http://toc.ui.ac.ir

Transactions on Combinatorics
ISSN (print): 2251-8657, ISSN (on-line): 2251-8665
Vol. 12 No. 4 (2023), pp. 227-238.

www.ui.ac.ir

GENERAL SUM-CONNECTIVITY INDEX OF TREES WITH GIVEN NUMBER OF BRANCHING VERTICES

TOMÁŠ VETRÍK

Abstract. In 2015, Borovićanin presented trees with the smallest first Zagreb index among trees with given number of vertices and number of branching vertices. The first Zagreb index is obtained from the general sum-connectivity index if $a = 1$. For $a \in \mathbb{R}$, the general sum-connectivity index of a graph G is defined as $\chi_a(G) = \sum_{uv \in E(G)}[d_G(u) + d_G(v)]^a$, where $E(G)$ is the edge set of G and $d_G(v)$ is the degree of a vertex v in G. We show that the result of Borovićanin cannot be generalized for the general sum-connectivity index (χ_a index) if $0 < a < 1$ or $a > 1$. Moreover, the sets of trees having the smallest χ_a index are not the same for $0 < a < 1$ and $a > 1$. Among trees with given number of vertices and number of branching vertices, we present all the trees with the smallest χ_a index for $0 < a < 1$ and $a > 1$. Since the hyper-Zagreb index is obtained from the χ_a index if $a = 2$, results on the hyper-Zagreb index are corollaries of our results on the χ_a index for $a > 1$.

1. Introduction

The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The set of vertices and the number of vertices adjacent to $v \in V(G)$ are the neighborhood $N_G(v)$ and the degree $d_G(v)$ of v in G, respectively. A nonincreasing sequence of degrees of all the vertices of G is called a degree sequence of G.

A connected graph without cycles is called a tree. The star S_n is a tree with n vertices containing a vertex of degree $n-1$. A vertex of a tree of degree at least 3 is a branching vertex and a vertex having degree 1 is called a pendant vertex. A segment S of a tree T is a subtree of T which is a path, its two terminal vertices have degrees different from 2 in T and all the other vertices of S have degree 2 in T.

Communicated by Ali Reza Ashrafi.
MSC(2010): Primary: 05C09; Secondary: 05C05, 05C07.
Keywords: Degree, extremal graph, hyper-Zagreb index.
Article Type: Research Paper.
Received: 07 May 2022, Accepted: 21 October 2022.
http://dx.doi.org/10.22108/TOC.2022.133548.1987 .

An internal segment is a segment whose both terminal vertices are branching vertices. A pendant path is a segment whose one terminal vertex is a pendant vertex. For $k_1 \geq k_2 \geq k_3 \geq 1$, let $P(k_1, k_2, k_3)$ be the tree which consists of three paths of lengths k_1, k_2, k_3, that share one terminal vertex. Note that $P(k_1, k_2, k_3)$ has $k_1 + k_2 + k_3 + 1$ vertices; see Figure 1.

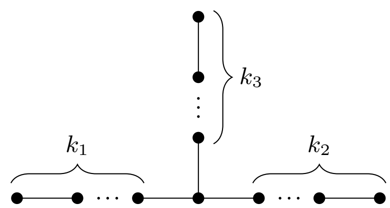

FIGURE 1. A tree $P(k_1, k_2, k_3)$.

Indices of graphs have been studied due to their wide applications, particularly in chemistry. For $a \in \mathbb{R}$, the general sum-connectivity index

$$\chi_a(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^a,$$

of a graph G was introduced by Zhou and Trinajstić [16]. We obtain the first Zagreb index if $a = 1$, the hyper-Zagreb index if $a = 2$ and the classical sum-connectivity index if $a = -\frac{1}{2}$. Moreover, if $a = -1$, then $2\chi_{-1}(G)$ is the harmonic index.

A lot of chemical structures can be represented by trees, therefore trees have a special importance in the study of indices. Trees of given order with the minimum and maximum χ_a index were given in [6] and [16]. Among trees with given order and diameter, and among trees with given order and number of pendant vertices, trees having the minimum χ_a index for $-1 \leq a < 0$ were given by Tomescu and Kanwal [13]. Trees with given order and number of pendant vertices having the maximum χ_a index for $-1 \leq a < 0$ were obtained by Cui and Zhong [5]. Trees with given order and independence number having the maximum χ_a index for $a \geq 1$ were presented by Tomescu and Jamil [12]. Trees with given order and maximum degree having the largest general sum-connectivity index were presented in [2] and [8]. Trees with given order and matching number having the minimum χ_a index for $-1 \leq a < 0$ were given by Jamil and Tomescu [9]. Trees with given matching number were studied also in [17] and [18]. Related indices were investigated for example in [1], [7], [10], [11] and [14]. A survey paper on the general sum-connectivity index and its special cases was given by Ali, Zhong and Gutman [3].

For $n, b \in \mathbb{Z}$, where $b \geq 0$ and $n \geq 2b+2$, let $T_{n,b}$ be the set of all trees with n vertices and b branching vertices. Borovićanin [4] studied Zagreb indices and proved that among trees in $T_{n,b}$, every tree with the degree sequence $(\underbrace{3, \dots, 3}_{b}, \underbrace{2, \dots, 2}_{n-2b-2}, \underbrace{1, \dots, 1}_{b+2})$ has the smallest first Zagreb index. The first Zagreb index is obtained from the general sum-connectivity index if $a = 1$. It is interesting that the result of Borovićanin cannot be generalized for the general sum-connectivity index (χ_a index) if $0 < a < 1$ or $a > 1$. Moreover, the sets of trees having the smallest χ_a index are not the same for $0 < a < 1$ and $a > 1$. Among trees in $T_{n,b}$, we present all the trees with the smallest χ_a index for $0 < a < 1$ and $a > 1$.

2. Preliminary results

Lemma 2.1 was presented in [15]. We also give its proof, since it is short.

Lemma 2.1. *Let $x \geq 1$ and $c > 0$. The function*

$$f(x) = (x + c)^a - x^a$$

is strictly increasing if $a > 1$, and it is strictly decreasing if $0 < a < 1$.

Proof. The derivative

$$f'(x) = a[(x + c)^{a-1} - x^{a-1}].$$

If $a > 1$, we get $(x + c)^{a-1} > x^{a-1}$, thus $f'(x) > 0$, so $f(x)$ is strictly increasing. If $0 < a < 1$, we have $(x + c)^{a-1} < x^{a-1}$, thus $f'(x) < 0$, so $f(x)$ is strictly decreasing. □

The proof of Lemma 2.2 is divided into two cases, because we need to use a more complicated transformation for the case $0 < a < 1$.

Lemma 2.2. *Let $a > 0$. Among trees in $T_{n,b}$, where $n \geq 6$, $b \geq 1$ and $n \geq 2b + 2$, a tree with the minimum χ_a index has each vertex of degree at most 3.*

Proof. We prove Lemma 2.2 by contradiction. Assume that a tree T' which has the minimum χ_a index contains a vertex of degree at least 4.

Note that T' is not S_n, since we have another tree $P(n - 3, 1, 1)$ having one branching vertex with a smaller χ_a index:

$$\chi_a(P(n - 3, 1, 1)) = (n - 3)4^a + 5^a + 3^a < (n - 1)n^a = \chi_a(S_n).$$

Let v_0 be a vertex of maximum degree in T'. We have $d_{T'}(v_0) = z \geq 4$. Let $u_s u_{s-1} \cdots u_1 v_0 v_1 v_2 \cdots v_r$ be a longest path containing v_0. We also use the notation u_0 for the vertex v_0. Since T' is not S_n, we can assume that $r \geq 2$ and $s \geq 1$. We have $d_{T'}(u_{s-1}) = p$, where $2 \leq p \leq z$. We distinguish two cases:

Case 1: $a > 1$.
We have $d_{T'}(v_1) = t$, where $2 \leq t \leq z$. We define T_1 with $V(T_1) = V(T')$ and $E(T_1) = \{u_s v_1\} \cup E(T') \setminus \{v_0 v_1\}$. The branching vertices of T' and T_1 are the same vertices. The vertices v_0, u_s are the only ones with different degrees in T' and T_1. Note that if $s \neq 1$, then $u_s \notin N_{T'}(v_0)$, which means that $N_{T'}(v_0) \setminus \{v_1, u_s\} = N_{T'}(v_0) \setminus \{v_1\}$. We get

$$\begin{aligned}
\chi_a(T') - \chi_a(T_1) &= [d_{T'}(u_{s-1}) + d_{T'}(u_s)]^a - [d_{T_1}(u_{s-1}) + d_{T_1}(u_s)]^a \\
&\quad + [d_{T'}(u_s) + d_{T'}(v_1)]^a - [d_{T_1}(v_0) + d_{T_1}(v_1)]^a \\
&\quad + \sum_{v \in N_{T'}(v_0) \setminus \{v_1, u_s\}} ([d_{T'}(v_0) + d_{T'}(v)]^a - [d_{T_1}(v_0) + d_{T_1}(v)]^a) \\
&> [d_{T'}(u_{s-1}) + d_{T'}(u_s)]^a - [d_{T_1}(u_{s-1}) + d_{T_1}(u_s)]^a \\
&\quad + [d_{T'}(u_s) + d_{T'}(v_1)]^a - [d_{T_1}(v_0) + d_{T_1}(v_1)]^a \\
&= (p + 1)^a - (p + 2)^a + (t + z)^a - (t + 2)^a.
\end{aligned}$$

Let $p = 2$. We have $z \geq 4$. So

$$\begin{aligned}\chi_a(T') - \chi_a(T_1) &> 3^a - 4^a + (t+4)^a - (t+2)^a \\ &\geq 3^a - 4^a + 6^a - 4^a \\ &> 3^a - 4^a + 5^a - 4^a \\ &> 0,\end{aligned}$$

since by Lemma 2.1,

$$5^a - 4^a > 4^a - 3^a \quad \text{and} \quad (t+4)^a - (t+2)^a \geq 6^a - 4^a$$

for $t \geq 2$. Hence $\chi_a(T') > \chi_a(T_1)$.

Let $p \geq 3$. Since $t \geq 2$, by Lemma 2.1, we have $(t+z)^a - (t+2)^a \geq (z+2)^a - 4^a$. So

$$\chi_a(T') - \chi_a(T_1) > (p+1)^a - (p+2)^a + (z+2)^a - 4^a \geq 0,$$

since

$$(p+1)^a - 4^a \geq 0 \quad \text{and} \quad (z+2)^a - (p+2)^a \geq 0$$

for $p \geq 3$ and $p \leq z$, respectively. Thus $\chi_a(T') > \chi_a(T_1)$, which means that T' does not have the minimum χ_a index.

Case 2: $0 < a < 1$.

Let v_i be a furthest vertex from v_0 on the path between v_0 and v_r having degree greater than 3. We have $0 \leq i \leq r-1$, $d_{T'}(v_i) = t$, where $4 \leq t \leq z$, and $d_{T'}(v_{i+1}) = t'$, where $1 \leq t' \leq 3$. We define T_2 with $V(T_2) = V(T')$ and $E(T_2) = \{u_s v_{i+1}\} \cup E(T') \setminus \{v_i v_{i+1}\}$. The vertices v_i, u_s are the only ones with different degrees in T' and T_2. Note that if $i \neq 0$ or $s \neq 1$, then $u_s \notin N_{T'}(v_i)$, which means that $N_{T'}(v_i) \setminus \{v_{i+1}, u_s\} = N_{T'}(v_i) \setminus \{v_{i+1}\}$. Then

$$\begin{aligned}\chi_a(T') - \chi_a(T_2) &= [d_{T'}(u_{s-1}) + d_{T'}(u_s)]^a - [d_{T_2}(u_{s-1}) + d_{T_2}(u_s)]^a \\ &\quad + [d_{T'}(u_s) + d_{T'}(v_{i+1})]^a - [d_{T_2}(v_i) + d_{T_2}(v_{i+1})]^a \\ &\quad + \sum_{v \in N_{T'}(v_i) \setminus \{v_{i+1}, u_s\}} ([d_{T'}(v_i) + d_{T'}(v)]^a - [d_{T_2}(v_i) + d_{T_2}(v)]^a) \\ &> [d_{T'}(u_{s-1}) + d_{T'}(u_s)]^a - [d_{T_2}(u_{s-1}) + d_{T_2}(u_s)]^a \\ &\quad + [d_{T'}(u_s) + d_{T'}(v_{i+1})]^a - [d_{T_2}(v_i) + d_{T_2}(v_{i+1})]^a \\ &= (p+1)^a - (p+2)^a + (t'+t)^a - (t'+2)^a \\ &\geq (p+1)^a - (p+2)^a + (t'+4)^a - (t'+2)^a \\ &> 3^a - 4^a + 8^a - 6^a \\ &= (4^a - 3^a)(2^a - 1) \\ &> 0,\end{aligned}$$

since by Lemma 2.1, we have

$$(t'+4)^a - (t'+2)^a > 8^a - 6^a \quad \text{and} \quad (p+2)^a - (p+1)^a \le 4^a - 3^a$$

for $1 \le t' \le 3$ and $p \ge 2$, respectively.

We get $\chi_a(T') > \chi_a(T_2)$, so T' does not have the minimum χ_a index. We have a contradiction. □

3. Main results

The path P_n is the unique tree in $T_{n,0}$, thus we consider trees in $T_{n,b}$ for $b \ge 1$. It is known that for any tree T, we have $b \le \frac{n}{2} - 1$ (equivalently $n \ge 2b + 2$). If $n = 4$, then S_4 is the unique tree having a branching vertex. If $n = 5$, we have two trees with a branching vertex: $P(2,1,1)$ and S_5. For $a > 0$,

$$\chi_a(P(2,1,1)) = 5^a + 2(4^a) + 3^a < 4(5^a) = \chi_a(S_5).$$

Let us study tress in $T_{n,b}$ with the minimum χ_a index for $n \ge 6$. First, we consider the case $b = 1$.

Theorem 3.1. *Let T be a tree in $T_{n,1}$. Then for $a > 1$ and $n \ge 6$,*

$$\chi_a(T) \ge (n-3)4^a + 5^a + 3^a$$

with equality if and only if T is $P(n-3,1,1)$. For $0 < a < 1$ and $n \ge 7$,

$$\chi_a(T) \ge (n-7)4^a + 3(5^a + 3^a)$$

with equality if and only if T is $P(k_1,k_2,k_3)$, where $k_1 \ge k_2 \ge k_3 \ge 2$ and $k_1 + k_2 + k_3 = n - 1$.

Proof. Among trees in $T_{n,1}$, let us denote a tree having the minimum χ_a index by T'. From Lemma 2.2, we know that the degree of the branching vertex of T' is 3. So, T' has the form $P(k_1,k_2,k_3)$, for some k_1, k_2, k_3, where $k_1 \ge k_2 \ge k_3 \ge 1$ and $k_1 + k_2 + k_3 = n - 1$.

Let $a > 1$. We show that $k_2 = 1$ (which implies that $k_3 = 1$). Suppose to the contrary that $k_2 \ge 2$ (which implies that $k_1 \ge 2$). We construct T_1 from T' by replacing the paths of lengths k_1 and k_2 by the paths of lengths $k_1 + k_2 - 1$ and 1, respectively. We get

$$\chi_a(T') - \chi_a(T_1) = 5^a + 3^a - 2(4^a) > 0,$$

since by Lemma 2.1, we have $5^a - 4^a > 4^a - 3^a$. Thus $\chi_a(T') > \chi_a(T_1)$, which is a contradiction. We have $k_2 = k_3 = 1$, which means that T' is $P(n-3,1,1)$ and

$$\chi_a(P(n-3,1,1)) = (n-3)4^a + 5^a + 3^a.$$

Let $0 < a < 1$. We show that $k_3 \ge 2$. Suppose to the contrary that $k_3 = 1$. Since $n \ge 7$, we have $k_1 \ge 3$. We construct T_2 from T' by replacing the paths of lengths k_1 and $k_3 = 1$ by the paths of lengths $k_1 - 1$ and 2, respectively. We get

$$\chi_a(T') - \chi_a(T_2) = 2(4^a) - 5^a - 3^a > 0,$$

since by Lemma 2.1, we have $4^a - 3^a > 5^a - 4^a$. Thus $\chi_a(T') > \chi_a(T_2)$, which is a contradiction. We have $k_3 \geq 2$, which means that T' is $P(k_1, k_2, k_3)$, where $k_1 \geq k_2 \geq k_3 \geq 2$ and

$$\chi_a(P(k_1, k_2, k_3)) = (n-7)4^a + 3(5^a + 3^a).$$

□

For $b = 1$, it remains to consider the case $0 < a < 1$ and $n = 6$. By Lemma 2.2, a tree with the minimum χ_a index for $b = 1$ and $n = 6$ is $P(3, 1, 1)$ or $P(2, 2, 1)$. We have

$$\chi_a(P(3,1,1)) - \chi_a(P(2,2,1)) = 5^a + 3(4^a) + 3^a - [2(5^a) + 4^a + 2(3^a)] = 2(4^a) - 5^a - 3^a > 0,$$

since by Lemma 2.1, $4^a - 3^a > 5^a - 4^a$ for $0 < a < 1$. Thus $\chi_a(P(3,1,1)) > \chi_a(P(2,2,1))$, so $P(2,2,1)$ has the minimum χ_a index.

We define the sets $S_i(n, b)$ for $b \geq 2$, where $i = 1, 2, 3, 4$. For $n \geq 3b+1$, $T \in S_1(n, b)$ if and only if T has n vertices and b branching vertices of degree 3, all the pendant paths of T have length 1 and the length of every internal segment is at least 2; see Figure 2.

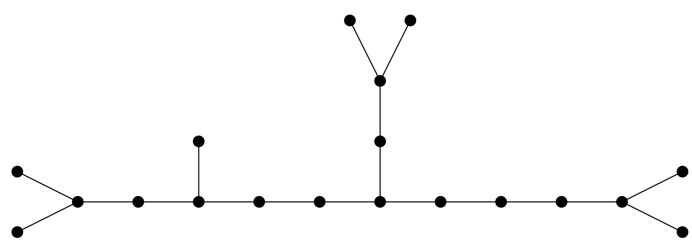

FIGURE 2. A tree in $S_1(19, 5)$.

For $2b + 2 \leq n \leq 3b$, $T \in S_2(n, b)$ if and only if T has n vertices and b branching vertices of degree 3, all the pendant paths of T have length 1 and the length of every internal segment is at most 2.

For $n \geq 3b + 4$, $T \in S_3(n, b)$ if and only if T has n vertices and b branching vertices of degree 3, all the internal segments of T have length 1 and the length of every pendant path is at least 2; see Figure 3.

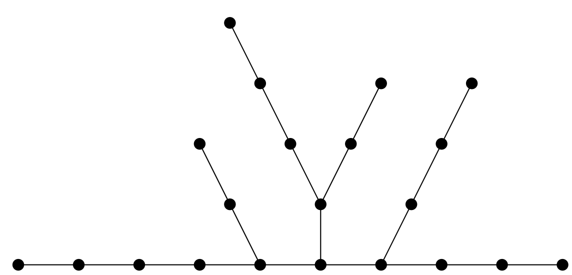

FIGURE 3. A tree in $S_3(21, 4)$.

For $2b+2 \leq n \leq 3b+3$, $T \in S_4(n, b)$ if and only if T has n vertices and b branching vertices of degree 3, all the internal segments of T have length 1 and the length of every pendant path is at most 2.

We present lower bounds on the χ_a index for trees in $T_{n,b}$, where $b \geq 2$ and $a > 1$.

Theorem 3.2. *Let $a > 1$ and let T be a tree in $T_{n,b}$, where $b \geq 2$. Then for $n \geq 3b + 1$,*

$$\chi_a(T) \geq 2(b-1)5^a + (n-2b+1)4^a$$

with equality if and only if $T \in S_1(n, b)$. For $2b + 2 \leq n \leq 3b$,

$$\chi_a(T) \geq (3b-n+1)6^a + 2(n-2b-2)5^a + (b+2)4^a$$

with equality if and only if $T \in S_2(n, b)$.

Proof. Among trees in $T_{n,b}$, let us denote a tree having the minimum χ_a index by T'. Let us denote the number of vertices having degree j in T' by p_j, where $j = 1, 2, 3$. From Lemma 2.2, we know that $n = p_1 + p_2 + p_3$ and $p_3 = b$. From the Handshaking lemma, we get

$$2|E(T')| = 2(n-1) = \sum_{v \in V(T')} d_{T'}(v) = p_1 + 2p_2 + 3p_3,$$

thus $p_1 = b + 2$ and $p_2 = n - 2b - 2$.

Claim 1. *Each pendant path of T' has length 1.*

We prove Claim 1 by contradiction. Suppose that T' contains a pendant path P of length $t \geq 2$. Let S be any internal segment of length $s \geq 1$ in T'. We replace the path P of length t by a single edge (path of length 1) and the segment S of length s by an internal segment of length $s + t - 1$ to obtain T_1 from T'.

If $s = 1$, we obtain

$$\chi_a(T') - \chi_a(T_1) = 6^a + 3^a - 5^a - 4^a > 0,$$

since by Lemma 2.1, we have $6^a - 5^a > 4^a - 3^a$.

If $s \geq 2$, we obtain

$$\chi_a(T') - \chi_a(T_1) = 5^a + 3^a - 2(4^a) > 0,$$

since by Lemma 2.1, we have $5^a - 4^a > 4^a - 3^a$. Thus $\chi_a(T') > \chi_a(T_1)$, which is a contradiction.

Claim 2. *If $n \geq 3b + 1$, then every internal segment of T' has length at least 2.*

Suppose to the contrary that T' contains an internal segment, say S, of length 1. Since we have $p_1 = b + 2$ pendant vertices and each pendant path has length 1 (by Claim 1), the number of edges contained in internal segments is

$$n - 1 - (b+2) \geq (3b+1) - 1 - (b+2) = 2(b-1).$$

It follows that if $b = 2$, then the only internal segment of T' has length at least 2, so Claim 2 holds.

Let $b \geq 3$. Since T' has $b - 1$ internal segments and the length of S is 1, there must be an internal segment, say S' of length $t \geq 3$. We replace the segment S of length 1 by an internal segment of length 2 and the segment S' of length t by an internal segment of length $t - 1$ to obtain T_2 from T'. Then

$$\chi_a(T') - \chi_a(T_2) = 6^a + 4^a - 2(5^a) > 0,$$

since by Lemma 2.1, we have $6^a - 5^a > 5^a - 4^a$. So $\chi_a(T') > \chi_a(T_2)$, which is a contradiction.

http://dx.doi.org/10.22108/TOC.2022.133548.1987

Claim 3. *If $2b+2 \leq n \leq 3b$, then every internal segment of T' has length at most 2.*

If $b=2$, then $n=6$ and the only internal segment of T' has length 1, so Claim 3 holds.

Let $b \geq 3$. Suppose to the contrary that T' contains an internal segment, say S, of length $s \geq 3$. Since we have $p_1 = b+2$ pendant vertices and each pendant path has length 1, the number of edges contained in internal segments is

$$n-1-(b+2) \leq 3b-1-(b+2) = 2(b-1)-1.$$

Since T' has $b-1$ internal segments and the length of S is 1, there must be an internal segment, say S' of length 1. We replace the segment S of length s by an internal segment of length $s-1$ and the segment S' of length 1 by an internal segment of length 2 to obtain T_3 from T'. Then

$$\chi_a(T') - \chi_a(T_3) = 6^a + 4^a - 2(5^a) > 0,$$

since by Lemma 2.1, we have $6^a - 5^a > 5^a - 4^a$. So $\chi_a(T') > \chi_a(T_3)$, which is a contradiction.

Note that the tree T' has $p_1 = b+2$ pendant vertices. By Claim 1, each of the $b+2$ pendant paths of T' has length 1. So T' has $b+2$ edges having the weight $(3+1)^a$.

If $n \geq 3b+1$, then by Claim 2, each of the $b-1$ internal segments of T' has length at least 2, thus T' contains $2(b-1)$ edges having weight is $(3+2)^a$ and all the other $n-1-2(b-1)-(b+2) = n-3b-1$ edges have weight is $(2+2)^a$. It follows that

$$\chi_a(T) \geq 2(b-1)5^a + (n-3b-1)4^a + (b+2)4^a = 2(b-1)5^a + (n-2b+1)4^a$$

with equality if and only if $T \in S_1(n,b)$.

If $2b+2 \leq n \leq 3b$, then by Claim 3, each internal segment of T' has length at most 2, thus each of the $p_2 = n-2b-2$ vertices having degree 2 is adjacent to two vertices having degree 3. Therefore, T' has $2(n-2b-2)$ edges having weight $(3+2)^a$. All the other $n-1-2(n-2b-2)-(b+2) = 3b-n+1$ edges have weight $(3+3)^a$. It follows that

$$\chi_a(T) \geq (3b-n+1)6^a + 2(n-2b-2)5^a + (b+2)4^a$$

with equality if and only if $T \in S_2(n,b)$. □

Let us give lower bounds on the χ_a index for trees in $T_{n,b}$, where $b \geq 2$ and $0 < a < 1$.

Theorem 3.3. *Let $0 < a < 1$ and let T be a tree in $T_{n,b}$, where $b \geq 2$. Then for $n \geq 3b+4$,*

$$\chi_a(T) \geq (b-1)6^a + (b+2)5^a + (n-3b-4)4^a + (b+2)3^a$$

with equality if and only if $T \in S_3(n,b)$. For $2b+2 \leq n \leq 3b+3$,

$$\chi_a(T) \geq (b-1)6^a + (n-2b-2)5^a + (3b-n+4)4^a + (n-2b-2)3^a$$

with equality if and only if $T \in S_4(n,b)$.

Proof. Among trees in $T_{n,b}$, let us denote a tree having the minimum χ_a index by T'. As in the proof of Theorem 3.2, we have

$$n = p_1 + p_2 + p_3, \quad p_1 = b + 2, \quad p_2 = n - 2b - 2 \text{ and } p_3 = b,$$

where p_j is the number of vertices having degree j in T'; $j = 1, 2, 3$.

Claim 4. *Each internal segment of T' has length 1.*

We prove Claim 4 by contradiction. Suppose that T' contains an internal segment S of length $s \geq 2$. Let P be any pendant path of length $t \geq 1$ in T'. We replace the segment S of length s by a single edge (path of length 1) and the path P of length t by a pendant path of length $s + t - 1$ to obtain T_1 from T'.

If $t = 1$, we obtain

$$\chi_a(T') - \chi_a(T_1) = 5^a + 4^a - 6^a - 3^a > 0,$$

since by Lemma 2.1, we have $4^a - 3^a > 6^a - 5^a$.

If $t \geq 2$, we obtain

$$\chi_a(T') - \chi_a(T_1) = 2(5^a) - 6^a - 4^a > 0,$$

since by Lemma 2.1, we have $5^a - 4^a > 6^a - 5^a$. Thus $\chi_a(T') > \chi_a(T_1)$, which is a contradiction.

Claim 5. *If $n \geq 3b + 4$, then every pendant path of T' has length at least 2.*

Suppose to the contrary that T' contains a pendant path, say P, of length 1. Since T' contains b branching vertices and each internal segment has length 1 (by Claim 4), there are $b - 1$ edges contained in internal segments. Thus, the number of edges contained in pendant paths is

$$n - 1 - (b - 1) \geq (3b + 4) - 1 - (b - 1) = 2(b + 2).$$

Since T' has $p_1 = b + 2$ pendant paths and the length of P is 1, there must be a pendant path, say P' of length $t \geq 3$. We replace the path P of length 1 by a pendant path of length 2 and the path P' of length t by a pendant path of length $t - 1$ to obtain T_2 from T'. Then

$$\chi_a(T') - \chi_a(T_2) = 2(4^a) - 5^a - 3^a > 0,$$

since by Lemma 2.1, we have $4^a - 3^a > 5^a - 4^a$. So $\chi_a(T') > \chi_a(T_2)$, which is a contradiction.

Claim 6. *If $2b + 2 \leq n \leq 3b + 3$, then every pendant path of T' has length at most 2.*

Suppose to the contrary that T' contains a pendant path, say P, of length $t \geq 3$. Since T' contains b branching vertices and each internal segment has length 1 (by Claim 4), there are $b - 1$ edges contained in internal segments. Thus, the number of edges contained in pendant paths is

$$n - 1 - (b - 1) \leq (3b + 3) - 1 - (b - 1) = 2(b + 2) - 1.$$

Since T' has $p_1 = b+2$ pendant paths and the length of P is $t \geq 3$, there must be a pendant path, say P' of length 1. We replace the path P of length t by a pendant path of length $t-1$ and the path P' of length 1 by a pendant path of length 2 to obtain T_3 from T'. Then

$$\chi_a(T') - \chi_a(T_3) = 2(4^a) - 5^a - 3^a > 0,$$

since by Lemma 2.1, we have $4^a - 3^a > 5^a - 4^a$. So $\chi_a(T') > \chi_a(T_3)$, which is a contradiction.

Since T' has b branching vertices, each of its $b-1$ internal segments has length 1 by Claim 4, so T' has $b-1$ edges having the weight $(3+3)^a$.

If $n \geq 3b+4$, then by Claim 5, every pendant path of T' has length at least 2. Since T' has $b+2$ pendant vertices, it has $b+2$ pendant paths, so T' has $b+2$ edges having weight $(3+2)^a$ and $b+2$ edges having weight $(2+1)^a$. All the other $n-1-(b-1)-2(b+2) = n-3b-4$ edges have weight is $(2+2)^a$. It follows that

$$\chi_a(T) \geq (b-1)6^a + (b+2)5^a + (n-3b-4)4^a + (b+2)3^a$$

with equality if and only if $T \in S_3(n,b)$.

If $2b+2 \leq n \leq 3b+3$, then by Claim 6, every pendant path of T' has length at most 2. Each of the $p_2 = n-2b-2$ vertices having degree 2 is adjacent to one vertex of degree 3 and one vertex of degree 1. Therefore, T' has $n-2b-2$ edges having weight $(3+2)^a$ and $n-2b-2$ edges having weight $(2+1)^a$. All the other $n-1-2(n-2b-2)-(b-1) = 3b-n+4$ edges have weight $(3+1)^a$. It follows that

$$\chi_a(T) \geq (b-1)6^a + (n-2b-2)5^a + (3b-n+4)4^a + (n-2b-2)3^a$$

with equality if and only if $T \in S_4(n,b)$. □

4. Conclusion

The hyper-Zagreb index is obtained from the general sum-connectivity index when $a = 2$. Thus we use $a = 2$ in Theorems 3.1 and 3.2 to get Corollaries 4.1 and 4.2 on the hyper-Zagreb index.

Corollary 4.1. *Let T be a tree in $T_{n,1}$. Then for $n \geq 6$,*

$$\chi_2(T) \geq 2(8n-7)$$

with equality if and only if T is $P(n-3,1,1)$.

Corollary 4.2. *Let T be a tree in $T_{n,b}$, where $b \geq 2$. Then for $n \geq 3b+1$,*

$$\chi_2(T) \geq 2(8n+9b-17)$$

with equality if and only if $T \in S_1(n,b)$. For $2b+2 \leq n \leq 3b$,

$$\chi_2(T) \geq 2(7n+12b-16)$$

with equality if and only if $T \in S_2(n,b)$.

http://dx.doi.org/10.22108/TOC.2022.133548.1987

We suggest to study upper bounds on the general sum-connectivity index of trees in $T_{n,b}$ for $a > 0$, and lower and upper bounds for $a < 0$.

Problem 4.3. *For $a > 0$, find trees in $T_{n,b}$ with the maximum χ_a index.*

Problem 4.4. *For $a < 0$, find trees in $T_{n,b}$ with the minimum or maximum χ_a index.*

These problems are open for further research.

Acknowledgments

This work is based on the research supported by the National Research Foundation of South Africa (Grant Number 129252).

References

[1] M. Aghel, A. Erfanian and A.R. Ashrafi, On the first and second Zagreb indices of quasi unicyclic graphs, *Trans. Comb.* **8** no. 3 (2019) 29–39.

[2] S. Ahmed, On the maximum general sum-connectivity index of trees with a fixed order and maximum degree, *Discrete Math. Algorithms Appl.* **13** no. 4 (2021) 13 p.

[3] A. Ali, L. Zhong and I. Gutman, Harmonic index and its generalizations: Extremal results and bounds, *MATCH Commun. Math. Comput. Chem.* **81** no. 2 (2019) 249–311.

[4] B. Borovićanin, On the extremal Zagreb indices of trees with given number of segments or given number of branching vertices, *MATCH Commun. Math. Comput. Chem.* **74** no. 1 (2015) 57–79.

[5] Q. Cui and L. Zhong, On the general sum-connectivity index of trees with given number of pendent vertices, *Discrete Appl. Math.* **222** (2017) 213–221.

[6] Z. Du, B. Zhou and N. Trinajstić, On the general sum-connectivity index of trees, *Appl. Math. Lett.* **24** no. 3 (2011) 402–405.

[7] I. Gutman, D. Hanyuan, S. Balachandran, S.K. Ayyaswamy and Y.B. Venkatakrishnan, On the average eccentricity, the harmonic index and the largest signless Laplacian eigenvalue of a graph, *Trans. Comb.* **6** no. 4 (2017) 43–50.

[8] M.K. Jamil and I. Tomescu, General sum-connectivity index of trees and unicyclic graphs with fixed maximum degree, *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.* **20** no. 1 (2019) 11–17.

[9] M.K. Jamil and I. Tomescu, Minimum general sum-connectivity index of trees and unicyclic graphs having a given matching number, *Discrete Appl. Math.* **222** (2017) 143–150.

[10] M. Liang, B. Cheng and J. Liu, Solution to the minimum harmonic index of graphs with given minimum degree, *Trans. Comb.* **7** no. 2 (2018) 25–33.

[11] H. Liu and Z. Tang, Maximal hyper-Zagreb index of trees, unicyclic and bicyclic graphs with a given order and matching number, *Discrete Math. Lett.* **4** (2020) 11–18.

[12] I. Tomescu and M.K. Jamil, Maximum general sum-connectivity index for trees with given independence number, *MATCH Commun. Math. Comput. Chem.* **72** no. 3 (2014) 715–722.

[13] I. Tomescu and S. Kanwal, Ordering trees having small general sum-connectivity index, *MATCH Commun. Math. Comput. Chem.* **69** no. 3 (2013) 535–548.

[14] T. Vetrík, General Randić index of trees with given number of branching vertices, *Discrete Math. Algorithms Appl.*, in press.

[15] T. Vetrík and M. Masre, Generalized eccentric connectivity index of trees and unicyclic graphs, *Discrete Appl. Math.* **284** (2020) 301–315.

[16] B. Zhou and N. Trinajstić, On general sum-connectivity index, *J. Math. Chem.* **47** (2010) 210–218.

[17] L. Zhong and Q. Qian, The minimum general sum-connectivity index of trees with given matching number, *Bull. Malays. Math. Sci. Soc.*, **43** no. 2 (2020) 1527–1544.

[18] Z. Zhu and W. Zhang, Trees with a given order and matching number that have maximum general sum-connectivity index, *Ars Combin.*, **128** (2016) 439–446.

Tomáš Vetrík

Department of Mathematics and Applied Mathematics, University of the Free State, P. O. Box 339, Bloemfontein, 9300, South Africa

Email: vetrikt@ufs.ac.za

http://dx.doi.org/10.22108/TOC.2022.133548.1987