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SEMI SQUARE STABLE GRAPHS AND EFFICIENT DOMINATING SETS

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Abstract. A graph G is called semi square stable if $\alpha(G^2) = i(G)$ where $\alpha(G^2)$ is the independence number of G^2 and $i(G)$ is the independent dominating number of G. A subset S of the vertex set of a graph G is an efficient dominating set if S is an independent set and every vertex of G is either in S or adjacent to exactly one vertex of S.

In this paper, we show that every square stable graph has an efficient dominating set and if a graph has an efficient dominating set, then it is semi square stable. We characterize when the join and the corona product of two disjoint graphs are semi square sable graphs and when they have efficient dominating sets.

1. Introduction

In this paper, by a graph G we mean a finite simple undirected graph. A set S of vertices in G is called an independent set (or a stable set) if every two distinct vertices in S are not adjacent in G. An independent set S in G is called a maximal independent set in G if $S \cup \{u\}$ is not an independent set in G for all $u \in V(G) \backslash S$. The independence number of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set in G and any maximal independent set of G with cardinality $\alpha(G)$ is called an α-set of G. A subset D of the vertex set of a graph G is called a dominating set if every vertex of G is either in D or adjacent to a vertex of D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The independent dominating

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number of a graph G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of G and any independent dominating set of G with cardinality $i(G)$ is called an i-set of G. It can be easily verified that independent dominating sets are exactly the maximal independent sets in G. More details on the independence number, the domination number and the independent dominating number of a graph can be found in [2],[5],[7] and [10]. For all $v \in V(G)$, the open neighborhood of v in G is $N_G(v) = \{u \in V(G) : u$ and v are adjacent in $G\}$, the closed neighborhood of v in G is $N_G[v] = N_G(v) \cup \{v\}$ and $N_k^G(v) = \{u \in V(G) : d_G(u, v) = k\}$ where $k \in \mathbb{N}$. A dominating set S in G is called an efficient dominating set if $|S \cap N_G[v]| = 1$ for all $v \in V(G)$. For more details on efficient dominating sets see [4] and [7]. Consider a graph G, the power graph G^k, where $k \in \mathbb{N}$, is the graph with $V(G^k) = V(G)$ and two vertices u and v are adjacent in G^k if $1 \leq d_G(u, v) \leq k$. In [9], Randerath et al., gave the following inequalities:

$$\alpha(G^2) \leq \gamma(G) \leq i(G) \leq \alpha(G) \tag{1}$$

A graph G is said be square stable if $\alpha(G^2) = \alpha(G)$, this definition was introduced in [8]. A graph G is said be semi square stable if $\alpha(G^2) = i(G)$, this definition was introduced in [1]. A vertex v in a graph G is called a branch vertex if $\deg_G(v) \geq 3$.

In this paper, we show that every square stable graph has an efficient dominating set. Also, we show that if a graph has an efficient dominating set, then this graph is semi square stable. We study when the graph join and the corona product of two disjoint graphs are semi square sable graphs and when they have efficient dominating sets. Finally, we prove that if T is a semi square stable tree and we replace a vertex v in T by a path P_4, then the new tree is also semi square stable. For undefined notions and terminology, the reader is referred to [3] and [6].

2. Semi square stable graphs and efficient dominating sets

From definition of efficient dominating sets, one can easily deduce the following Lemma.

Lemma 1. *Let G be a graph. A set S is an efficient dominating set if and only if S is a maximal independent set in G with $d_G(a, b) \geq 3$, for any distinct vertices $a, b \in S$.*

In the following theorem, we prove that every square stable graph has an efficient dominating set.

Theorem 2. *Let G be square stable. Then, G has an efficient dominating set.*

Proof. Assume $\alpha\left(G^2\right) = n = \alpha(G)$. Let S be a maximum independet set in G^2. Then, $d_G(a, b) \geq 3$, for any distinct vertices $a, b \in S$ and S is independent in G with $|S| = n$. Hence S is a maximal independent set in G. Using Lemma 1 S is an efficient domnating set in G. □

The following example shows that the converse of Theorem 2 is not correct.

Example 3. *A path P_3 has an efficient dominating set but it is not square stable.*

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In the following Theorem, we prove that any graph that has an efficient dominating set is a semi square stable graph.

Theorem 4. *If G has an efficient dominating set, then G is semi square stable.*

Proof. Let S be an efficient domnating set in G. Using Lemma 1 S is a maximal independent set in G and hence

$$|S| \geq i\,(G) \tag{2}$$

Also, $d_G\,(a,b) \geq 3$, for any distinct vertices $a, b \in S$. Therefore $d_{G^2}\,(a,b) \geq 2$, for any distinct vertices $a, b \in S$. So S is independent in G^2 and hence

$$|S| \leq \alpha\left(G^2\right) \tag{3}$$

From (2) and (3), we get $i\,(G) \leq \alpha\left(G^2\right)$. But $\alpha\left(G^2\right) \leq i(G)$ by (1). So G is semi square stable. □

But the converse of Theorem 4 is not true, consider the following example.

Example 5. *Consider the following graph G*

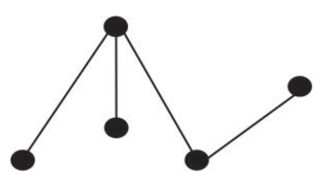

Figure 1. G

Note that $\alpha\left(G^2\right) = 2 = i\,(G)$ and hence G is semi square stable but G has no efficient dominating set.

Theorem 6. *If a graph G has two adjacent branch vertices such that both of them are adjacent to at least two endvertices, then G is not semi square stable.*

Proof. Let S be an $i\,(G)$-set and let u and v be two adjacent branch vertices such that both of them are adjacent to at least two endvertices. Then, $u \notin S$ or $v \notin S$. Without loss of generality assume $u \notin S$. So the endvertices which are adjacent to u must belong to S. But the set $D = (S - \{x \in V\,(G) : \deg_G(x) = 1 \text{ and } xu \in E\,(G)\}) \cup \{u\}$ is a dominating set in G with $|D| < |S| = i\,(G)$. Therefore $\alpha\left(G^2\right) \leq \gamma(G) \leq |D| < i\,(G)$ and hence G is not semi square stable. □

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3. Graph operations

Now, we will study when graphs obtained by some graph operations on two disjoint graphs are semi square stable. Also, we will check when these graphs have efficient dominating sets.

We start by the graph join. The graph join of two disjoint graphs G and H, denoted by $G+H$, is the graph obtained by taking G and H, and then making each vertex of G adjacent to each vertex of H.

Theorem 7. *Let G and H be disjoint graphs. Then, the following are equivalent:*

1- *$G+H$ has an efficient dominating set.*

2- *$G+H$ is semi square stable.*

3- $\min\{i(G), i(H)\} = 1$.

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 4.
(2) $\Longrightarrow$ (3) Since $d_{G+H}(u,v) \leq 2$, for all $u, v \in V(G+H)$. Then, $(G+H)^2$ is a complete graph and hence $\alpha\left((G+H)^2\right) = 1$. Also, every maximal independent set in $G+H$ is a maximal independent set in G or a maximal independent set in H. Thus $i(G+H) = \min\{i(G), i(H)\}$. But $G+H$ is a semi square stable graph and $\alpha\left((G+H)^2\right) = 1$. So, $\min\{i(G), i(H)\} = 1$.
(3) $\Longrightarrow$ (1) Without loss of generality suppose $i(G) = 1$. So there exists $v \in V(G)$ such that $\{v\}$ is a maximal independent set in G. So $\{v\}$ is a maximal independent set in $G+H$ and hence $\{v\}$ is an efficient domnating set in $G+H$. □

From the previous Theorem, we can conclude the following two corollaries.

Corollary 8. *Every wheel $W_{1,n}$ is semi square stable and has an efficient dominating set.*

Proof. Since $W_{1,n} = C_n + K_1$ and $i(K_1) = 1$, then by Theorem 7 $W_{1,n}$ is semi square stable and has an efficient domnating set. □

Corollary 9. *Let $K_{n,m}$ be any complete bipartite graph. Then, the following are equivalent:*

1- *$K_{n,m}$ has an efficient dominating set.*

2- *$K_{n,m}$ is semi square stable.*

3- *$K_{n,m}$ is a star.*

Proof. $K_{n,m} = N_n + N_m$, where N_n and N_m are the null graphs with n and m vertices respectively. Also, $i(N_n) = n$ and $i(N_m) = m$. Thus by Theorem 7 $K_{n,m}$ has an efficient domnating set if and only if $K_{n,m}$ is semi square stable if and only if $\min\{n, m\} = 1$ iff it is a star. □

Now, the will study the Corona product of two disjoint graphs. Let G and H be two disjoint graphs and for each $v \in V(G)$, let H_v be a copy of H. Then, the Corona product of G and H, denoted by $G \circ H$, is the graph obtained by taking G and H_v, and then making v adjacent to each vertex in H_v, for all $v \in V(G)$.

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Theorem 10. *Let G and H be disjoint graphs. Then, the following are equivalent:*

1- *$G \circ H$ has an efficient dominating set.*

2- *$G \circ H$ is semi square stable.*

3- *G is a null graph or $i(H) = 1$.*

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 4.
(2) $\Longrightarrow$ (3) For any $v \in V(G)$, $d_{G \circ H}(u, w) \leq 2$, for all $u, w \in V(H_v)$. So any two vertices of H_v are adjacent in $(G \circ H)^2$. Therefore $(H_v + \{v\})^2$ is a clique in $(G \circ H)^2$. Now take one vertex from each copy of H in $G \circ H$. Then the set consisting of these vertices is a maximum independent set in $(G \circ H)^2$. Thus $\alpha((G \circ H)^2) = |V(G)|$ and so $i(G \circ H) = |V(G)|$ because $G \circ H$ is semi square stable. Let S be an $i(G \circ H)$-set. Then, $S \cap (V(H_v) \cup \{v\}) \neq \emptyset$ for all $v \in V(G)$. So

$$|V(G)| = |S| =_{v \in V(G)} |S \cap (V(H_v) \cup \{v\})|$$

Thus $|S \cap (V(H_v) \cup \{v\})|$ must equal 1 for all $v \in V(G)$. We want to show $S \cap (V(H_v) \cup \{v\}) = \{v\}$ for all $v \in V(G)$, when $i(H) \geq 2$.
If $S \cap (V(H_v) \cup \{v\}) \neq \{v\}$ for some $v \in V(G)$, then there exists $u \in V(H_v)$ such that $S \cap (V(H_v) \cup \{v\}) = \{u\}$. Since $i(H_v) \geq 2$, there exists $w \in V(H_v)$ such that u and w are not adjacent in $G \circ H$. So $S \cup \{w\}$ is an independent set in $G \circ H$ and this contradicts the fact that S a maximal independent set in $G \circ H$. Thus

$$S = \bigcup_{v \in V(G)} (S \cap (V(H_v \cup \{v\})) = \bigcup_{v \in V(G)} \{v\} = V(G)$$

Therefore $V(G)$ is an independent set in $G \circ H$ and G is a null graph.
(3) $\Longrightarrow$ (1) Case(1) G is a null graph. In this case, it is clear that $V(G)$ is an efficient domnating set in $G \circ H$.
Case(2) $i(H) = 1$. Then for all $v \in V(G)$ we have $i(H_v) = 1$. So there exists $u_v \in V(H_v)$ such is $\{u_v\}$ is a amximal independent set in H_v. Let $S = \bigcup_{v \in V(G)} \{u_v\}$. Then S is a maximal independent set in $G \circ H$ and each vertex in the set $V(G \circ H) \backslash S$ is adjacent to a unique vertex in S. Thus S is an efficient domnating set in $G \circ H$. □

In the last part of this paper, we prove that if we replace a vertex in a semi square stable tree by a path P_4, then the new tree is semi square stable.

Theorem 11. *Let v be a vertex in a graph G and let G_1 be a graph obtained by replacing a vertex v by a path $P_4 : u_1u_2u_3u_4$ such that $\deg_{G_1}(u_2) = \deg_{G_1}(u_3) = 2$. Then, $\alpha\left(G_1^2\right) = \alpha\left(G^2\right) + 1$.*

Proof. Firstly, we want to prove that $\alpha\left(G_1^2\right) \geq \alpha\left(G^2\right) + 1$. Let S be maximum independent in G^2. Then, we have three cases:

Case(1) $v \in S$, then $N_k^G(v) \cap S = \phi$, for $k = 1, 2$. So, let $S_1 = S \backslash \{v\} \cup \{u_1, u_4\}$. Then, S_1 is a maximal independent set in G_1^2.

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Case(2) $v \notin S$ and $N_G(v) \cap S = \phi$, then there exists $u \in N_2^G(v)$ such that $u \in S$ and hence let $S_1 = S \cup \{u_2\}$. Then, S_1 is a maximal independent set in G_1^2.

Case(3) $v \notin S$ and $N_G(v) \cap S \neq \phi$. Since $d(x, y) \leq 2$ for all $x, y \in N_G(v)$. Then, there exists unique $u \in N_G(v) \cap S$. So, either $d_{G_1}(u, u_4) = 1$ and hence $S_1 = S \cup \{u_2\}$ is a maximal independent set in G_1^2 or $d_{G_1}(u, u_1) = 1$ and hence $S_2 = S \cup \{u_3\}$ is a maximal independent set in G_1^2

Therefore in all cases, we have

$$\alpha\left(G_1^2\right) \geq \alpha\left(G^2\right) + 1 \tag{4}$$

Now, we want to prove that $\alpha\left(G_1^2\right) \leq \alpha\left(G^2\right) + 1$. Let S_1 be maximum independent in G_1^2. Then, we have three cases:

Case(1) $u_1, u_4 \in S_1$. Let $S = (S_1 \backslash \{u_1, u_4\}) \cup \{v\}$. Then, S is a maximal independent set in G^2.

Case(2) $S_1 \cap \{u_1, u_2, u_3, u_4\} = \phi$. Then, there exists a unique vertex in G_1, say x such that $x \in N_G(u_1) \cap S_1$. Let $S = S_1 \backslash \{x\}$. Then S is independent in G^2.

Case(3) $|S_1 \cap \{u_1, u_2, u_3, u_4\}| = 1$. Let $S = S_1 \backslash \{u_1, u_2, u_3, u_4\}$. Then, S is independent in G^2.

Therefore in all cases, we have

$$\alpha\left(G^2\right) \geq \alpha\left(G_1^2\right) - 1 \tag{5}$$

From (4) and (5), we get

$$\alpha\left(G_1^2\right) = \alpha\left(G^2\right) + 1$$

□

Lemma 12. *Let v be a vertex in a tree T and let T_1 be the tree obtained by replacing the vertex v by the path $P_4 : u_1u_2u_3u_4$. Then, $i(T_1) \leq i(T) + 1$.*

Proof. Let S be a maximal independent set in T with $|S| = i(T)$. If $v \in S$, then $S_1 = (S \backslash \{v\}) \cup \{u_1, u_4\}$ is a maximal independent set in T_1. If $v \notin S$, then $N_T(v) \cap S \neq \phi$ and hence we have two cases:

Case(1) $N_{T_1}(u_1) \cap S \neq \phi$. Then, $S_1 = S \cup \{u_3\}$ is a maximal independent set in T_1.

Case(2) $N_{T_1}(u_4) \cap S \neq \phi$. Then, $S_1 = S \cup \{u_2\}$ is a maximal independent set in T_1. So in all cases, we have

$$i(T_1) \leq |S_1| = i(T) + 1$$

□

Corollary 13. *Let T be a semi square stable tree, $v \in V(T)$ and T_1 be the tree obtained by replacing the vertex v by the path $P_4 : u_1u_2u_3u_4$. Then, T_1 is semi square stable with $i(T_1) = i(T) + 1$.*

Proof. Since T is a semi square stable tree, then

$$i(T) = \alpha\left(T^2\right) = \alpha\left(T_1^2\right) - 1 \leq i(T_1) - 1 \leq i(T) + 1 - 1 = i(T)$$

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So,

$$\alpha\left(T_1^2\right) = i\left(T_1\right)$$

Therefore T_1 is semi square stable and $i\left(T_1\right) = i\left(T\right) + 1$.

□

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