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ON LAPLACIAN RESOLVENT ENERGY OF GRAPHS

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Abstract. Let G be a simple connected graph of order n and size m. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix of G, where $D(G)$ and $A(G)$ are the degree diagonal matrix and the adjacency matrix, respectively. For the graph G, let $d_1 \geq d_2 \geq \cdots d_n$ be the vertex degree sequence and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ be the Laplacian eigenvalues. The Laplacian resolvent energy $RL(G)$ of a graph G is defined as $RL(G) = \sum\limits_{i=1}^{n} \frac{1}{n+1-\mu_i}$. In this paper, we obtain an upper bound for the Laplacian resolvent energy $RL(G)$ in terms of the order, size and the algebraic connectivity of the graph. Further, we establish relations between the Laplacian resolvent energy $RL(G)$ with each of the Laplacian-energy-Like invariant LEL, the Kirchhoff index Kf and the Laplacian energy LE of the graph.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where order $|V(G)| = n$ and size $|E(G)| = m$. The degree $d(v_i)$ or d_i of a vertex v_i is the number of edges incident on v_i. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. Let $max\{d_i : v_i \in V(G)\} = d_1 = \Delta$ and $min\{d_i : v_i \in V(G)\} = d_n = \delta$. For more definitions and other notations, see [21].

The adjacency matrix $A(G)$ associated with G is a square matrix defined as $A(G) = (a_{ij})$, where $a_{ij} = 1$, if vertex v_i is adjacent to vertex v_j, and 0 otherwise. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$

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forms the adjacency spectrum of G. The resolvent energy introduced by Gutman and Furtula [8] is defined as

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - \lambda_i}.$$

More on resolvent energy can be seen in [1, 3, 4, 8, 9, 25] and the references therein.

The Laplacian matrix $L(G)$ of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G) = diag\{d_1, d_2, \ldots, d_n\}$ is the vertex degree diagonal matrix of G and $A(G)$ is the adjacency matrix of G. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $L(G)$ forms the Laplacian spectrum of G. The Laplacian matrix is a real symmetric and positive semidefinite matrix. The Laplacian eigenvalues can be arranged in the non-increasing order as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$. We note that $\mu_n = 0$ with multiplicity equal to the number of the connected components of G. Also, $\mu_{n-1} > 0$ if and only if the graph G is connected.

Gutman and Zhou [6] defined the Laplacian energy $LE(G)$ of a graph G as the sum of the absolute deviations (that is, the distances from the average) of the Laplacian eigenvalues. That is,

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where G is a graph with n vertices and m edges and $\mu_1, \mu_2, \ldots, \mu_n$ are the Laplacian eigenvalues.

Cafure et al. [2] defined the Laplacian resolvent energy $RL(G)$ of a graph G as

$$RL(G) = \sum_{i=1}^{n} \frac{1}{n + 1 - \mu_i}.$$

Liu and Liu [11] defined the Laplacian-energy-like invariant (LEL) as

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$$

More on LEL can be seen in [12, 14, 18, 19, 20] and in references therein.

The Wiener Index $W(G)$ is a topological index and is defined as

$$W(G) = \sum_{i<j} d_{ij},$$

where d_{ij} is the number of edges in the shortest path between the vertices i and j in G. Wiener [24] investigated the Wiener index and found the correlation between the boiling points of paraffin and the structure of the molecules. Analogous to the Wiener index, Klein and Randic [10] defined the Kirchhoff index $Kf(G)$, as

$$Kf(G) = \sum_{i<j} r_{ij},$$

where r_{ij} is the resistance distance between the vertices i and j of G. That is, r_{ij} is equal to the resistance between two equivalent points on an associated electronic network, obtained by replacing

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each edge of G by a unit (1 Ohm) resistor. Gutman and Mohar [7] proved that the Kirchhoff index can be represented in terms of the Laplacian eigenvalues as

$$Kf = Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}$$

Matejic et al. [13] obtained a lower bound for μ_1 in terms of the number of vertices n, the number of edges m and k, where k is an arbitrary real number such that $\mu_1 \geq k \geq \frac{2m}{n-1}$.

Theorem 1.1. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any k, where k is an arbitrary real number such that $\mu_1 \geq k \geq \frac{2m}{n-1}$,holds*

$$RL(G) \geq \frac{1}{n+1} + \frac{1}{n+1-k} + \frac{(n-2)^2}{(n+1)(n-2)-2m+k}$$

Equality holds if and only if $k = n$ and $G \cong K_n$,or $k = n$ and $G \cong K_{1,n-1}$, or $k = n$ and $G \cong K_{\frac{n}{2},\frac{n}{2}}$, n is even.

The following inequality for the positive real numbers is due to Rennie [23].

Lemma 1.2. . *Let $p = (p_i)$ and $a = (a_i)$, $i \in 1, 2, \ldots, n-1$, be two sequences of positive real numbers with the properties $p_1 + p_2 + \cdots + p_{n-1} = 1$ and $0 < r \leq a_i \leq R < +\infty$. Then the following inequality holds:*

$$\sum_{i=1}^{n-1} p_i a_i + rR\sum_{i=1}^{n-1}\frac{p_i}{a_i} \leq r + R.$$

In this paper, we obtain an upper bound for the Laplacian resolvent energy $RL(G)$ with the methods similar to used in [13]. Further, we establish relations between Laplacian resolvent energy $RL(G)$ with each of the Laplacian-energy-Like invariant LEL, the Kirchhoff index Kf, the Laplacian energy LE.

2. On Laplacian resolvent energy

We start with the following lemma.

Lemma 2.1. [22] (*Radon Inequality*). *Let $a_i, x_i > 0$, $n > 0$, $i \in 1, 2, \ldots, n$. Then for all real r, $r \geq 0$, the following inequality holds*

$$\sum_{i=1}^{n}\frac{x_i^{r+1}}{a_i^r} \geq \frac{(\sum_{i=1}^{n} x_i)^{r+1}}{(\sum_{i=1}^{n} a_i)^r}$$

Equality holds if and only if $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Now, we obtain a sharp upper bound for the Laplacian resolvent energy in terms of the number of vertices n, the number of edges m and the algebraic connectivity μ_{n-1}.

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Theorem 2.2. *If G is a connected graph with n vertices and m edges having algebraic connectivity μ_{n-1}, then*

$$RL(G) \leq \frac{1}{n+1} + \frac{(n+1)(n-1) - (n+\mu_{n-1})(n-1) + 2m}{(n+1-\mu_{n-1})}$$

Equality, if and only if $G \cong K_n$, or for any $s, 1 \leq s \leq n-2$, holds $\mu_1 = \mu_2 = \cdots = \mu_s \geq \mu_{s+1} = \cdots = \mu_{n-1}$.

Proof. In Lemma 1.2, for $i = 1, 2, \ldots, n-1$, setting

$$p_i = \frac{1}{n-1},\ a_i = n+1-\mu_i,\ R = n+1-\mu_{n-1},\ r = n+1-\mu_1,$$

we have,

$$\sum_{i=1}^{n-1} \frac{1}{n-1}(n+1-\mu_i) + (n+1-\mu_1)(n+1-\mu_{n-1}) \sum_{i=1}^{n-1} \frac{1}{(n-1)(n+1-\mu_i)}$$
$$\leq (n+1-\mu_1) + (n+1-\mu_{n-1}),$$

or,

$$\sum_{i=1}^{n-1} (n+1-\mu_i) + (n+1-\mu_1)(n+1-\mu_{n-1}) \sum_{i=1}^{n-1} \frac{1}{(n+1-\mu_i)}$$
$$\leq (2(n+1) - \mu_1 - \mu_{n-1})(n-1),$$

that is,

$$(n+1)\sum_{i=1}^{n-1} 1 - \sum_{i=1}^{n-1} \mu_i + (n+1-\mu_1)(n+1-\mu_{n-1}) \left(RL(G) - \frac{1}{n+1} \right)$$
$$\leq (2(n+1) - \mu_1 - \mu_{n-1})(n-1).$$

On simplification, this gives

$$\begin{aligned} RL(G) - \frac{1}{n+1} &\leq \frac{(2(n+1) - \mu_1 - \mu_{n-1})(n-1) - (n+1)(n-1) + 2m}{(n+1-\mu_1)(n+1-\mu_{n-1})} \\ &= \frac{2(n+1)(n-1) - (\mu_1 - \mu_{n-1})(n-1) - (n+1)(n-1) + 2m}{(n+1-\mu_1)(n+1-\mu_{n-1})} \\ &= \frac{(n+1)(n-1) - (\mu_1 - \mu_{n-1})(n-1) + 2m}{(n+1-\mu_1)(n+1-\mu_{n-1})} \end{aligned}$$

So,

$$RL(G) \leq \frac{1}{n+1} + \frac{(n+1)(n-1) - (\mu_1 - \mu_{n-1})(n-1) + 2m}{(n+1-\mu_1)(n+1-\mu_{n-1})}.$$

Now, let

$$f(x) = \frac{(n^2-1) + 2m - (n-1)(x+\mu_{n-1})}{(n+1-x)(n+1-\mu_{n-1})}.$$

On differentiating both sides with respect to x, we have

$$f'(x) = \frac{2m - (n-1)\mu_{n-1}}{(n+1-x)^2(n+1-\mu_{n-1})}.$$

Since $\mu_{n-1} \leq \frac{2m}{n} < \frac{2m}{n-1}$, therefore $f'(x) \geq 0$, which implies that $f(x)$ is an increasing function of x. This further implies that $f(x) \leq f(n)$. So,

$$\begin{aligned} f(x) &\leq \frac{1}{n+1} + \frac{(n+1)(n-1) - (n+\mu_{n-1})(n-1) + 2m}{(n+1-n)(n+1-\mu_{n-1})} \\ &= \frac{1}{n+1} + \frac{(n+1)(n-1) - (n+\mu_{n-1})(n-1) + 2m}{(n+1-\mu_{n-1})} \end{aligned}$$

Hence

$$RL(G) \leq \frac{1}{n+1} + \frac{(n+1)(n-1) - (n+\mu_{n-1})(n-1) + 2m}{(n+1-\mu_{n-1})}$$

The equality case can be proved as in [15, lemma 3.1],completes the proof. □

Now, we establish a relation between the Laplacian resolvent energy $RL(G)$ and the Laplacian-energy-like invariant $LEL(G)$.

Theorem 2.3. *If G is a connected graph with n vertices and m edges, then*

$$(2mn - M_1)\left(RL(G) - \frac{1}{n+1}\right) \geq (LEL(G))^2$$

The equality holds if and only if $G \cong K_n$.

Proof. In Lemma 2.1 (Radon inequality), setting

$$x_i = \sqrt{\mu_i},\ a_i = \frac{1}{n+1-\mu_i}, i = 1, 2, \ldots, n-1,$$

we have,

$$\begin{aligned} \sum_{i=1}^{n-1} \mu_i(n+1-\mu_i) &\geq \frac{(\sum_{i=1}^{n-1}\sqrt{\mu_i})^2}{\sum_{i=1}^{n-1}\frac{1}{n+1-\mu_i}} \\ \text{or } \sum_{i=1}^{n-1}((n+1)\mu_i - \mu_i^2) &\geq \frac{(LEL(G))^2}{(RL(G) - \frac{1}{n-1})} \\ \text{or } (n+1)\sum_{i=1}^{n-1}\mu_i - \sum_{i=1}^{n-1}\mu_i^2 &\geq \frac{(LEL(G))^2}{(RL(G) - \frac{1}{n-1})} \\ \text{or } (n+1)2m - (M_1 + 2m) &\geq \frac{(LEL(G))^2}{(RL(G) - \frac{1}{n-1})} \\ \text{or } 2mn - M_1 &\geq \frac{(LEL(G))^2}{(RL(G) - \frac{1}{n-1})} \\ \text{or } (RL(G) - \frac{1}{n-1})(2mn - M_1) &\geq (LEL(G))^2. \end{aligned}$$

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For the equality, the ratio's $\frac{x_i}{a_i}$ must be identical for $i = 1, 2, \ldots, n-1$. This is possible if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$. Therefore, equality holds in the theorem if and only if the graph G is the complete graph K_n, completing the proof. □

The following theorem gives a relation between the Laplacian resolvent energy $RL(G)$ of G and the Kirchhoff index $Kf(\overline{G})$ of the complement graph $\overline{G}$ of G.

Theorem 2.4. *If G is a connected graph with n vertices and m edges and $\overline{G}$ is the complement of G, then*

$$RL(G) \geq \frac{1}{n+1} + \frac{n-1}{4} + \frac{Kf(\overline{G})}{4n}.$$

Proof. We have

$$\begin{aligned} RL(G) &= \sum_{i=1}^{n} \frac{1}{n+1-\mu_i} = \frac{1}{n+1} + \sum_{i=1}^{n-1} \frac{1}{n+1-\mu_i} \\ &= \frac{1}{n+1} + \sum_{i=1}^{n-1} \frac{1}{(n-\mu_i)+1} = \frac{1}{n+1} + \sum_{i=1}^{n-1} \frac{1}{\overline{\mu_i}+1} \end{aligned}$$

The arithmetic mean-harmonic mean inequality $\frac{a+b}{2} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}$ implies that $\frac{1}{a+b} \leq \frac{1}{4}(\frac{1}{a}+\frac{1}{b})$. Applying AM-HM inequality, we have

$$\begin{aligned} RL(G) &= \frac{1}{n+1} + \sum_{i=1}^{n-1} \frac{1}{\overline{\mu_i}+1} \leq \frac{1}{n+1} + \sum_{i=1}^{n-1} \frac{1}{4}(\frac{1}{1} + \frac{1}{\overline{\mu_i}}) \\ &= \frac{1}{n+1} + \frac{1}{4}\sum_{i=1}^{n-1} \frac{1}{1} + \frac{1}{\overline{\mu_i}} = \frac{1}{n+1} + \frac{1}{4}\sum_{i=1}^{n-1}(1) + \frac{1}{4}\sum_{i=1}^{n-1}\frac{1}{\overline{\mu_i}} \\ &= \frac{1}{n+1} + \frac{n-1}{4} + \frac{Kf(\overline{G})}{4n} \end{aligned}$$

completing the proof. □

Now, we establish a relation between Laplacian resolvent energy (RL) and Laplacian energy (LE) of G.

Theorem 2.5. *If G is a connected graph with n vertices and m edges, then*

$$\frac{\left(LE(G) - \frac{2m}{n}\right)^2}{\left(RL(G) - \frac{1}{n+1}\right)} \geq$$

$$\left[(M_1 + 2m)\left(n+1+\frac{4m}{n}\right) - (F + 3M_1 - 6C_3) - \left(\frac{2m(n+1)}{n}\right)^2 - \frac{8m^3}{n^2}\right]$$

where $M_1 = \sum_{i=1}^{n} d_i^2$, $F = \sum_{i=1}^{n} d_i^3$ and C_3 is the number of triangles in the graph. Equality occurs if and only if $G \cong K_n$ or $\mu_1 = \mu_2 = \cdots = \mu_p$, $\mu_{p+1} = \mu_{p+2} = \cdots = \mu_{n-1}$, $(1 \leq p \leq n-2)$ with $n(\mu_1 + \mu_{n-1}) = 2m + n(n+1)$

Proof. In Lemma 2.1 (Radon inequality), set

$$x_i = \left|\mu_i - \frac{2m}{n}\right|, \ a_i = \frac{1}{n+1-\mu_i}, \ r = 1, i = 1, 2, \dots, n-1,$$

we have

$$\sum_{i=1}^{n-1}(n+1-\mu_i)\left(\mu_i - \frac{2m}{n}\right)^2 \geq \frac{\left(\sum_{i=1}^{n-1}\left|\mu_i - \frac{2m}{n}\right|\right)^2}{\sum_{i=1}^{n-1}\frac{1}{n+1-\mu_i}}$$
$$or \ \sum_{i=1}^{n-1}(n+1-\mu_i)\left(\mu_i - \frac{2m}{n}\right)^2 \geq \frac{(LE(G) - \frac{2m}{n})^2}{RL(G) - \frac{1}{n+1}}$$

For the Laplacian,

$$\sum_{i=1}^{n-1}\mu_i = trace(D-A) = \sum_{i=1}^{n} d_i = 2m,$$
$$\sum_{i=1}^{n-1}\mu_i^2 = trace(D-A)^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m,$$
$$\sum_{i=1}^{n-1}\mu_i^3 = trace(D-A)^3 = trace(D^3 - 3D^2A + 3DA^2 - A^3) = F + 3M_1 - 6C_3,$$

where $M_1 = \sum\limits_{i=1}^{n} d_i^2, \ F = \sum\limits_{i=1}^{n} d_i^3$ and C_3 is the number of triangles in the graph.
Now, we have

$$\sum_{i=1}^{n-1}(n+1-\mu_i)\left(\mu_i - \frac{2m}{n}\right)^2$$
$$= (n+1)\sum_{i=1}^{n-1}\left(\mu_i - \frac{2m}{n}\right)^2 - \sum_{i=1}^{n-1}\mu_i\left(\mu_i - \frac{2m}{n}\right)^2$$
$$= (n+1)\sum_{i=1}^{n-1}\left(\mu_i^2 + \frac{4m^2}{n^2} - \frac{4m}{n}\mu_i\right) - \sum_{i=1}^{n-1}\mu_i\left(\mu_i^2 + \frac{4m^2}{n^2} - \frac{4m}{n}\mu_i\right)$$
$$= (n+1)\left(M_1 + 2m - \frac{4m^2}{n} - \frac{4m^2}{n^2}\right) - \sum_{i=1}^{n-1}\left(\mu_i^3 + \frac{4m^2}{n^2}\mu_i - \frac{4m}{n}\mu_i^2\right)$$
$$= (n+1)\left(M_1 + 2m - \frac{4m^2}{n} - \frac{4m^2}{n^2}\right) - (F + 3M_1 - 6C_3) - \frac{8m^3}{n^2}$$
$$= (M_1 + 2m)\left(n + 1 + \frac{4m}{n}\right) - (F + 3M_1 - 6C_3)$$
$$- (n+1)\left(\frac{4m^2}{n} + \frac{4m^2}{n^2}\right) - \frac{8m^3}{n^2}$$

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Thus, we have

$$\frac{\left(LE(G)-\frac{2m}{n}\right)^2}{\left(RL(G)-\frac{1}{n+1}\right)} \geq$$
$$\left[(M_1+2m)\left(n+1+\frac{4m}{n}\right)-(F+3M_1-6C_3)-\left(\frac{2m(n+1)}{n}\right)^2-\frac{8m^3}{n^2}\right]$$

The equality can be easily proved in the same way as in [16, Theorem 3.1], completing the proof. □

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