



## BOUNDING THE DOMINATION NUMBER OF A TREE IN TERMS OF ITS ANNIHILATION NUMBER

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ABSTRACT. A set  $S$  of vertices in a graph  $G$  is a dominating set if every vertex of  $V - S$  is adjacent to some vertex in  $S$ . The domination number  $\gamma(G)$  is the minimum cardinality of a dominating set in  $G$ . The annihilation number  $a(G)$  is the largest integer  $k$  such that the sum of the first  $k$  terms of the non-decreasing degree sequence of  $G$  is at most the number of edges in  $G$ . In this paper, we show that for any tree  $T$  of order  $n \geq 2$ ,  $\gamma(T) \leq \frac{3a(T)+2}{4}$ , and we characterize the trees achieving this bound.

### 1. Introduction

In this paper,  $G$  is a simple graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . The order  $|V|$  of  $G$  is denoted by  $n = n(G)$ . For every vertex  $v \in V(G)$ , the *open neighborhood*  $N_G(v) = N(v)$  is the set  $\{u \in V(G) \mid uv \in E(G)\}$  and the *closed neighborhood* of  $v$  is the set  $N_G[v] = N[v] = N(v) \cup \{v\}$ . The *degree* of a vertex  $v \in V$  is  $\deg_G(v) = \deg(v) = |N(v)|$ . The *minimum degree* of a graph  $G$  is denoted by  $\delta = \delta(G)$ . We write  $P_n$  for a path of order  $n$ . For a subset  $S \subseteq V(G)$ , we let

$$\sum(S, G) = \sum_{v \in S} \deg_G(v).$$

A dominating set, abbreviated DS, of a graph  $G$  is a set  $S$  of vertices of  $G$  such that every vertex in  $V(G) - S$  is adjacent to at least one vertex in  $S$ . The domination number of  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality of a DS of  $G$ . A DS of cardinality  $\gamma(G)$  is called a  $\gamma(G)$ -set. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The literature

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on the subject of domination parameters in graphs has been surveyed and detailed in the two books [7, 8].

Let  $d_1, d_2, \dots, d_n$  be the degree sequence of a graph  $G$  arranged in non-decreasing order, and so  $d_1 \leq d_2 \leq \dots \leq d_n$ . The annihilation number of  $G$ , denoted  $a(G)$ , is the largest integer  $k$  such that the sum of the first  $k$  terms of the degree sequence is at most half the sum of the degrees in the sequence. Equivalently, the annihilation number is the largest integer  $k$  such that the

$$\sum_{i=1}^k d_i \leq \sum_{i=k+1}^n d_i.$$

It is clear from the definition that if  $G$  has  $m$  edges and annihilation number  $k$ , then  $\sum_{i=1}^k d_i \leq m$ . As an immediate consequence of the definition of the annihilation number, Larson and Pepper [10] observed that for any graph  $G$  of order  $n$ ,

$$(1.1) \quad a(G) \geq \lfloor \frac{n}{2} \rfloor.$$

The annihilation number was introduced by Pepper in [12] and has been studied by several authors (see for example [1, 2, 4, 5, 6, 9, 10, 13]). In [12] and [13], Pepper proved that the annihilation number is an upper bound on the independence number of a graph and in [10] the case for equality of the upper bound was characterized by Larson and Pepper. Since independence number is an upper bound on domination number, we deduce that for any graph  $G$ ,  $\gamma(G) \leq a(G)$ .

The relation between annihilation number and some graph parameters have been studied by several authors. For instance, DeLaViña et al. presented an upper bound on 2-domination number in terms of annihilation number for some classes of graphs [4], Aram et al. investigated the relation between the Roman domination number and the annihilation number of trees [1], Desormeaux et al. proved that for any tree  $T$ ,  $a(T) + 1$  is an upper bound on the total domination number [6].

Our purpose in this paper is to establish an upper bound on the domination number of a tree in terms of its annihilation number.

The domination and annihilation numbers are easy to compute for paths and we have the following Observations.

**Observation 1.1.** For  $n \geq 2$ ,

$$\gamma(P_n) = \lceil \frac{n}{3} \rceil.$$

**Observation 1.2.** For  $n \geq 2$ ,

$$a(P_n) = \lceil \frac{n}{2} \rceil.$$

**Proposition 1.3.** For  $n \geq 2$ ,  $\gamma(P_n) \leq \frac{3a(T)+2}{4}$  with equality if and only if  $n = 4$ . Furthermore,  $\gamma(P_n) = \frac{3a(T)+1}{4}$  if and only if  $n = 2$  or  $10$ .

## 2. Main results

A *subdivision* of an edge  $uv$  is obtained by removing the edge  $uv$ , adding a new vertex  $w$ , and adding edges  $uw$  and  $wv$ . The *subdivision graph*  $S(G)$  is the graph obtained from  $G$  by subdividing each edge of  $G$ . The subdivision star  $S(K_{1,t})$  for  $t \geq 2$ , is called a *healthy spider*  $S_t$ . A *wounded spider*  $S_t$  is the graph formed by subdividing at most  $t - 1$  of the edges of a star  $K_{1,t}$  for  $t \geq 2$ . Note that stars are wounded spiders. A *spider* is a healthy or wounded spider.

**Proposition 2.1.** If  $T$  is a spider different from  $P_4$ , then  $\gamma(T) \leq \frac{3a(T)+1}{4}$  with equality if and only if  $T$  is the wounded spider obtained from  $K_{1,4}$  by subdividing its exactly three edges.

*Proof.* Let  $T$  be a spider. If  $T = S_t$  is a healthy spider for some  $t \geq 2$ , then obviously  $\gamma(T) = t$  and  $a(T) = t + \lfloor \frac{t}{2} \rfloor$ , and hence  $\gamma(T) < \frac{3a(T)}{4}$ .

Now let  $T$  be a wounded spider obtained from  $K_{1,t}$  ( $t \geq 2$ ) by subdividing  $0 \leq s \leq t - 1$  edges. If  $s = 0$ , then  $T$  is a star and we have  $\gamma(T) = 1$  and  $a(T) = t$ . Hence  $\gamma(T) < \frac{3a(T)}{4}$ . Suppose  $s > 0$ . Since  $T \neq P_4$ , we have  $s \neq 1$  or  $t \neq 2$ . Then  $\gamma(T) = 1 + s$  and  $a(T) = s + \lfloor \frac{s}{2} \rfloor + (t - s)$ . If  $s = 3$  and  $t = 4$ , then clearly  $\gamma(T) = \frac{3a(T)+1}{4}$ . Otherwise, it is easy to see that  $\gamma(T) < \frac{3a(T)+1}{4}$ .

If  $T$  is the wounded spider obtained from  $K_{1,4}$  by subdividing its exactly three edges, then clearly  $\gamma(T) = 4$  and  $a(T) = 5$ . Hence  $\gamma(T) = \frac{3a(T)+1}{4}$ , and the proof is complete. □

A *leaf* of a tree  $T$  is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. A strong support vertex is said to be *end-strong support vertex* if all its neighbors except one of them are leaves. For  $r, s \geq 1$ , a double star  $S(r, s)$  is a tree with exactly two vertices that are not leaves, with one adjacent to  $r$  leaves and the other to  $s$  leaves. For a vertex  $v$  in a rooted tree  $T$ , let  $C(v)$  denote the set of children of  $v$ . Let  $D(v)$  denote the set of descendants of  $v$  and  $D[v] = D(v) \cup \{v\}$ . The *maximal subtree* at  $v$  is the subtree of  $T$  induced by  $D[v]$ , and is denoted by  $T_v$ . In the sequel, we denote by  $T - T_v$ , the tree obtained from a rooted tree  $T$  by deleting all vertices of  $D[v]$ .

**Theorem 2.2.** If  $T$  is a tree of order  $n \geq 2$ , then  $\gamma(T) \leq \frac{3a(T)+2}{4}$ .

*Proof.* The proof is by induction on  $n$ . The statement holds for all trees of order  $n \leq 4$ . For the inductive hypothesis, let  $n \geq 5$  and suppose that for every nontrivial tree  $T$  of order less than  $n$  the result is true. Let  $T$  be a tree of order  $n$ . If  $T$  is a path, then the result follows by Proposition 1.3. So, assume  $T$  is not a path. If  $\text{diam}(T) = 2$ , then  $T$  is a star and it follows from Proposition 2.1 that  $\gamma(T) < \frac{3a(T)}{4}$ . If  $\text{diam}(T) = 3$ , then  $T$  is a double star  $S(r, s)$ . In this case,  $a(T) = r + s \geq 3$  and  $\gamma(T) = 2$ . Hence  $\gamma(T) < \frac{3a(T)}{4}$ . Thus, we may assume that  $\text{diam}(T) \geq 4$ .

In what follows, we will consider trees  $T'$  formed from  $T$  by removing a set of vertices. For such a tree  $T'$  of order  $n'$ , let  $d'_1, d'_2, \dots, d'_{n'}$  be the non-decreasing degree sequence of  $T'$ , and let  $S'$  be a set of vertices corresponding to the first  $a(T')$  terms in the degree sequence of  $T'$ . In fact, if  $u_1, u_2, \dots, u_{n'}$  are the vertices of  $T'$  such that  $\text{deg}(u_i) = d'_i$  for each  $i$ , then  $S' = \{u_1, u_2, \dots, u_{a(T')}\}$ . We denote the size of  $T'$  by  $m'$ . We proceed further with a series of claims that we may assume satisfied by the tree.

**Claim 1.**  $T$  has no end-strong support vertex.

Let  $T$  have a end-strong support vertex  $u$  and let  $u_1, u_2$  be the two leaves adjacent to  $u$  and let  $w$  be a vertex of  $T$  with maximum distance from  $u$ . Root  $T$  at  $w$  and let  $v$  be the parent of  $u$ . Assume  $T' = T - T_u$ . Then obviously  $\gamma(T) \leq \gamma(T') + 1$ . If  $v \notin S'$ , then  $\sum(S', T) = \sum(S', T')$  and if  $v \in S'$ , then  $\sum(S', T) = \sum(S', T') + 1$ . Thus,  $\sum(S', T) - 1 \leq \sum(S', T') \leq m' = m - 3$ , and hence  $\sum(S', T) \leq m - 2$ . Let  $S = S' \cup \{u_1, u_2\}$ . Then  $\sum(S, T) = \sum(S', T) + 2 \leq m$  implying that  $a(T) \geq a(T') + 2$ . By inductive hypothesis, we obtain

$$\gamma(T) \leq \gamma(T') + 1 \leq \frac{3a(T') + 2}{4} + 1 \leq \frac{3(a(T) - 2) + 2}{4} + 1 = \frac{3a(T)}{4},$$

as desired. (■)

Let  $v_1 v_2 \dots v_D$  be a diametral path in  $T$  and root  $T$  at  $v_D$ . By Claim 1, we have  $\deg(v_2) = 2$  and all neighbors of  $v_3$ , except  $v_4$ , are leaves or support vertices of degree 2. Similarly, by rooting  $T$  at  $v_1$ , we may assume  $\deg(v_{D-1}) = 2$  and all neighbors of  $v_{D-2}$ , except  $v_{D-3}$ , are leaves or support vertices of degree 2. If  $\text{diam}(T) = 4$ , then  $T$  is a spider and the result follows by Proposition 2.1. Assume  $\text{diam}(T) \geq 5$ .

**Claim 2.**  $\deg_T(v_3) \leq 3$ .

Let  $\deg_T(v_3) \geq 4$ . First let  $v_3$  be adjacent to a support vertex, say  $w_2$ , not in  $\{v_2, v_4\}$ . Suppose  $w_1$  is the leaf adjacent to  $w_2$  and let  $T' = T - \{v_1, v_2, w_1, w_2\}$ . Then every dominating set of  $T'$  can be extended to a dominating set of  $T$  by adding  $v_1, w_1$  and hence  $\gamma(T) \leq \gamma(T') + 2$ . Suppose  $v_3 \notin S'$ . Then  $\sum(S', T) = \sum(S', T')$ . In this case, let  $S = S' \cup \{v_1, v_2, w_1\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w_1) \leq m$  implying that  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ . It follows from inductive hypothesis that

$$\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T') + 2}{4} + 2 \leq \frac{3(a(T) - 3) + 2}{4} + 2 < \frac{3a(T) + 2}{4}.$$

Now assume  $v_3 \in S'$ . Then  $\sum(S', T) = \sum(S', T') + 1$ . In this case, let  $S = (S' - \{v_3\}) \cup \{v_1, v_2, w_1, w_2\}$ . Since  $\deg_T(w_2) \leq \deg_{T'}(v_3)$ , we have that  $\sum(S, T) = \sum(S', T) - \deg_{T'}(v_3) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w_1) + \deg_T(w_2) \leq m$ . Therefore,  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$  and the result follows as above.

Now let all neighbors of  $v_3$ , except  $v_2, v_4$ , are leaves. Assume  $T' = T - T_{v_3}$ . Then every dominating set of  $T'$  can be extended to a domination set of  $T$  by adding  $v_1, v_3$  and hence  $\gamma(T) \leq \gamma(T') + 2$ . Suppose  $z_1, z_2$  are two leaves adjacent to  $v_3$  and let  $S = S' \cup \{v_1, z_1, z_2\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(z_1) + \deg_T(z_2) \leq m$  implying that  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$  and the result follows as above. (■)

**Claim 3.**  $\deg_T(v_3) = 2$ .

Assume  $\deg_T(v_3) = 3$ . First let  $v_3$  be adjacent to a support vertex  $x_2$  of degree 2, not in  $\{v_2, v_4\}$ . Suppose  $x_1$  is the leaf adjacent to  $x_2$  and let  $T' = T - T_{v_3}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $v_2, x_2$  and hence  $\gamma(T) \leq \gamma(T') + 2$ . If  $v_4 \notin S'$ , then  $\sum(S', T) = \sum(S', T')$  and if  $v_4 \in S'$ , then  $\sum(S', T) = \sum(S', T') + 1$ . Thus,  $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 4$ . Let  $S = S' \cup \{v_1, v_2, x_1\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(x_1) \leq m$  implying

that  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ . It follows from inductive hypothesis that  $\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T')+2}{4} + 2 \leq \frac{3(a(T)-3)+2}{4} + 2 < \frac{3a(T)+2}{4}$ .

Now let  $v_3$  be adjacent to a leaf  $w$ . We consider the following cases.

**Case 3.1.**  $\deg_T(v_4) \geq 4$ .

Let  $T' = T - T_{v_3}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $v_2, v_3$ . Hence  $\gamma(T) \leq \gamma(T') + 2$ . Suppose that  $v_4 \notin S'$ . In this case, let  $S = S' \cup \{v_1, v_2, w\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) = \sum(S', T') + 4 \leq m' + 4 = m$ , implying that  $a(T) \geq a(T') + 3$ . By inductive hypothesis we have  $\gamma(T) < \frac{3a(T)+2}{4}$ .

Now let  $v_4 \in S'$ . Assume  $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3, w\}$ . Then  $\sum(S, T) = \sum(S', T') - \deg_{T'}(v_4) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(v_3) + \deg_T(w) \leq m$  and hence  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ . By inductive hypothesis, we obtain  $\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T')+2}{4} + 2 \leq \frac{3(a(T)-3)+2}{4} + 2 < \frac{3a(T)+2}{4}$ .

**Case 3.2.**  $\deg_T(v_4) = 2$ .

Let  $T' = T - T_{v_4}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $v_1, v_3$  and so  $\gamma(T) \leq \gamma(T') + 2$ . If  $v_5 \notin S'$ , then  $\sum(S', T) = \sum(S', T')$  and if  $v_5 \in S'$ , then  $\sum(S', T) = \sum(S', T') + 1$ . Thus,  $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 4$ . Let  $S = S' \cup \{v_1, v_2, w\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) \leq m$  implying that  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ . By inductive hypothesis, we obtain  $\gamma(T) < \frac{3a(T)+2}{4}$ .

**Case 3.3.**  $\deg_T(v_4) = 3$  and there exists a path  $v_4 z_3 z_2 z_1$  in  $T$  such that  $\deg_T(z_3) = 2, \deg_T(z_1) = 1$  and  $z_3 \notin \{v_3, v_5\}$ .

By Claim 1, we have  $\deg_T(z_2) = 2$ . Let  $T' = T - T_{z_3}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $z_2$  and so  $\gamma(T) \leq \gamma(T') + 1$ . Assume that  $v_4 \notin S'$ . In this case, let  $S = S' \cup \{z_1, z_2\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(z_1) + \deg_T(z_2) \leq m' + 3 = m$ , implying that  $a(T) \geq |S| = |S'| + 2 = a(T') + 2$ . Applying inductive hypothesis we obtain  $\gamma(T) \leq \gamma(T') + 1 \leq \frac{3a(T')+2}{4} + 1 \leq \frac{3(a(T)-2)+2}{4} + 1 \leq \frac{3a(T)}{4} < \frac{3a(T)+2}{4}$ .

Now suppose  $v_4 \in S'$ . In this case, let  $S = (S' - \{v_4\}) \cup \{z_1, z_2, z_3\}$ . Since  $\deg_T(z_3) \leq \deg_{T'}(v_4)$ , we have  $\sum(S, T) = \sum(S', T') - \deg_{T'}(v_4) + \deg_T(z_1) + \deg_T(z_2) + \deg_T(z_3) \leq \sum(S', T') + 3 \leq m' + 3 \leq m$ . Therefore,  $a(T) \geq |S| = |S'| + 2 = a(T') + 2$  and the result follows by inductive hypothesis as above.

**Case 3.4.**  $\deg_T(v_4) = 3$  and there exists a path  $z_4 z_3 z_2 z_1$  in  $T$  such that  $v_4 z_3 \in E(T)$ , all neighbors of  $z_3$ , except  $z_2, v_4$ , are leaves,  $\deg(z_1) = \deg(z_4) = 1$  and  $z_3 \notin \{v_3, v_5\}$ .

By Claim 1, we may assume  $\deg_T(z_2) = 2$ . If  $\deg_T(z_3) \geq 4$ , then the result follows as Claim 2. Thus, we assume  $\deg_T(z_3) = 3$ . Let  $T' = T - T_{v_4}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $z_3, z_1, v_3, v_1$ , implying that  $\gamma(T) \leq \gamma(T') + 4$ . If  $v_5 \notin S'$ , then  $\sum(S', T) = \sum(S', T')$  and if  $v_5 \in S'$ , then  $\sum(S', T) = \sum(S', T') + 1$ . Thus,  $\sum(S', T) \leq m - 8$ . Let  $S = S' \cup \{v_1, v_2, w, z_1, z_2, z_4\}$ . Then  $\sum(S, T) = \sum(S', T) + 8 \leq m$  implying that  $a(T) \geq |S| = a(T') + 6$ . By inductive hypothesis, we have  $\gamma(T) \leq \gamma(T') + 4 \leq \frac{3a(T')+2}{4} + 4 \leq \frac{3(a(T)-6)+2}{4} + 4 \leq \frac{3a(T)}{4} < \frac{3a(T)+2}{4}$ .

**Case 3.5.**  $\deg(v_4) = 3$  and  $v_4$  is adjacent to a leaf, say  $w'$ .

Assume  $T' = T - T_{v_4}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $v_1, v_3, v_4$  and so  $\gamma(T) \leq \gamma(T') + 3$ . As above, we have  $\sum(S', T) \leq m - 5$ . Let  $S = S' \cup \{v_1, v_2, w, w'\}$ .

Then  $\sum(S, T) \leq m$  and hence  $a(T) \geq |S| = a(T') + 4$ . Applying inductive hypothesis we obtain  $\gamma(T) \leq \gamma(T') + 3 \leq \frac{3a(T')+2}{4} + 3 \leq \frac{3(a(T)-4)+2}{4} + 3 = \frac{3a(T)+2}{4}$ .

**Case 3.6.**  $\deg(v_4) = 3$  and  $v_4$  is adjacent to a support vertex  $z_2 \neq v_5$ .

By Claim 1, we may assume  $\deg_T(z_2) = 2$ . Let  $z_1$  be the leaf adjacent to  $z_2$  and let  $T' = T - T_{v_4}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $z_2, v_1, v_3$  and so  $\gamma(T) \leq \gamma(T') + 3$ . Clearly  $\sum(S', T) \leq \sum(S', T') + 1 \leq m' + 1 = m - 6$ . Let  $S = S' \cup \{v_1, v_2, w, z_1\}$ . Then  $\sum(S, T) = \sum(S', T) + \deg_T(v_1) + \deg_T(v_2) + \deg_T(w) + \deg_T(z_1) \leq m$  and hence  $a(T) \geq |S| = |S'| + 4 = a(T') + 4$ . Applying inductive hypothesis we obtain  $\gamma(T) \leq \gamma(T') + 3 \leq \frac{3a(T')+2}{4} + 3 \leq \frac{3(a(T)-4)+2}{4} + 3 = \frac{3a(T)+2}{4}$ . (■)

Similarly, by rooting  $T$  at  $v_1$ , we may assume that  $\deg(v_{D-2}) = 2$ .

We now return to the proof of theorem. If  $\text{diam}(T) = 5$  or  $6$  then  $T = P_6$  or  $P_7$ , respectively, and the result is immediate by Proposition 1.3. Let  $\text{diam}(T) \geq 7$  and  $T' = T - \{v_1, v_2, v_3, v_D, v_{D-1}, v_{D-2}\}$ . Then every  $\gamma(T')$ -set can be extended to a dominating set of  $T$  by adding  $v_2, v_{D-1}$  and hence  $\gamma(T) \leq \gamma(T') + 2$ . Suppose  $S = S' \cup \{v_1, v_2, v_D\}$ . Then  $\sum(S, T) \leq m$  implying that  $a(T) \geq |S| = |S'| + 3 = a(T') + 3$ . Applying inductive hypothesis, we obtain  $\gamma(T) \leq \gamma(T') + 2 \leq \frac{3a(T')+2}{4} + 2 \leq \frac{3(a(T)-3)+2}{4} + 2 < \frac{3a(T)+2}{4}$ . This completes the proof. □

**Theorem 2.3.** Let  $T$  be a tree of order  $n \geq 2$ . Then  $\gamma(T) = \frac{3a(T)+2}{4}$  if and only if  $T = P_4$ .

*Proof.* If  $T = P_4$ , then clearly  $\gamma(T) = \frac{3a(T)+2}{4}$ .

Conversely, let  $\gamma(T) = \frac{3a(T)+2}{4}$ . By Proposition 1.3, we have  $n \geq 4$ . Suppose to the contrary that  $T \neq P_4$ . Among all trees with these properties, let  $T$  be chosen so that its order is minimum. Let  $v_1 v_2 \dots v_D$  be a diametral path in  $T$  and root  $T$  at  $v_D$ . By the proof of Theorem 2.2, we may assume  $\text{diam}(T) \geq 5$  and we need to consider two cases.

**Case 1.**  $\deg(v_2) = 2, \deg(v_3) = \deg(v_4) = 3, v_3$  is adjacent to a leaf  $w$  and  $v_4$  is adjacent to a leaf  $w'$ . Let  $T' = T - T_{v_4}$ . By the Case 3.5, we have  $\gamma(T) \leq \gamma(T') + 3$  and  $a(T) \geq a(T') + 4$ . It follows from Theorem 2.2 that

$$(2.1) \quad \gamma(T) \leq \gamma(T') + 3 \leq \frac{3a(T') + 2}{4} + 3 \leq \frac{3(a(T) - 4) + 2}{4} + 3 = \frac{3a(T) + 2}{4}.$$

Since  $\gamma(T) = \frac{3a(T)+2}{4}$ , the inequalities occurring in (2.1) become equalities. In particular, we have  $\gamma(T) = \gamma(T') + 3$  and  $\gamma(T') = \frac{3a(T')+2}{4}$ . By the choice of  $T$ , we deduce that  $T' = P_4$ . If  $v_4$  is adjacent to a leaf of  $T' = P_4$ , then clearly  $\gamma(T) = \gamma(T') + 2 < \gamma(T') + 3$ , a contradiction. If  $v_4$  is adjacent to a support vertex of  $T' = P_4$ , then it is easy to see that  $\gamma(T) = 5$  and  $a(T) = 7$  and hence  $\gamma(T) = 5 < \frac{3a(T)+2}{4}$ , which is a contradiction.

**Case 2.**  $\deg(v_2) = 2, \deg(v_3) = \deg(v_4) = 3$  and  $v_3$  is adjacent to a leaf  $w$  and  $v_4$  is adjacent to a support vertex  $z_2$  of degree 2.

Assume  $T' = T - T_{v_4}$ . An argument similar to that described in Case 1, shows that  $T' = P_4$ . It is easy to see that  $\gamma(T) = 5$  and  $a(T) = 7$ . Hence  $\gamma(T) = 5 < \frac{3a(T)+2}{4}$ , a contradiction. This completes the proof. □

We conclude this paper with two open problems.

**Problem 1.** Characterize the trees  $T$  for which  $\gamma(T) = \frac{3a(T)+1}{4}$ .

If  $G$  is a connected graph of order  $n$  with minimum degree at least three, then it is known ([14]) that  $\gamma(G) \leq \frac{3n}{8}$ . Hence if  $G$  is a connected graph of order  $n$  with minimum degree at least 3, then it follows from (1.1) that  $\gamma(G) \leq \frac{3a(T)+1}{4}$ .

Cockayne, Ko and Shepherd [3] proved that if a connected graph  $G$  of order  $n$ , is  $K_{1,3}$ -free and  $K_{30}K_1$ -free then  $\gamma(G) \leq \lceil \frac{n}{3} \rceil$ . Using (1.1), we deduce that if  $G$  is a connected  $K_{1,3}$ -free and  $K_{30}K_1$ -free graph of order  $n$ , then  $\gamma(G) \leq \frac{3a(T)+2}{4}$ . Hence we propose the following conjecture.

**Conjecture 2.** For any connected graph  $G$ ,  $\gamma(G) \leq \frac{3a(G)+2}{4}$ .

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