https://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 12 No. 3 (2023), pp. 165-171.

www.ui.ac.ir

ENERGY OF STRONG RECIPROCAL GRAPHS

MARYAM GHAHREMANI, ABOLFAZL TEHRANIAN*, HAMID RASOULI AND
MOHAMMAD ALI HOSSEINZADEH

Abstract. The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined as the sum of absolute values of all eigenvalues of G. A graph G is called reciprocal if $\frac{1}{\lambda}$ is an eigenvalue of G whenever λ is an eigenvalue of G. Further, if λ and $\frac{1}{\lambda}$ have the same multiplicities, for each eigenvalue λ, then it is called strong reciprocal. In (MATCH Commun. Math. Comput. Chem. 83 (2020) 631–633), it was conjectured that for every graph G with maximum degree $\Delta(G)$ and minimum degree $\delta(G)$ whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph. Here, we prove the validity of this conjecture for some strong reciprocal graphs. Moreover, we show that if G is a strong reciprocal graph, then $\mathcal{E}(G) \geq \Delta(G) + \delta(G) - \frac{1}{2}$. Recently, it has been proved that if G is a reciprocal graph of order n and its spectral radius, ρ, is at least $4\lambda_{min}$, where λ_{min} is the smallest absolute value of eigenvalues of G, then $\mathcal{E}(G) \geq n + \frac{1}{2}$. In this paper, we extend this result to almost all strong reciprocal graphs without the mentioned assumption.

1. Introduction

Let G be a graph of order n with minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. As usual, a path and a complete graph of order n are denoted by P_n and K_n, respectively. Also, a complete bipartite graph with part sizes a and b is denoted by $K_{a,b}$. By tK_2 we mean t copies of K_2, for some positive integer t. A $\{1,2\}$-*factor* of G is a spanning subgraph of G which is a disjoint union of a matching and a 2-regular subgraph of G. The *line graph*, $L(G)$, of a graph G is the graph whose

Communicated by Ali Reza Ashrafi.

MSC(2010): 05C50, 15A18.

Keywords: Graph energy, Strong reciprocal graph, Non-singular graph.

Received: 04 July 2022, Accepted: 14 August 2022.

Article Type: Research Paper.

*Corresponding author.

http://dx.doi.org/10.22108/TOC.2022.134259.1999 .

vertices are the edges of G, with two vertices adjacent whenever the corresponding edges have exactly one common vertex. A *cocktail party graph*, $CP(n)$, is the regular graph on $2n$ vertices of degree $2n-2$. A *generalized line graph* $L(G; a_1, \ldots, a_n)$ is defined for graph G with n vertices $\{v_1, \ldots, v_n\}$ and non-negative integers $(a_1, \ldots, a_n)$ by taking the graphs $L(G)$ and $CP(a_i)$ and adding extra edges. A vertex in $L(G)$ is joined to one in $CP(a_i)$, $i = 1, \ldots, n$, if vertex v_i is an end point of the vertex in $L(G)$ (viewed as an edge of G). Special cases include an ordinary line graph ($a_1 = \cdots = a_n = 0$) and the cocktail party graph $CP(n)$ ($n = 1$ and $a_1 = n$), see [7, Ch. 1]. The *adjacency matrix* of a graph G of order n, $A(G) = [a_{ij}]$, is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$, otherwise. The eigenvalues of G will be referred to the *eigenvalues* of $A(G)$. The adjacency spectrum $Spec(G)$ of a graph G is the multiset of eigenvalues of G. A graph G is *non-singular* if its adjacency matrix is *non-singular*. The largest eigenvalue of $A(G)$ is called the *spectral radius* of G and denoted by ρ. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of G. The *energy* $\mathcal{E}(G)$ of a graph G, introduced by Ivan Gutman [13], is defined as the sum of the absolute values of all eigenvalues of G, that is, $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. For more details on the energy of graphs we refer the reader to [15].

It is well-known that a graph G is bipartite if and only if the symmetry of each eigenvalue of G is also an eigenvalue of G, see [8, Theorem 3.11]. Furthermore, if G is a non-singular bipartite graph, then the order of G must be even. Thus if G is non-singular and has order $n = 2k$, then G has exactly k positive and k negative eigenvalues. In contrast to the plus-minus pairs of eigenvalues of bipartite graphs, Barik et al. [5], have introduced the notion of *reciprocal* graphs, which have the property that $\frac{1}{\lambda}$ is an eigenvalue of G, whenever λ is an eigenvalue of G. If for each eigenvalue λ, λ and $\frac{1}{\lambda}$ have the same multiplicities, then G is called a *strong reciprocal* graph. Therefore, the energy of a strong reciprocal graph G is as follows:

$$\mathcal{E}(G) = \lambda_1 + |\lambda_2| + \cdots + |\lambda_n| = \frac{1}{\lambda_1} + \frac{1}{|\lambda_2|} + \cdots + \frac{1}{|\lambda_n|}.$$

It has been proved in [5] that when a graph G is obtained by taking the corona of a bipartite graph with an isolated vertex, then it is strong reciprocal and then all strong reciprocal trees were characterized in terms of corona of trees. Also in [6] it was shown that the notions of reciprocal and strong reciprocal coincide for trees.

In [16], Ma proved that for a graph G, $\mathcal{E}(G) \geq 2\delta(G)$. Akbari and Hosseinzadeh [4] improved this lower bound to $\mathcal{E}(G) \geq 2\bar{d}$, where $\bar{d}$ is the average degree of G. Also they proposed the following conjecture:

Conjecture 1.1. [4] *For every graph G whose adjacency matrix is non-singular, $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$ and the equality holds if and only if G is a complete graph.*

Recently, the validity of the conjecture has been shown for planar graphs, triangle-free graphs and quadrangle-free graphs in [3] and for line graphs in [1]. In this paper, we prove this conjecture for

http://dx.doi.org/10.22108/TOC.2022.134259.1999

some strong reciprocal graphs. In particular, we show that for each strong reciprocal graph G,

$$\mathcal{E}(G) \geq \Delta(G) + \delta(G) - \frac{1}{2}.$$

In [11, Proposition 3], Filipovski and Jajcay obtained a lower bound for the energy of a reciprocal graph G under the condition $\rho \geq 4\lambda_{min}$ and $\rho = max\{|\lambda_1|, \ldots, |\lambda_n|\}$ and $\lambda_{min} = min\{|\lambda_1|, \ldots, |\lambda_n|\}$. By eliminating this assumption, we generalize this result in Theorem 2.2 to almost all strong reciprocal graphs. More results on the energy of strong reciprocal graphs can be found in [12, 14].

2. Lower Bounds for Strong Reciprocal Graph Energy

In this section, some lower bounds for the energy of a strong reciprocal graph are obtained. We remove some restrictions in [11, Proposition 3]. To this end, first we recall the following:

Lemma 2.1. [2, Theorem 2] *Let G be a graph of order n. If G has a $\{1, 2\}$-factor, then $\mathcal{E}(G) \geq n$.*

Note that the eigenvalues of P_4 are $\frac{1\pm\sqrt{5}}{2}$ and $\frac{-1\pm\sqrt{5}}{2}$. Thus P_4 is strong reciprocal. Also since zero is an eigenvalue of each bipartite graph of odd order, there is no strong reciprocal bipartite graph of odd order.

Theorem 2.2. *Let $G \notin \{P_4, tK_2, P_4 \cup tK_2\}$ be a strong reciprocal graph of order n, for some positive integer t. Then $\mathcal{E}(G) \geq n + \frac{1}{2}$.*

Proof. Suppose that $\rho = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G. Thus

$$\mathcal{E}(G) = \rho + \frac{1}{\rho} + \sum_{\lambda \in Spec(G)\setminus\{\lambda_1\}, |\lambda|>1} (|\lambda| + \frac{1}{|\lambda|}) + 1 + \cdots + 1 + |-1| + \cdots + |-1|.$$

Assume $l \geq 1$ is the number of eigenvalues with absolute value more than 1 and $s, t \geq 0$ are the multiplicities of eigenvalues 1 and -1, respectively. If $\Delta(G) \geq 4$, then $\rho \geq \sqrt{\Delta(G)} \geq 2$, see [9, p. 257], and so $\rho + \frac{1}{\rho} \geq 2.5$. Note that $n = 2 + 2(l-1) + s + t$. Now, by the inequality $|x| + \frac{1}{|x|} \geq 2$, we have

$$\mathcal{E}(G) \geq 2.5 + 2(l-1) + s + t = n + \frac{1}{2},$$

as desired. Also if $\Delta(G) \leq 1$, since G is not a union of K_2, then one can find that G is not strong reciprocal. Let $\Delta(G) = 2$. Thus G is a union of paths and cycles. Since G is strong reciprocal, it contains no cycle in this case, because $\frac{1}{2}$ is not an algebraic integer. Hence, assume that G is a union of paths. Since P_n for odd n has eigenvalue zero, it is not reciprocal and so G contains no odd-path component. Note that $\mathcal{E}(P_4) \approx 4.47$. Also, if $G \simeq P_n$ for an even integer $n \geq 6$, then by [15, p. 26],

$$\mathcal{E}(P_n) = \frac{2}{\sin(\frac{\pi}{2(n+1)})} - 2 \geq \frac{4(n+1)}{\pi} - 2 > n + \frac{1}{2}.$$

Since no component of G is an odd-path, G has a $\{1, 2\}$-factor. Hence, if G contains a path of even order $n \geq 6$, then by Lemma 2.1, the proof is complete. Note that if G has at least two components isomorphic to P_4, then the assertion also holds.

Now, suppose that $\Delta(G) = 3$. So G contains one of the graphs H, depicted in Figure 1, or $K_4 - e$ or $K_{1,3}$ as an induced subgraph.

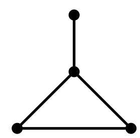

FIGURE 1. H

If H or $K_4 - e$ is an induced subgraph of G, since the spectral radius of these two graphs are greater than 2.17, then by [9, p. 229], $\rho \geq 2.17$. Note that the function $f(x) = |x| + \frac{1}{|x|} \geq 2$ is increasing for $|x| > 1$, so we have

$$\mathcal{E}(G) = \rho + \frac{1}{\rho} + \sum_{\lambda \in Spec(G)\setminus\{\lambda_1\}, |\lambda|>1} (|\lambda| + \frac{1}{|\lambda|}) + 1 + \cdots + 1 + |-1| + \cdots + |-1| \geq n + \frac{1}{2}.$$

Also, if $K_{1,3}$ is an induced subgraph of G, then by Interlacing Theorem [9, p. 17], ρ, $|\lambda_n| \geq \sqrt{3}$. Therefore, $\rho + \frac{1}{\rho} + |\lambda_n| + \frac{1}{|\lambda_n|} \geq 4.622$ and the inequality $|x| + \frac{1}{|x|} \geq 2$ yield the result. □

The next theorem gives a lower bound for the energy of strong reciprocal graphs, a slightly weaker than that of Conjecture 1.1.

By an *exceptional graph* we mean a connected graph with least eigenvalue greater than or equal to -2 which is not a generalized line graph.

Theorem 2.3. *Let G be a strong reciprocal graph. Then*

$$\mathcal{E}(G) \geq \Delta(G) + \delta(G) - \frac{1}{2}.$$

Proof. Assume that the order of G is n and $\rho = \lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of G. Thus,

$$\mathcal{E}(G) = \rho + \frac{1}{\rho} + 1 + \cdots + 1 + \sum_{\lambda \in Spec(G)\setminus\{\lambda_1\}, |\lambda|>1} (|\lambda| + \frac{1}{|\lambda|}). \tag{2.1}$$

First note that the order of each exceptional graph is at most 8, see [10]. By a computer search which described in the Appendix, one can find that Conjecture 1.1 holds for graphs of order at most 8. Moreover, the least eigenvalue of any generalized line graph is greater than or equal to -2, see [9, p. 9]. By [1], Conjecture 1.1 holds for generalized line graphs whose a_i's, $(1 \leq i \leq n)$, are zero. Also if there is a non-zero a_i in a generalized line graph, then there are two non-adjacent vertices with the same neighbors in this graph and so the generalized line graph is singular which is a contradiction. Hence, assume that the least eigenvalue of G, λ_n, is not greater than -2 and so $|\lambda_n| + \frac{1}{|\lambda_n|} \geq 2.5$. Using [8, Theorem 3.8], $\rho \geq \delta(G)$. Also for each λ with $|\lambda| > 1$, $|\lambda| + \frac{1}{|\lambda|} \geq 2$. Hence, by Equation (1), $\mathcal{E}(G) \geq \delta(G) + n - 2 + \frac{1}{2}$ which yields the result. □

In the next theorem, Conjecture 1.1 is proved for strong reciprocal graphs whose average degrees are bounded by a multiply smaller than one of their orders.

http://dx.doi.org/10.22108/TOC.2022.134259.1999

Theorem 2.4. *Let G be a strong reciprocal graph on n vertices and $\overline{d} \leq \frac{n}{4} - 1$, then $\mathcal{E}(G) \geq \Delta(G) + \delta(G)$.*

Proof. Let G be a strong reciprocal graph of order n whose average degree, $\overline{d}$, does not exceed $\frac{n}{4} - 1$. Also let $d_1 \geq \cdots \geq d_n$ be degree sequence of G. By Equation (1) in the proof of Theorem 2.3, if $\rho > \delta(G) + 1$ or $\Delta(G) \leq n - 2$, then $\mathcal{E}(G) > \delta(G) + 1 + n - 2 \geq \Delta(G) + \delta(G)$ which gives the result. Now, due to $\rho \geq \overline{d}$ [8, Theorem 3.8], suppose that $\rho \leq \delta(G) + 1$ and $\Delta(G) = n - 1$. Thus,

$$\begin{aligned} 2\left(1+\delta(G)\right)\left(\rho-\overline{d}\right) &\geq 2\rho\left(\rho-\overline{d}\right) \geq \rho^2 - \overline{d}^2 \\ &\geq \frac{1}{n}\sum_{i=1}^{n} d_i^2 - \overline{d}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2. \end{aligned}$$

Therefore by the assumption,

$$\rho - \overline{d} \geq \frac{\frac{1}{n}\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2}{2\left(1+\delta(G)\right)} \geq \frac{2\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2}{n^2}.$$

On the other hand, since $\overline{d} \leq \frac{n}{4} - 1$,

$$\sum_{i=1}^{n}\left(d_i-\overline{d}\right)^2 > \left(n-1-\overline{d}\right)^2 \geq \frac{9}{16}n^2.$$

Thus $\rho > \overline{d} + 1$, which contradicts the assumption $\rho \leq \delta(G) + 1$ and then the proof is complete. □

Remark 2.5. *A graph G is said to be anti-reciprocal if $\frac{-1}{\lambda}$ is an eigenvalue of G, whenever λ is an eigenvalue of G. Further, if λ and $\frac{-1}{\lambda}$ have the same multiplicities for each eigenvalue λ of G, then G is called a strong anti-reciprocal graph. One can find that the energy of a strong anti-reciprocal graph G is equal to the energy of a strong reciprocal graph whose absolute eigenvalues are the same as absolute eigenvalues of G (if exist). Thus all results of this paper also hold for the energy of strong anti-reciprocal graphs.*

References

[1] S. Akbari, A. Alazemi, M. Anđelić and M. A. Hosseinzadeh, On the energy of line graphs, *Linear Algebra Appl.*, **636** (2022) 143–153.

[2] S. Akbari, M. Ghahremani, I. Gutman and F. Koorepazan Moftakhar, Orderenergetic graphs, *MATCH Commun. Math. Comput. Chem.*, **84** (2020) 325–334.

[3] S. Akbari, M. Ghahremani, M. A. Hosseinzadeh, S. Khalashi-Ghezelahmad, H. Rasouli and A. Tehranian, A Lower bound for graph energy in terms of minimum and maximum degrees, *MATCH Commun. Math. Comput. Chem.*, **86** (2021) 549–558.

[4] S. Akbari and M. A. Hosseinzadeh, A short proof for graph energy is at least twice of minimum degree, *MATCH Commun. Math. Comput. Chem.*, **83** (2020) 631–633.

[5] S. Barik, S. Pati and B. K. Sarma, The spectrum of the corona of two graphs, *SIAM J. Discrete Math.*, **21** (2007) 47–56.

[6] S. Barik, M. Neumann and S. Pati, On nonsingular trees and a reciprocal eigenvalue property, *Linear and Multilinear Algebra*, **54** (2006) 453–456.

http://dx.doi.org/10.22108/TOC.2022.134259.1999

[7] D. Cvetković, M. Doob, I. Gutman and A. Torgašev, *Recent results in the theory of graph spectra*, Annals of Discrete Mathematics, **36**, North-Holland Publishing Co., Amsterdam, 1988.

[8] D. Cvetkovic', M. Doob and H. Sachs, *Spectra of graphs*, Academic Press, New York, 1980.

[9] D. Cvetković, P. Rowlinson and S. K. Simić, *An introduction to the theory of graph spectra*, London Mathematical Society Student Texts, **75**, Cambridge University Press, Cambridge, 2010.

[10] M. Doob and D. Cvetković, On spectral characterizations and embedding of graphs, *Linear Algebra Appl.*, **27** (1979) 17–26.

[11] S. Filipovski and R. Jajcay, Bounds for the energy of graphs, *Mathematics*, **9** (2021) .

[12] S. Filipovski and R. Jajcay, New upper bounds for the energy and spectral radius of graphs, *MATCH Commun. Math. Comput. Chem.*, **84** (2020) 335–343.

[13] I. Gutman, The energy of a graph, *Ber. Math-Statist. Sekt. Forschungsz. Graz.*, **103** (1978) 1–22.

[14] G. Indulal and A. Vijayakumar, Reciprocal graphs, *Malaya J. Mat.*, **4** (2016) 380–387.

[15] X. Li, Y. Shi and I. Gutman, *Graph energy*, Springer, New York, 2012.

[16] X. Ma, A low bound on graph energy in terms of minimum degree, *MATCH Commun. Math. Comput. Chem.*, **81** (2019) 393–404.

Appendix

Using the following code in the Sage Mathematics Software System (SageMath, the Sage Mathematics Software System (Version 9.0), The Sage Developers, 2021, http://www.sagemath.org), one can verify the validity of Conjecture 1.1 for graphs of order n.

```
def E(G):
    s=0
    for x in G.spectrum():
        s = s + abs(x)
    return s

gen = graphs.nauty_geng("n -c")
List=[ ]
for G in gen:
    if G.adjacency_matrix().determinant()! = 0:
        if E(G) <= max(G.degree_sequence())+min(G.degree_sequence()):
            List.append(G)
for G in List:
    G.plot(figsize=2).show()
```

Maryam Ghahremani

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Email: maryam.ghahremani@srbiau.ac.ir

Abolfazl Tehranian

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Email: tehranian@srbiau.ac.ir

Hamid Rasouli

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran

Email: hrasouli@srbiau.ac.ir

Mohammad Ali Hosseinzadeh

Faculty of Engineering Modern Technologies, Amol University of Special Modern Technologies, Amol, Iran

Email: hosseinzadeh@ausmt.ac.ir