) = 2$ . Then every member of a non-isolated basis of  $G$  is on the cycle.*

*Proof.* Suppose, on the contrary, there exists a non-isolated basis  $B$  of  $G$  and  $v \in B$  is not on the cycle. Let  $u$  be the nearest vertex of the cycle to  $v$  and  $u_1, u_2$  be the neighbours of  $u$  on the cycle. Since  $B$  is non-isolated basis, for every  $b \in B$  we have  $d(u_1, b) = d(u_2, b)$ . This contradiction completes the proof.  $\square$

By the next lemma, in a unicyclic graph with non-isolated resolving number two the degree of vertices not in the cycle is at most 2.

**Lemma 2.2.** *Let  $G$  be a unicyclic graph with  $nr(G) = 2$ . Then every vertex out of the cycle is of degree at most 2.*

*Proof.* Suppose, on the contrary, that there exists a vertex  $u$  out of the cycle with degree at least three. If  $B$  is a non-isolated basis of  $G$ , then by Lemma 2.1 every vertex in  $B$  is on the cycle. Since  $\deg(u) \geq 3$ ,  $u$  has three neighbours  $u_1, u_2, u_3$  such that for every  $b \in B$ ,  $d(u_2, b) = d(u_1, b) + 2 = d(u_3, b)$ , which is a contradiction. Therefore every vertex out of the cycle is of degree at most 2.  $\square$

**Lemma 2.3.** *Let  $G$  be a unicyclic graph with  $nr(G) = 2$ . Then  $\deg(v) \leq 3$ , for every  $v \in V(G)$ .*

*Proof.* By Lemma 2.2 it is sufficient to prove that all vertices on the cycle are of degree at most 3. If  $u$  is a vertex on the cycle of degree at least 4, then there are two neighbours  $u_1, u_2$  of  $u$ , out of the cycle. Note that by Lemma 2.1 all vertices of a non-isolated basis for  $G$  are on the cycle. But for each vertex  $v$  on the cycle we have  $d(u_1, v) = d(u, v) + 1 = d(u_2, v)$ . This contradiction implies that every vertex on the cycle is of degree at most 3.  $\square$

The following lemma implies that in a unicyclic graph with non-isolated resolving number 2, there are at most three vertices of degree more than 2.

**Lemma 2.4.** *Let  $G$  be a unicyclic graph with  $nr(G) = 2$  and  $C_m$  be the unique cycle of  $G$ . If  $m$  is odd, then there are at most three vertices of degree 3 in  $G$  and the distance between every two vertices of degree 3 is at most two. If  $m$  is even, then there are at most two vertices of degree 3 in  $G$  and they are adjacent.*

*Proof.* By Lemma 2.2, all vertices of degree 3 are on the cycle. Let  $C_m = (v_1, v_2, \dots, v_m, v_1)$  be the unique cycle of  $G$  and  $B = \{v_1, v_2\}$  be a non-isolated basis of  $G$ . Clearly for every  $1 \leq i, j \leq m$ ,  $d(v_i, v_j) \leq \lfloor \frac{m}{2} \rfloor = d$ . Note that if for some  $1 \leq i \leq m$  both entries of  $r(v_i|B) = (r_1, r_2)$  are less than  $d$ , then  $v_i$  has a neighbour on  $C_m$  with metric representation  $(r_1 + 1, r_2 + 1)$ . On the other hand, if  $u_i \notin V(C_m)$  is adjacent to  $v_i$ , then  $r(u_i|B) = (r_1 + 1, r_2 + 1)$  with respect to  $B$ . Hence a vertex of  $C_m$  has a neighbour out of  $C_m$  if and only if at least one of the entries of its metric representation with respect to  $B$  is  $d$ . Therefore when  $m$  is odd all vertices of degree 3 must be in the set  $\{v_{d+1}, v_{d+2}, v_{d+3}\}$ , because vertices in this set are all vertices on  $C_m$  with distance  $d$  to  $v_1$  or  $v_2$ . When  $m$  is even all vertices of degree 3 must be in the set  $\{v_{d+1}, v_{d+2}\}$ , because vertices in this set are all vertices on  $C_m$  with distance  $d$  to  $v_1$  or  $v_2$ . Clearly the distance between each pair of these vertices is at most 2.  $\square$

To state our characterization we need the following definition.

**Definition 2.5.** If  $n \geq 3$  is an odd number, then define the family  $\mathcal{O}_n$  of graphs  $G$  obtained from a cycle  $C_n = (v_1, v_2, \dots, v_n, v_1)$  and paths  $P_r = (a_1, a_2, \dots, a_r)$ ,  $P_s = (b_1, b_2, \dots, b_s)$ , and  $P_t = (c_1, c_2, \dots, c_t)$ ,  $r, s, t \geq 0$ , by identifying vertices  $v_1$  to  $a_1$ ,  $v_2$  to  $b_1$  and  $v_n$  to  $c_1$ .

If  $n \geq 4$  is an even number, then define the family  $\mathcal{E}_n$  of graphs  $G$  obtained from a cycle  $C_n = (v_1, v_2, \dots, v_n, v_1)$ , and paths  $P_r = (a_1, a_2, \dots, a_r)$  and  $P_s = (b_1, b_2, \dots, b_s)$ ,  $r, s \geq 0$ , by identifying vertices  $v_1$  to  $a_1$  and  $v_2$  to  $b_1$ .

The next theorem characterizes all unicyclic graphs with non-isolated resolving number 2.

**Theorem 2.6.** Let  $G$  be a unicyclic graph. Then  $nr(G) = 2$  if and only if  $G$  belongs to  $\mathcal{O}_n$  or  $\mathcal{E}_n$ , for some integer  $n$ .

*Proof.* If  $G \in \mathcal{O}_n$ , let  $m = \frac{n-1}{2}$ . We prove that  $B = \{v_{m+1}, v_{m+2}\}$  is a non-isolated basis for  $G$ . Since  $v_{m+1}$  is adjacent to  $v_{m+2}$  it is enough to prove that  $B$  is a resolving set for  $G$ . Note that

$$d(v_i, v_{m+1}) = \begin{cases} i - (m + 1) & \text{if } m + 2 \leq i \leq n, \\ m + 1 - i & \text{if } 1 \leq i \leq m + 1. \end{cases}$$

$$d(v_i, v_{m+2}) = \begin{cases} i - (m + 2) & \text{if } m + 2 \leq i \leq n, \\ m + 2 - i & \text{if } 1 < i \leq m + 1, \\ m & \text{if } i = 1. \end{cases}$$

Hence, if for some  $1 \leq i < j \leq n$ , we have  $d(v_i, v_{m+1}) = d(v_j, v_{m+1})$ , then  $i = 2m + 2 - j$ . Therefore  $i \leq m$  and  $m + 2 \leq j \leq n$ . This implies  $d(v_j, v_{m+2}) = j - (m + 2)$ . if  $i = 1$ , clearly  $d(v_i, v_{m+2}) = m \neq j - (m + 2)$ . Hence  $i \neq 1$  and  $d(v_i, v_{m+2}) = i - (m + 2)$ . This means  $d(v_i, v_{m+2}) \neq d(v_j, v_{m+2})$ , otherwise  $i = 2m + 4 - j$ , which contradicts  $i = 2m + 2 - j$ . Therefore  $r(v_i|B) \neq r(v_j|B)$ , for  $1 \leq i \neq j \leq n$ . On the other hand,

$$r(a_i|B) = (i + m - 1, i + m - 1), \quad 1 \leq i \leq r,$$

$$r(b_i|B) = (i + m - 2, i + m - 1), \quad 1 \leq i \leq s,$$

$$r(c_i|B) = (i + m - 1, i + m - 2), \quad 1 \leq i \leq t.$$

Therefore  $r(x|B) \neq r(y|B)$  for all  $x, y \in V(G) \setminus V(C_n)$ . Also if  $v_i \in V(C_n)$  and  $r(v_i|B) = (r_1, r_2)$ , then  $r_1 + r_2 \leq 2m$ , while for every  $u \in V(G) \setminus V(C_n)$  the summation of entries of  $r(u|B)$  is at least  $2m + 1$ , because  $a_1, b_1, c_1 \in V(C_n)$ . Hence  $B$  is a non-isolated basis of  $G$ .

If  $G \in \mathcal{E}_n$ , let  $m = \frac{n}{2}$ . We prove that  $B = \{v_{m+1}, v_{m+2}\}$  is a non-isolated basis for  $G$ . Since  $v_{m+1}$  is adjacent to  $v_{m+2}$  it is enough to prove that  $B$  is a resolving set for  $G$ . Note that

$$d(v_i, v_{m+1}) = \begin{cases} i - (m + 1) & \text{if } m + 2 \leq i \leq n, \\ m + 1 - i & \text{if } 1 \leq i \leq m + 1. \end{cases}$$

$$d(v_i, v_{m+2}) = \begin{cases} i - (m + 2) & \text{if } m + 2 \leq i \leq n, \\ m + 2 - i & \text{if } 1 < i \leq m + 1, \\ m - 1 & \text{if } i = 1. \end{cases}$$

Hence, if for some  $1 \leq i < j \leq n$ , we have  $d(v_i, v_{m+1}) = d(v_j, v_{m+1})$ , then  $i = 2m + 2 - j$ . Therefore  $i \leq m$  and  $m + 2 \leq j \leq n$ . This implies  $d(v_j, v_{m+2}) = j - (m + 2)$ . if  $i = 1$ , clearly  $d(v_i, v_{m+2}) = m - 1 \neq j - (m + 2)$ . Hence  $i \neq 1$  and  $d(v_i, v_{m+2}) = i - (m + 2)$ . This means  $d(v_i, v_{m+2}) \neq d(v_j, v_{m+2})$ , otherwise  $i = 2m + 4 - j$ , which contradicts  $i = 2m + 2 - j$ . Therefore  $r(v_i|B) \neq r(v_j|B)$ , for  $1 \leq i \neq j \leq n$ . On the other hand,

$$r(a_i|B) = (i + m - 1, i + m - 2), \quad 1 \leq i \leq r,$$

$$r(b_i|B) = (i + m - 2, i + m - 1), \quad 1 \leq i \leq s,$$

Therefore  $r(x|B) \neq r(y|B)$  for all  $x, y \in V(G) \setminus V(C_n)$ . Also if  $v_i \in V(C_n)$  and  $r(v_i|B) = (r_1, r_2)$ , then  $r_1 + r_2 \leq 2m - 1$ , while for every  $u \in V(G) \setminus V(C_n)$  the summation of entries of  $r(u|B)$  is at least  $2m + 1$ , because  $a_1, b_1 \in V(C_n)$ . Hence  $B$  is a non-isolated basis of  $G$ .

For the converse let  $G$  be a unicyclic graph with  $nr(G) = 2$ , Let  $C_n = (v_1, v_2, \dots, v_n, v_1)$  be the unique cycle of  $G$ . By Lemma 2.2,  $G$  consists of  $C_n$  and some paths that connect to  $C_n$  by identifying one of their leaves and a vertex of  $C_n$ . By Lemma 2.3 at most one path can connect to each vertex of  $C_n$ . If  $n$  is odd, then Lemma 2.4 concludes the number of paths is at most 3 and they must connect to vertices with distance at most 2 on  $C_n$ . Hence by a relabeling vertices of  $C_n$  we can let paths are connected to some vertices in the set  $\{v_n, v_1, v_2\}$ . This means for odd numbers  $n$ ,  $G$  belongs to  $\mathcal{O}_n$ . If  $n$  is even, then Lemma 2.4 implies that the number of paths is at most 2 and they must connect to vertices with distance at most 1 on  $C_n$ . Hence by a relabeling vertices of  $C_n$  we can let paths are connected to some vertices in the set  $\{v_1, v_2\}$ . This means  $G$  belongs to  $\mathcal{E}_n$ . □

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