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DISTANCE (SIGNLESS) LAPLACIAN SPECTRUM OF DUMBBELL GRAPHS

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Abstract. In this paper, we determine the distance Laplacian and distance signless Laplacian spectrum of generalized wheel graphs and a new class of graphs called dumbbell graphs.

1. Introduction

We consider an undirected and connected graph $G = (V, E)$, where V is the vertex set and E is the edge set, on n vertices. The adjacency matrix of a graph G is a square matrix $A(G) = (a_{ij})$, in which $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$, otherwise. The eigenvalues of the adjacency matrix $A(G)$, say $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, forms the adjacency spectrum of G. A graph G is *regular* if every vertex has the same degree. The maximum of all distance between any two vertices of a graph G is called the *diameter* of a graph G. The *complement* of G is the graph whose vertex set is same as that of G and two vertices are adjacent in $\overline{G}$ if only if they are not adjacent in G. The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph whose vertex set is $V(G_1) \cup V(G_2)$ and the edge set is $E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, denoted by $G_1 \nabla G_2$ is the graph obtained from $G_1 \cup G_2$ by adding all possible edges from the vertices of G_1 to those in G_2. As usual, we denote by K_n the complete graph and by C_n the cycle on n vertices.

The *distance matrix* $D(G)$ of G is an $n \times n$ matrix $(d_{i,j})$ such that $d_{i,j} = d(v_i, v_j)$, the shortest distance between two vertices v_i and v_j in G. It is a real, symmetric matrix with zeroes on the main

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diagonal. The *transmission degree* of a vertex v, denoted by $Tr_G(v)$ is defined to be the sum of the distances from v to all other vertices in G, i.e.,

$$Tr_G(v) = \sum_{u \in V} d(u, v)$$

Let $Tr_i = Tr_G(v_i)$. Then the sequence $\{Tr_1, Tr_2, \ldots, Tr_n\}$ is said to be the *transmission degree sequence*. The diagonal matrix $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ characterizes the vertex transmissions of G.

For a connected graph G, M. Aouchiche and P. Hansen [17] introduced the distance Laplacian and distance signless Laplacian matrices. They also derived the spectra of these matrices. The *distance Laplacian matrix* of a connected graph G is the matrix $D^{\mathbb{L}}(G) = Tr(G) - D(G)$. The *distance signless Laplacian matrix* of a connected graph G is the matrix $D^{\mathbb{Q}}(G) = Tr(G) + D(G)$. The eigenvalues of $D^{\mathbb{L}}(G)$ and $D^{\mathbb{Q}}(G)$ are called $\mathbb{L}$ - eigenvalues, $\rho_1^{\mathbb{L}} \geq \rho_2^{\mathbb{L}} \geq \cdots \geq \rho_n^{\mathbb{L}} = 0$ and $\mathbb{Q}$ - eigenvalues, $\rho_1^{\mathbb{Q}} \geq \rho_2^{\mathbb{Q}} \geq \cdots \geq \rho_n^{\mathbb{Q}}$, respectively. The *distance Laplacian spectrum* and *the distance signless Laplacian spectrum* of G are called $\mathbb{L}$ - spectrum, $\{\rho_1^{\mathbb{L}}, \rho_2^{\mathbb{L}}, \ldots, \rho_n^{\mathbb{L}}\}$ and $\mathbb{Q}$ - spectrum, $\left\{\rho_1^{\mathbb{Q}}, \rho_2^{\mathbb{Q}}, \ldots, \rho_n^{\mathbb{Q}}\right\}$, respectively.

Recently, the study of $\mathbb{L}(\mathbb{G})$ and $\mathbb{Q}(G)$ - spectra of graphs and some of their properties have been established [2, 3, 4, 16, 15, 14]. A. Alhevaz et al. [5] described the distance signless Laplacian spectrum of the joined union of regular graphs in terms of their adjacency spectrum and the eigenvalues of an auxiliary matrix determined by the graph G. A. Alhevaz et al. [1] presented some inequalities relating the vertex transmissions, distance eigenvalues, distance Laplacian eigenvalues and distance signless Laplacian eigenvalues of graphs. They also obtained new results and bounds for the distance signless Laplacian energy of graphs. Subsequently, A. Alhevaz et al. [6] have determined some new upper and lower bounds on the distance signless Laplacian spectral radius of G and characterized the extremal graphs attaining these bounds. A. Alhevaz et al. [7] have obtained novel sharp bounds for the distance signless Laplacian spectral radius and the bounds are expressed through graph diameter, vertex covering number, edge covering number, clique number, independence number, domination number as well as extremal transmission degrees.

2. **Preliminaries**

We now provide the lemmas and theorems used in our study.

Lemma 2.1. [12] *Let* $T = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$ *be a symmetric 2×2 block matrix. Eigenvalues of* T *are the union of eigenvalues of the sum and difference matrices of* A *and* B*.*

Lemma 2.2. [10] *Let* $K = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ *be a symmetric 2×2 block matrix and* $det(P) \neq 0$*, then* $det(K) = det(P)\ det\left(S - RP^{-1}Q\right)$*. If* P *and* R *commute,* $det(K) = det(PS - RQ)$*.*

Theorem 2.3. [18] *Let* G_1 *and* G_2 *be* r_1 *and* r_2 *- regular graphs with* n_1 *and* n_2 *vertices, respectively. Let the eigenvalues of adjacency matrices* A_{G_1} *and* A_{G_2} *be* $\lambda_{i,1} = r_i \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,n_i}$*, for* $i = 1, 2$*,* *respectively. The distance signless Laplacian spectrum of $G_1 \nabla G_2$ consists of the eigenvalues $2n - n_2 - r_1 - 4 - \lambda_{1,s}$ and $2n - n_1 - r_2 - 4 - \lambda_{2,t}$ with multiplicities $n_1 - 1$ and $n_2 - 1$, respectively, where $n = n_1 + n_2$, $2 \leq s \leq n_1$ and $2 \leq t \leq n_2$. The remaining two eigenvalues are the zeros of the characteristic polynomial of the following equitable quotient matrix*

$$\begin{pmatrix} n + 3n_1 - 2r_1 - 4 & n_2 \\ n_1 & n + 3n_2 - 2r_2 - 4 \end{pmatrix}.$$

Theorem 2.4. [9] *The adjacency spectrum of the cycle graph $C_n, n \geq 3$ is*

$$\left\{ 2cos\left(\frac{2\pi k}{n}\right); k = 0, 1, \ldots, n-1 \right\}.$$

This paper is organised as follows. In section 3, we derive the distance Laplacian spectra of the generalized wheel and dumbbell graphs. We also provide an example for obtaining the distance Laplacian spectrum of a dumbbell graph. In section 4, we derive the distance signless Laplacian spectra of the generalized wheel and dumbbell graphs. We also provide an example for obtaining the distance signless Laplacian spectrum of a dumbbell graph.

3. Distance Laplacian spectrum of generalized wheel and dumbbell graph

In this section, we derive the distance Laplacian spectrum of generalized wheel graph $W_{m,n}, m \geq 2, n \geq 3$ and dumbbell graph.

In 1988, Fred Buckley and Frank Harary [13] defined the generalized wheel graph $W_{m,n}$ as the join $\overline{K_m} \nabla C_n,\ m \geq 2,\ n \geq 3$, where, $\overline{K_m}$ is the complement of the complete graph on m vertices and C_n is the cycle graph on n vertices.

Theorem 3.1. *The distance Laplacian spectrum of generalized wheel graph $W_{m,n}$ is*

$$\left\{ 0, \quad m+n, \quad (2m+n)^{m-1}, \quad 2n + m - 2 + 2cos\left(\frac{2k\pi}{n}\right);\ k = 1, 2, \ldots, n-1 \right\}.$$

Proof. Consider $W_{m,n} = \overline{K_m} \nabla\ C_n, m \geq 2,\ n \geq 3$, the join of empty graph with m vertices and 2 - regular cycle graph with n vertices. Clearly, the diameter of $W_{m,n}$ is two.
The distance Laplacian matrix of $W_{m,\ n}$ can be written as

$$D^{\mathbb{L}}(W_{m,n}) = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}$$

where

$$X = (2m + n - 2)I_{m \times m} - 2\,(J - I)_{m \times m}, \qquad Y = -J_{m \times n},$$
$$W = (2n + m - 4)I_{n \times n} - 2\,(J - I)_{n \times n} + A\,(C_n), \qquad Z = -J_{n \times m}$$

and $J_{m \times n}$ is an all ones matrix, I_n is the identity matrix of order n.

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Let $V\left(\overline{K_m}\right)=\{u_1,u_2,\ldots,u_m\}$ and $V\left(C_n\right)=\{v_1,v_2,\ldots,v_n\}$ be the vertex sets of graphs $\overline{K_m}$ and C_n, respectively. The vertex set of $W_{m,n}$, $V\left(W_{m,n}\right)=V\left(\overline{K_m}\right)\cup V(C_n)$.

Consider the labelled vertices of $W_{m,\ n}$, such that the first m vertices are from $\overline{K_m}, m\geq 2$ and n vertices are from $C_n, n\geq 3$. For all $u_i\in V\left(\overline{K_m}\right), i=1,2,\ldots,m$, we have $Tr\left(u_i\right)=2m+n-2$ and for all $v_i\in V(C_n), i=1,2,\ldots,n$, we have $Tr\left(v_i\right)=2n+m-4$.

Since $\overline{K_m}, m\geq 2$ is an empty graph of order m, in $D^{\mathbb{L}}\left(W_{m,n}\right)$, the shortest distance between the vertices $u_i\in V\left(\overline{K_m}\right); i=1,\ 2,\ldots,m$ is two . Let $\mathbf{v_m}$ be all ones vector of length m say, $\mathbf{v_m}=(1,1,\ldots,1)^T$. In $D^L(W_{m,n})$, the block matrix X has an eigenvector $\begin{bmatrix}\mathbf{v_m} & {0_n}^T\end{bmatrix}^T_{1\times(m+n)}$ corresponding to the eigenvalue n while the remaining eigenvectors corresponding to the eigenvalue $2m+n$ with algebraic multiplicity $m-1$ are orthogonal to $\begin{bmatrix}\mathbf{v_m} & {0_n}^T\end{bmatrix}^T_{1\times(m+n)}$.

$D\left(C_n\right)$ has all ones vector $\mathbf{v_n}=(1,1,\ldots,1)^T$ as an eigenvector corresponding to the eigenvalue 2; remaining eigenvectors are orthogonal to $\mathbf{v_n}$. In $D^{\mathbb{L}}\left(W_{m,n}\right)$, the block matrix W has an eigenvector $\begin{bmatrix}{0_m}^T & \mathbf{v_n}\end{bmatrix}^T_{1\times(m+n)}$ corresponding to the eigenvalue m while the remaining eigenvectors corresponding to the eigenvalues $m+2n-2+2cos\left(\frac{2k\pi}{n}\right); k=1,2,\ldots,n-1$, are orthogonal to $\begin{bmatrix}{0_m}^T & \mathbf{v_n}\end{bmatrix}^T_{1\times(m+n)}$. The other two distance Laplacian eigenvalues are $\{0,m+n\}$ which are the zeros of the characteristic polynomial of the following quotient matrix

$$\begin{bmatrix} n & -n\\ -m & m\end{bmatrix}.$$

This completes proof. □

Corollary 3.2. *The distance Laplacian spectrum of $W_{1,n}, n\geq 3$ consists of the eigenvalues*

$$\left\{0,\quad n+1,\quad 2n-1+2cos\left(\frac{2k\pi}{n}\right); k=1,2,\ldots,n-1\right\}.$$

Proof. In Theorem 3.1, by substituting $m=1$, we obtain the distance Laplacian spectrum of $W_{1,n},\ n\geq 3$. Hence the result. □

Now, we present a new class of graphs called dumbbell graph and derive the distance Laplacian spectrum of dumbbell graph.

Definition 3.3. *A dumbbell graph, denoted by $\boldsymbol{DB}\left(W_{m,n}\right)$, on $2\left(m+n\right)$ vertices is obtained by connecting m - vertices at the centres of two generalized wheel graphs $W_{m,n}, m\geq 2, n\geq 3$ through m - edges. The diameter of the $\boldsymbol{DB}\left(W_{m,n}\right)$ graph is three.*

Lemma 3.4. *Let $M=\begin{bmatrix}(3J-2I)_{m\times m} & 2J_{m\times n}\\ 2J_{n\times m} & 3J_{n\times n}\end{bmatrix}_{(m+n)\times(m+n)}$ be a symmetric 2×2 block matrix. The characteristic polynomial of M is*

$$P(x)=x^{n-1}\left(x+2\right)^{m-1}\left[x^2-\left(3m+3n-2\right)x+\left(5mn-6n\right)\right].$$

The eigenvalues of M are

$$\left\{\frac{1}{2}\left[(3m+3n-2)\pm\sqrt{(3m+3n-2)^2-20mn+24n}\right],\quad 0^{n-1},\quad (-2)^{m-1}\right\}.$$

Proof. Let $M=\begin{bmatrix}(3J-2I)_{m\times m} & 2J_{m\times n}\\ 2J_{n\times m} & 3J_{n\times n}\end{bmatrix}_{(m+n)\times(m+n)}$.

By Lemma 2.2, we obtain the characteristic polynomial of M as

$$P(x)=x^{n-1}\left(x+2\right)^{m-1}\left[x^2-\left(3m+3n-2\right)x+\left(5mn-6n\right)\right]$$

and its eigenvalues are

$$\left\{\frac{1}{2}\left[(3m+3n-2)\pm\sqrt{(3m+3n-2)^2-20mn+24n}\right],\quad 0^{n-1},\quad (-2)^{m-1}\right\}.$$

Hence the result. □

Theorem 3.5. *The distance Laplacian spectrum of dumbbell graph* $\mathbf{DB}\left(W_{m,n}\right)$ *consists of the eigenvalues*

$$\left\{0,\quad 3m+3n,\quad (5m+3n)^{m-1},\quad (5m+3n-4)^{m-1},\quad \left\{(3m+5n-2)^{n-1}+2cos\left(\frac{2k\pi}{n}\right)\right\}^2;\right.$$

$$\left. k=1,2,\ldots,n-1,\quad \frac{1}{2}\left[(9m+9n-4)\pm\sqrt{(3m-3n-4)^2+4mn}\right]\right\}.$$

Proof. The distance Laplacian matrix of dumbbell graph $\mathbf{DB}\ (W_{m,n})$, on $2\left(m+n\right)$ vertices is

$$D^{\mathbb{L}}\left(\mathbf{DB}\left(W_{m,n}\right)\right)=\begin{bmatrix}D^{\mathbb{L}}\left(W_{m,n}\right) & M\\ M & D^{L}\left(W_{m,n}\right)\end{bmatrix}.$$

Using Lemma 2.1, Theorem 3.1 and Lemma 3.4, we get the distance Laplacian spectrum of dumbbell graph $\mathbf{DB}\ (W_{m,n})$, on $2\left(m+n\right)$ vertices. Hence the proof. □

Corollary 3.6. *The distance Laplacian spectrum of dumbbell graph* $\boldsymbol{DB}\ (W_{1,n})$*, on* $2\left(1+n\right)$ *vertices is*

$$\left\{0,\quad 3n+3,\quad \frac{1}{2}\left[(9n+5)\pm\sqrt{9n^2+10n+1}\right],\left\{(5n+1)^{n-1}+2cos\left(\frac{2k\pi}{n}\right)\right\}^2;k=1,2,\ldots,n-1\right\}.$$

Proof. The distance Laplacian matrix of dumbbell graph $D^{L}\left(\mathbf{DB}\left(W_{1,n}\right)\right)$, on $2\left(1+n\right)$ vertices is

$$D^{\mathbb{L}}\left(\mathbf{DB}\left(W_{1,n}\right)\right)=\begin{bmatrix}D^{\mathbb{L}}\left(W_{1,n}\right) & M\\ M & D^{L}\left(W_{1,n}\right)\end{bmatrix}.$$

Using Corollary 3.2 and Lemma 3.4, we obtain the eigenvalues of $D^{\mathbb{L}}(W_{1,n})$ and M, respectively.

By Lemma 2.1, we obtain the distance Laplacian spectrum of $\mathbf{DB}\ (W_{1,n})$, on $2(1+n)$ vertices. Hence the result. □

Example 3.7. *Consider the dumbbell graph* $\mathbf{DB}(W_{2,3})$.

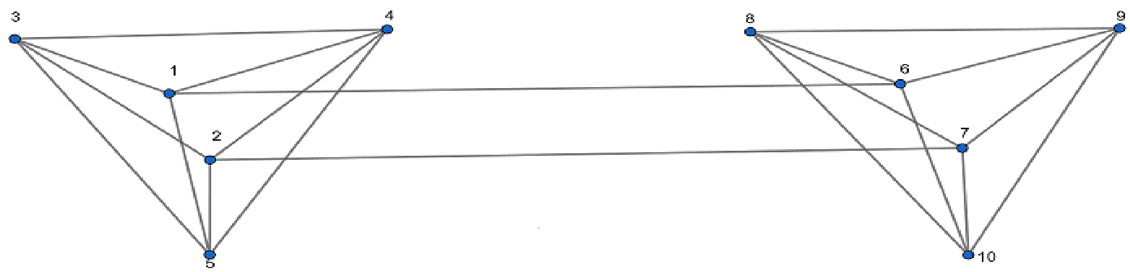

Figure 1. $\mathbf{DB}(W_{2,3})$

The distance Laplacian matrix of dumbbell graph $\mathbf{DB}(W_{2,3})$ *is*

$$D^{\mathbb{L}}(\mathbf{DB}(W_{2,3})) = \begin{bmatrix} D^{\mathbb{L}}(W_{2,3}) & M_{5\times5} \\ M_{5\times5} & D^{\mathbb{L}}(W_{2,3}) \end{bmatrix}.$$

$$D^{\mathbb{L}}(\mathbf{DB}(W_{2,3})) = \begin{bmatrix} 15 & -2 & -1 & -1 & -1 & -1 & -3 & -2 & -2 & -2 \\ -2 & 15 & -1 & -1 & -1 & -3 & -1 & -2 & -2 & -2 \\ -1 & -1 & 17 & -1 & -1 & -2 & -2 & -3 & -3 & -3 \\ -1 & -1 & -1 & 17 & -1 & -2 & -2 & -3 & -3 & -3 \\ -1 & -1 & -1 & -1 & 17 & -2 & -2 & -3 & -3 & -3 \\ -1 & -3 & -2 & -2 & -2 & 15 & -2 & -1 & -1 & -1 \\ -3 & -1 & -2 & -2 & -2 & -2 & 15 & -1 & -1 & -1 \\ -2 & -2 & -3 & -3 & -3 & -1 & -1 & 17 & -1 & -1 \\ -2 & -2 & -3 & -3 & -3 & -1 & -1 & -1 & 17 & -1 \\ -2 & -2 & -3 & -3 & -3 & -1 & -1 & -1 & -1 & 17 \end{bmatrix}$$

Using Theorem 3.5, we obtain the distance Laplacian spectrum of the dumbbell graph $\mathbf{DB}(W_{2,3})$ *as* $\{0,\ 15^2,\ 16.2280,\ 18^4,\ 19,\ 24.7720\}$.

4. Distance signless Laplacian spectrum of generalized wheel and dumbbell graph

In this section, we derive the distance signless Laplacian spectrum of generalized wheel graph $W_{m,n}, m \geq 2, n \geq 3$ and dumbbell graph.

Theorem 4.1. *The distance signless Laplacian spectrum of generalized wheel graph* $W_{m,n}$, *is*

$$\left\{(2m+n-4)^{m-1}, \quad m+2n-6-2cos\left(\frac{2k\pi}{n}\right); k=1,2,\ldots,n-1,\right.$$

$$\left.\frac{1}{2}\left[(5m+5n-12)\pm\sqrt{(3m-3n+4)^2+4mn}\right]\right\}.$$

Proof. Consider $W_{m,n} = \overline{K_m}\nabla\, C_n, m \geq 2,\ n \geq 3$, the join of empty graph with m vertices and 2 - regular cycle graph with n vertices. Clearly, diameter of $W_{m,n}$ is 2.
The distance signless Laplacian matrix of $W_{m,\,n}$ can be written as

$$D^{\mathbb{Q}}(W_{m,n}) = \begin{bmatrix} S & T \\ U & V \end{bmatrix}$$

where

$$S = (2m+n-2)I_{m\times m} + 2\left(J-I\right)_{m\times m}, \qquad T = J_{m\times n},$$
$$V = (2n+m-4)I_{n\times n} + 2\left(J-I\right)_{n\times n} - A\left(C_n\right), \qquad U = J_{n\times m}$$

Let $V\left(\overline{K_m}\right) = \{u_1, u_2, \ldots, u_m\}$ and $V\left(C_n\right) = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of graphs $\overline{K_m}$ and C_n, respectively. The vertex set of $W_{m,n}$ is $V\left(W_{m,n}\right) = V\left(\overline{K_m}\right) \cup V(C_n)$.

Consider the labelled vertices of $W_{m,\,n}$, such that the first m vertices are from $\overline{K_m}, m \geq 2$ and n vertices are from $C_n, n \geq 3$. For all $u_i \in V\left(\overline{K_m}\right), i = 1,2,\ldots,m$, we have $Tr\left(u_i\right) = 2m+n-2$ and for all $v_i \in V(C_n), i = 1,2,\ldots,n$, we have $Tr\left(v_i\right) = 2n+m-4$.

In $D^{\mathbb{Q}}(W_{m,n})$, the shortest distance between the vertices $u_i \in V\left(\overline{K_m}\right); i = 1,2,\ldots,m$ is two. Let $\mathbf{v_m}$ be all ones vector of length m. say, $\mathbf{v_m} = (1,1,\ldots,1)^T$. In $D^{\mathbb{Q}}(W_{m,n})$, the block matrix S has an eigenvector $\begin{bmatrix}\mathbf{v_m} & {0_n}^T\end{bmatrix}^T_{1\times(m+n)}$ corresponding to the eigenvalue $4m+n-4$ while the remaining eigenvectors corresponding to the eigenvalue $2m+n-4$ with algebraic multiplicity $m-1$ are orthogonal to $\begin{bmatrix}\mathbf{v_m} & {0_n}^T\end{bmatrix}^T_{1\times(m+n)}$.

$D\left(C_n\right)$ has all ones vector $\mathbf{v_n} = (1,1,\ldots,1)^T$ as an eigenvector corresponding to the eigenvalue 2, remaining eigenvectors are orthogonal to $\mathbf{v_n}$. In $D^{\mathbb{Q}}(W_{m,n})$, the block matrix V has an eigenvector $\begin{bmatrix}{0_m}^T & \mathbf{v_n}\end{bmatrix}^T_{1\times(m+n)}$ corresponding to the eigenvalue $4n+m-8$ while the remaining eigenvectors corresponding to the eigenvalues $m+2n-6-2cos\left(\frac{2k\pi}{n}\right); k = 1,2,\ldots,n-1$, are orthogonal to $\begin{bmatrix}{0_m}^T & \mathbf{v_n}\end{bmatrix}^T_{1\times(m+n)}$. The other two distance signless Laplacian eigenvalues are the zeros of the characteristic polynomial of the following quotient matrix

$$\begin{bmatrix} 4m+n-4 & n \\ m & 4n+m-8 \end{bmatrix}.$$

From the above quotient matrix, we obtain the eigenvalues as

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$$\rho^S = \frac{1}{2}\left[(5m+5n-12) \pm \sqrt{(3m-3n+4)^2+4mn}\right].$$

□

Corollary 4.2. *The distance signless Laplacian spectrum of $W_{1,n}, n \geq 3$ consists of the eigenvalues*

$$\left\{2n-5-2cos\left(\frac{2k\pi}{n}\right); k=1,2,\ldots,n-1, \quad \frac{1}{2}\left[(5n-7) \pm \sqrt{9n^2-38n+49}\right]\right\}.$$

Proof. In Theorem 4.1, by substituting $m=1$, we obtain the distance signless Laplacian spectrum of wheel graph $W_{1,n}, n \geq 3$. Hence the result. □

Theorem 4.3. *The distance signless Laplacian spectrum of dumbbell graph* $\mathbf{DB}(W_{m,n})$ *is*

$$\left\{(5m+3n-8)^{m-1}, \quad (5m+3n-4)^{m-1}, \quad \left\{(3m+5n-6)^{n-1} - 2cos\left(\frac{2k\pi}{n}\right)\right\}^2; k=1,2,\ldots,n-1,\right.$$

$$\left.\frac{1}{2}\left[(13m+13n-16) \pm \sqrt{49m^2+49n^2-62mn}\right], \quad \frac{1}{2}\left[(7m+7n-12) \pm \sqrt{(m+n-4)^2+16m}\right]\right\}.$$

Proof. The distance signless Laplacian matrix of dumbbell graph $\mathbf{DB}\,(W_{m,n})$, on $2\,(m+n)$ vertices is

$$D^{\mathbb{Q}}\left(\mathbf{DB}\left(W_{m,n}\right)\right) = \begin{bmatrix} D^{\mathbb{Q}}\left(W_{m,n}\right) & M \\ M & D^{\mathbb{Q}}\left(W_{m,n}\right) \end{bmatrix}.$$

Using Lemma 2.1, Theorem 4.1 and Lemma 3.4, we obtain the distance signless Laplacian spectrum of dumbbell $\mathbf{DB}\,(W_{m,n})$, on $2\,(m+n)$ vertices. Hence the proof. □

Corollary 4.4. *The distance signless Laplacian spectrum of dumbbell graph* $\boldsymbol{DB}\,(W_{1,n})$*, on* $2\,(1+n)$ *vertices is,*

$$\left\{\left\{(5n-3)^{n-1} - 2cos\left(\frac{2k\pi}{n}\right)\right\}^2; k=1,2,\ldots,n-1, \quad \frac{1}{2}\left[(13n-3) \pm \sqrt{49n^2-62n+49}\right],\right.$$

$$\left.\frac{1}{2}\left[(7n-5) \pm \sqrt{(n-3)^2+16}\right]\right\}.$$

Proof. The distance signless Laplacian matrix of dumbbell graph $\mathbf{DB}\,(W_{1,n})$, on $2\,(1+n)$ vertices is

$$D^{\mathbb{Q}}\left(\mathbf{DB}\left(W_{1,n}\right)\right) = \begin{bmatrix} D^{\mathbb{Q}}\left(W_{1,n}\right) & M \\ M & D^{\mathbb{Q}}\left(W_{1,n}\right) \end{bmatrix}.$$

Using Corollary 4.2 and Lemma 3.4, we obtain the eigenvalues of $D^{\mathbb{Q}}\,(W_{1,n})$ and M, respectively.

By Lemma 2.1, we obtain the distance signless Laplacian spectrum of $\mathbf{DB}\,(W_{1,n})$, on $2\,(1+n)$ vertices. Hence the result. □

Example 4.5. *Consider the dumbbell graph* $\mathbf{DB}\left(W_{2,3}\right)$ *depicted in Figure* 1. *The distance signless Laplacian matrix of dumbbell graph* $\mathbf{DB}\left(W_{2,3}\right)$ *is*

$$D^{\mathbb{Q}}\left(\mathbf{DB}\left(W_{2,3}\right)\right)=\begin{bmatrix} D^{\mathbb{Q}}\left(W_{2,3}\right) & M_{5\times 5} \\ M_{5\times 5} & D^{\mathbb{Q}}\left(W_{2,3}\right) \end{bmatrix}.$$

$$D^{\mathbb{Q}}\left(\mathbf{DB}\left(W_{2,3}\right)\right)=\begin{bmatrix} 15 & 2 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 2 \\ 2 & 15 & 1 & 1 & 1 & 3 & 1 & 2 & 2 & 2 \\ 1 & 1 & 17 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 17 & 1 & 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 17 & 2 & 2 & 3 & 3 & 3 \\ 1 & 3 & 2 & 2 & 2 & 15 & 2 & 1 & 1 & 1 \\ 3 & 1 & 2 & 2 & 2 & 2 & 15 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 & 3 & 1 & 1 & 17 & 1 & 1 \\ 2 & 2 & 3 & 3 & 3 & 1 & 1 & 1 & 17 & 1 \\ 2 & 2 & 3 & 3 & 3 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Using Theorem 4.3, we obtain the distance signless Laplacian spectrum of the dumbbell graph $\mathbf{DB}\left(W_{2,3}\right)$ *as* $\left\{8.6277, \quad 11, \quad 14.3723, \quad 15, \quad 16^4, \quad 16.3605, \quad 32.6394\right\}$.

Conclusion

We have been obtained the distance Laplacian and distance signless Laplacian spectrum of generalized wheel $\left(W_{m,n}\right)$ and dumbbell graphs $\left(DB\left(W_{m,n}\right)\right)$. We have illustrated our results through examples. We also obtained the distance Laplacian and signless Laplacian spectrum of $DB\left(W_{1,n}\right)$.

The matrix $D_\alpha\left(G\right)=\alpha Tr\left(G\right)+\left(1-\alpha\right)D\left(G\right), 0\leq\alpha\leq 1$, is called generalized distance matrix of G, where $D\left(G\right)$ is the distance matrix and $Tr\left(G\right)$ is the diagonal matrix of the vertex transmissions. In our future work, we envisage to derive the generalized distance spectrum for these classes of graphs.

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