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ON GRAPHS WITH ANTI-RECIPROCAL EIGENVALUE PROPERTY

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Abstract. Let $\mathtt{A}(\mathtt{G})$ be the adjacency matrix of a simple connected undirected graph $\mathtt{G}$. A graph $\mathtt{G}$ of order n is said to be non-singular (respectively singular) if $\mathtt{A}(\mathtt{G})$ is non-singular (respectively singular). The spectrum of a graph $\mathtt{G}$ is the set of all its eigenvalues denoted by $spec(\mathtt{G})$. The anti-reciprocal (respectively reciprocal) eigenvalue property for a graph $\mathtt{G}$ can be defined as " Let $\mathtt{G}$ be a non-singular graph $\mathtt{G}$ if the negative reciprocal (respectively positive reciprocal) of each eigenvalue is likewise an eigenvalue of $\mathtt{G}$, then $\mathtt{G}$ has anti-reciprocal (respectively reciprocal) eigenvalue property ." Furthermore, a graph $\mathtt{G}$ is said to have strong anti-reciprocal eigenvalue property (resp. strong reciprocal eigenvalue property) if the eigenvalues and their negative (resp. positive) reciprocals are of same multiplicities. In this article, graphs satisfying anti-reciprocal eigenvalue (or property $(-\mathtt{R})$) and strong anti-reciprocal eigenvalue property (or property $(-\mathtt{SR})$) are discussed.

1. Introduction

In this article, we only consider simple connected undirected graphs. A graph $\mathtt{G}$ of order $\mathtt{n}$ is an ordered pair $\mathtt{G} = (\mathtt{V}, \mathtt{E})$ with vertex set $\mathtt{V}$ and edge set $\mathtt{E}$, respectively. The adjacency matrix of a graph $\mathtt{G}$ can be written as $\mathtt{A}(\mathtt{G}) = [a_{ij}]$ is an $\mathtt{n} \times \mathtt{n}$ square symmetric matrix with $ij - th$ entry 1 if there is an edge between the vertices i and j and 0, otherwise. We call a graph $\mathtt{G}$ to be non-singular (resp. singular) if $\mathtt{A}(\mathtt{G})$ is non-singular (resp. singular). The roots of the characteristic polynomial $\mathcal{P}_{\mathtt{G}}(\lambda)$ associated with $\mathtt{A}(\mathtt{G})$ are called the eigenvalues of graph $\mathtt{G}$. The fact that $\mathtt{A}(\mathtt{G})$ is symmetric

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implies all of its eigenvalues are real. Let $\lambda_1(\mathtt{G}) \geq \lambda_2(\mathtt{G}) \geq \cdots \geq \lambda_n(\mathtt{G})$ are the eigenvalues of the graph G. The spectrum of a graph G is the set consisting of all its eigenvalues denoted as, $spec(\mathtt{G})$.

$$spec(\mathtt{G}) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m(\lambda_1) & m(\lambda_2) & \cdots & m(\lambda_n) \end{pmatrix}$$

where $m(\lambda_i)$ is the multiplicity of λ_i, for $i = 1, 2, \ldots, n$. Pavithra and Rajkumar, [16] introduced the M-bicone product of graphs and discussed the spectra of these graphs. (for reader's interest [3], [4], [17])

The reciprocal eigenvalue property (or property (R)) is satisfied by G if for each eigenvalue λ there exist $\frac{1}{\lambda}$ in the spectrum of G. Note that, for a graph G with property (R), the multiplicities of eigenvalues and there reciprocals may or may not be equal but if there exist at least one eigenvalue λ such that $m(\lambda) \neq m(\frac{1}{\lambda})$ then we say that graph strictly satisfies property (R). In addition, if λ and $\frac{1}{\lambda}$ have same multiplicities for each eigenvalue λ of G then the graph G satisfies strong reciprocal eigenvalue property (or property (SR)).

Barik et al. started the investigation on graphs having property (R) in [9]. In [12] and [10] authors explored the property (SR) for trees under the titles symmetric property and property C. The authors (in [12], [10]) established the concept that a non-singular tree satisfies property (SR) if and only if it is a simple corona graph, and they also proved in [7] that property (R) and property (SR) are equivalent for non-singular trees. In [6], authors studied unicyclic graphs with property (SR). In [8], authors presented some classes of non-bipartite graphs having property (R) but not property (SR).

In the year 2012, Lagrange proposed the concept of anti-reciprocal eigenvalue property (or property (−R)) and strong anti-reciprocal eigenvalue property (or property (−SR)) for the graphs [14]. For a nonsingular graph G, if for each eigenvalue λ there exist $-\frac{1}{\lambda}$ in the spectrum of G then that graph is said to have property (−R). Note that, for a graph G with property (−R) the multiplicities of eigenvalues and there negative reciprocals may or may not be equal but if there exist at least one eigenvalue λ such that $m(\lambda) \neq m(-\frac{1}{\lambda})$ then we say that graph strictly satisfies property (−R). Consequently, a nonsingular graph G is said to have strong anti-reciprocal eigenvalue property (or property (−SR)), if for each eigenvalue λ there exist $-\frac{1}{\lambda}$ in the spectrum of G along with $m(\lambda) = m(-\frac{1}{\lambda})$.

Lagrange [14] studied property (−SR) for the zero-divisor graphs of finite commutative rings with non-zero divisors. In [1], authors carried out investigation for the graphs with property (−SR). They investigated a class of connected simple weighted graphs with unique perfect matching M, denoted by $\mathtt{G}_M$ for property (−SR). It was shown that, a weighted graph $\mathtt{G}_w$ satisfies the property (−SR) for any $w \in W(\mathtt{G})$ if and only if it is a corona graph.

Later on, they also investigated the property (−SR) for different non-corona families of graphs [2]. In [5], authors further generalized these families and they also studied non-bipartite unicyclic graphs for property (−SR) and it was proved that they must be simple corona graphs. A family of graphs with unique perfect matching along with diagonal entries zero in the inverse of their adjacency matrix was investigated by Hameed et al. in [13]. Authors showed that this family does not satisfy

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property $(-\mathtt{SR})$ not even for a single weight function w. Non-bipartite graphs with property $(\mathtt{R})$ are investigated in [15] and [8]. A lot of work has been done on non-bipartite graphs with properties $(\mathtt{R})$, $(\mathtt{SR})$ and $(-\mathtt{SR})$. Until now, non-bipartite graphs with property $(-\mathtt{R})$ are not studied. So, the main motivation of this article is to investigate non-bipartite graphs with property $(-\mathtt{R})$ or property $(-\mathtt{SR})$. We begin with some basic concepts which will be used in our work.

Definition 1.1. [11] *Consider two simple connected graphs $\mathtt{G}_1$ and $\mathtt{G}_2$ of order n and m, respectively. The corona product $\mathtt{G}_1 \circ \mathtt{G}_2$ is a graph formed by one copy of graph $\mathtt{G}_1$ and n-copies of $\mathtt{G}_2$ and by connecting each vertex of jth copy of $\mathtt{G}_2$ with the jth vertex of $\mathtt{G}_1$, for $1 \leq j \leq n$.*

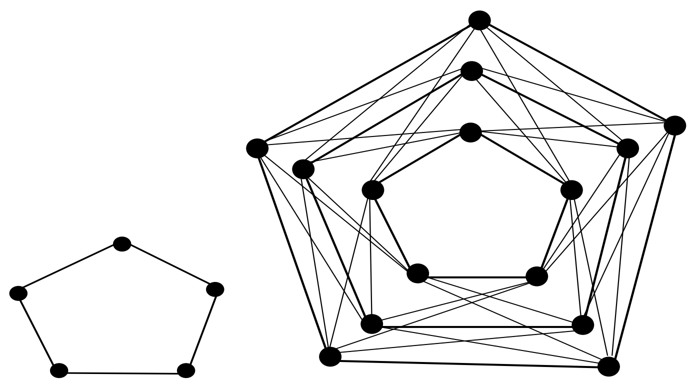

FIGURE 1. $\mathtt{C}_5$ and $\mathtt{D}_3(\mathtt{C}_5)$.

Definition 1.2. [18] *Let $\mathtt{G}$ be a simple connected graph, the shadow graph $\mathtt{D}_2(\mathtt{G})$ of graph $\mathtt{G}$ is constructed by taking two copies of $\mathtt{G}_1$ and $\mathtt{G}_2$ of $\mathtt{G}$ and joining each vertex u_1 of $\mathtt{G}_1$ to the neighbors of the corresponding vertex u_2 of $\mathtt{G}_2$.*

Definition 1.3. [18] *Let $\mathtt{G}_1, \mathtt{G}_2, \ldots, \mathtt{G}_m$ be m copies of a simple connected graph $\mathtt{G}$, the m-shadow graph $\mathtt{D}_\mathtt{m}(\mathtt{G})$ of graph $\mathtt{G}$ is constructed by joining each vertex v of $\mathtt{G}_i$ to the neighbors of the corresponding vertex u of $\mathtt{G}_j$ for $1 \leq i, j \leq m$.*

Lemma 1.4. [9] *Let $\mathtt{H}$ be a simple graph and $\mathtt{H}_1 = \mathtt{H} \circ \mathtt{K}_1$. Then, if μ is an eigenvalue of $\mathtt{H}$ afforded by the eigenvector $\boldsymbol{x}$, then $\lambda = \frac{\mu+\sqrt{\mu^2+4}}{2}$ and $-\frac{1}{\lambda} = \frac{\mu-\sqrt{\mu^2+4}}{2}$ are eigenvalues of $\mathtt{H}_1$ afforded by the eigenvectors $\begin{bmatrix} \boldsymbol{x} \\ \frac{1}{\lambda}\boldsymbol{x} \end{bmatrix}$ and $\begin{bmatrix} \boldsymbol{x} \\ -\lambda\boldsymbol{x} \end{bmatrix}$.*

2. Main Results

In this section, some new families of graphs namely, $\mathtt{Z}$ and $\mathtt{A}$ are presented and investigation is carried out whether these families satisfy property $(-\mathtt{SR})$ or strictly satisfy property $(-\mathtt{R})$. For this purpose we first construct a family of graphs in the following definition.

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Definition 2.1. *Let* $\mathtt{G}_1, \mathtt{G}_2, \ldots, \mathtt{G}_p$ *be* p *copies of a graph* $\mathtt{G}$*. Then* $\mathtt{D}_p(\mathtt{G})$, $p \geq 6$ *be the* p*-shadow graph obtained from* p *copies of* $\mathtt{G}$ *according to the Definition 1.3. Let* $\mathtt{Z} = \{\mathtt{Z}_p(\mathtt{G}) : p \in \mathbb{N}, p \geq 6\}$ *be the family of graphs which can be obtained from* $\mathtt{D}_p(\mathtt{G})$ *by deleting all edges of the* p*-shadow graph of* $\mathtt{D}_p(\mathtt{G})$ *except the joining edges of the following copies*

$$\begin{cases} \mathtt{G}_i \text{ and } \mathtt{G}_{i+1}, & \text{for } 1 \leq i < \frac{p}{2}; \\ \mathtt{G}_1 \text{ and } \mathtt{G}_{\frac{p}{2}}, & \text{for } i = \frac{p}{2}; \\ \mathtt{G}_i \text{ and } \mathtt{G}_{i-\frac{p}{2}}, & \text{for } \frac{p}{2} < i \leq p. \end{cases}$$

For example the graph $\mathtt{Z}_6(\mathtt{C}_4)$ *is shown in Figure 2, obtained from* 6 *copies of* $\mathtt{C}_4$ *and deleting all the edges in the* 6*-shadow graph* $\mathtt{D}_6(\mathtt{C}_4)$ *except the joining edges of the following copies*

$$\begin{cases} \mathtt{G}_i \text{ and } \mathtt{G}_{i+1}, & \text{for } 1 \leq i < 3; \\ \mathtt{G}_1 \text{ and } \mathtt{G}_3, & \text{for } i = 3; \\ \mathtt{G}_i \text{ and } \mathtt{G}_{i-3}, & \text{for } 3 < i \leq 6. \end{cases}$$

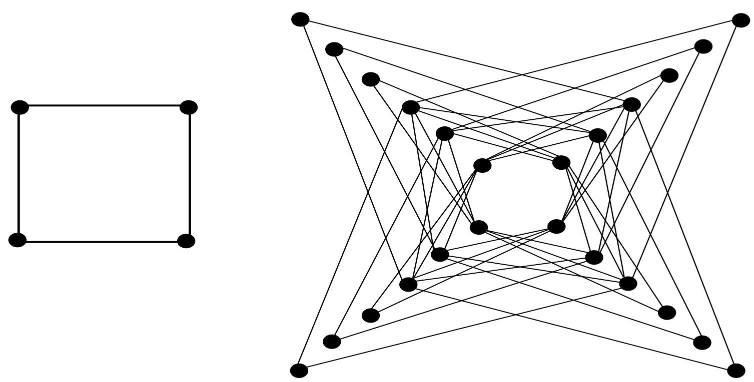

FIGURE 2. $\mathtt{C}_4$ and $\mathtt{Z}_6(\mathtt{C}_4)$.

The following result tells us that each graph in $\mathtt{Z}$ strictly satisfies property $(-\mathtt{R})$ for $p \cong 2 \ mod \ 4$, $p \geq 6$ and satisfy property $(-\mathtt{SR})$ for $p \cong 0 \ mod \ 4$, $p \geq 8$.

Theorem 2.2. *Let* $\mathtt{G}$ *be any graph which strictly satisfy property* $(\mathtt{R})$*. Then the graph* $\mathtt{Z}_p(\mathtt{G}) \in \mathtt{Z}$,

(1) *Satisfies property* $(-\mathtt{SR})$ *if* $p \cong 0 \ mod \ 4$ *and* $p \geq 8$,
(2) *Strictly satisfies property* $(-\mathtt{R})$ *if* $p \cong 2 \ mod \ 4$ *and* $p \geq 6$.

Proof. Consider p copies a graph $\mathtt{G}$ of order n strictly satisfying property $(\mathtt{R})$.

Case 1: When $p \cong 0 \ mod \ 4$, $p \geq 8$. Let $A = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ be the set of all those eigenvalues of $\mathtt{G}$ for which

$$m(\sigma_i) \neq m(\frac{1}{\sigma_i})$$

$i = 1, 2, \ldots, k$ where $k < n$ and $B = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$, $l < n$ be the set of all those eigenvalues such that

$$m(\lambda_j) = m(\frac{1}{\lambda_j})$$

where $j = 1, 2, \ldots, l$. Let

$$t_i = \frac{2cos\frac{2\pi i}{p/2} \pm \sqrt{\left(2cos\frac{2\pi i}{p/2}\right)^2 + 4}}{2}, for 1 \leq i \leq \frac{p}{2} - 1,\ i \neq \frac{p}{4},$$

and for $i = 0$ or $i = \frac{p}{4}$, t_i becomes $t_{0,p/4} = \pm 1 \pm \sqrt{2}$. For each $\sigma \in A$ and $\lambda \in B$ the spectrum of the graph $\mathtt{Z}_p(\mathtt{G})$ can be written as,

$$\begin{pmatrix} t_i\lambda = \begin{cases} t_i\lambda, for\ t_i \neq \pm\frac{1}{\lambda} \\ \pm 1, for\ t_i = \pm\frac{1}{\lambda} \end{cases} & -\frac{1}{t_i}\lambda = \begin{cases} -\frac{1}{t_i}\lambda, & for\ \frac{1}{t_i} \neq \pm\frac{1}{\lambda} \\ \pm 1, & for\ \frac{1}{t_i} = \pm\frac{1}{\lambda} \end{cases} \\ m(t_i\lambda) = \begin{cases} 2m(\lambda), & for\ t_i \neq \pm\frac{1}{\lambda} \\ 4m(\lambda), & for\ t_i = \pm\frac{1}{\lambda} \end{cases} & m(-\frac{1}{t_i}\lambda) = \begin{cases} 2m(\lambda), & for\ \frac{1}{t_i} \neq \pm\frac{1}{\lambda} \\ 4m(\lambda), & for\ \frac{1}{t_i} = \pm\frac{1}{\lambda} \end{cases} \\ t_i\sigma = \begin{cases} t_i\sigma, & for\ t_i \neq \pm\frac{1}{\sigma} \\ \pm 1, & for\ t_i = \pm\frac{1}{\sigma} \end{cases} & -\frac{1}{t_i}\sigma = \begin{cases} -\frac{1}{t_i}\sigma, & for\ \frac{1}{t_i} \neq \pm\frac{1}{\sigma} \\ \pm 1, & for\ \frac{1}{t_i} = \pm\frac{1}{\sigma} \end{cases} \\ m(t_i\sigma) = \begin{cases} 2[m(\sigma) + m(\frac{1}{\sigma})], & for\ t_i \neq \pm\frac{1}{\sigma} \\ 4[m(\sigma) + m(\frac{1}{\sigma})], & for\ t_i = \pm\frac{1}{\sigma} \end{cases} & m(-\frac{1}{t_i}\sigma) = \begin{cases} 2[m(\sigma) + m(\frac{1}{\sigma})], & for\ \frac{1}{t_i} \neq \pm\frac{1}{\sigma} \\ 4[m(\sigma) + m(\frac{1}{\sigma})], & for\ \frac{1}{t_i} = \pm\frac{1}{\sigma} \end{cases} \\ t_{0,p/4}\ \lambda & -\frac{1}{t_{0,p/4}}\lambda \\ m(\lambda) & m(\lambda) \\ t_{0,p/4}\ \sigma & -\frac{1}{t_{0,p/4}}\sigma \\ m(\sigma) + m(\frac{1}{\sigma}) & m(\sigma) + m(\frac{1}{\sigma}) \end{pmatrix}.$$

Let $\alpha \in spec(\mathtt{Z}_p(\mathtt{G}))$ and

$$\alpha = t_i\lambda = \frac{2cos\frac{2\pi i}{p/2} + \sqrt{\left(2cos\frac{2\pi i}{p/2}\right)^2 + 4}}{2}\lambda.$$

Then

$$-\frac{1}{\alpha} = -\frac{1}{t_i\lambda} = -\frac{2cos\frac{2\pi i}{p/2} - \sqrt{\left(2cos\frac{2\pi i}{p/2}\right)^2 + 4}}{2}(\frac{1}{\lambda}) \in spec(\mathtt{Z}_p(\mathtt{G}))$$

as $\mathtt{G}$ strictly satisfies property $(\mathtt{R})$. Moreover, $\lambda \in B$ then for each λ, $m(\lambda) = m(\frac{1}{\lambda})$ so we can conclude that $m(\alpha) = m(-\frac{1}{\alpha})$. Now let,

$$\beta = t_i\sigma = \frac{2cos\frac{2\pi i}{p/2} + \sqrt{\left(2cos\frac{2\pi i}{p/2}\right)^2 + 4}}{2}\sigma.$$

Then

$$-\frac{1}{\beta}=-\frac{1}{t_i\sigma}=-\frac{2cos\frac{2\pi i}{p/2}-\sqrt{(2cos\frac{2\pi i}{p/2})^2+4}}{2}(\frac{1}{\sigma})\in spec(\mathtt{Z}_p(\mathtt{G}))$$

as G strictly satisfies property (R). Moreover,

$$m(\beta)=m(t_i\sigma)=\begin{cases}2[m(\sigma)+m(\frac{1}{\sigma})], & for\ t_i\neq\pm\frac{1}{\sigma}\\ 4[m(\sigma)+m(\frac{1}{\sigma})], & for\ t_i=\pm\frac{1}{\sigma}\end{cases}$$

and

$$m(-\frac{1}{\beta})=m(-\frac{1}{t_i}\sigma)=\begin{cases}2[m(\sigma)+m(\frac{1}{\sigma})], & for\ \frac{1}{t_i}\neq\pm\frac{1}{\sigma}\\ 4[m(\sigma)+m(\frac{1}{\sigma})], & for\ \frac{1}{t_i}=\pm\frac{1}{\sigma}\end{cases}$$

which implies that $m(\beta)=m(-\frac{1}{\beta})$. Similarly, for $\gamma=t_{0,p/4}\ \lambda\in spec(\mathtt{Z}_p(\mathtt{G}))$ there exist $-\frac{1}{\gamma}=-\frac{1}{t_{0,p/4}\lambda}\in spec(\mathtt{Z}_p(\mathtt{G}))$ with $m(\gamma)=m(-\frac{1}{\gamma})$ and for $\nu=t_{0,p/4}\ \sigma\in spec(\mathtt{Z}_p(\mathtt{G}))$ there exist $-\frac{1}{\nu}=-\frac{1}{t_{0,p/4}\sigma}\in spec(\mathtt{Z}_p(\mathtt{G}))$ with same multiplicity. Thus, we conclude that each graph in Z satisfies property (−SR).

Case 2: When $p\cong 2\ mod\ 4$ and $p\geq 8$. Let

$$t_i=\frac{2cos\frac{2\pi i}{p/2}\pm\sqrt{(2cos\frac{2\pi i}{p/2})^2+4}}{2}, for\ 1\leq i\leq\frac{p}{2}-1,$$

and for $i=0$, t_i becomes $t_0=1\pm\sqrt{2}$. Then the spectrum of the graph $\mathtt{Z}_p(\mathtt{G})$ for each $\lambda\in spec(\mathtt{G})$ can be written as,

$$spec(\mathtt{Z}_p(\mathtt{G}))=\begin{pmatrix} t_i\lambda=\begin{cases}t_i\lambda, & for\ t_i\neq\frac{1}{\lambda}\\ \pm1, & for\ t_i=\pm\frac{1}{\lambda}\end{cases} & -\frac{1}{t_i}\lambda=\begin{cases}\frac{1}{t_i}\lambda, & for\ \frac{1}{t_i}\neq\frac{1}{\lambda}\\ \pm1, & for\ \frac{1}{t_i}=\pm\frac{1}{\lambda}\end{cases}\\ m(t_i\lambda)=\begin{cases}2m(\lambda), & for\ t_i\neq\frac{1}{\lambda}\\ 4m(\frac{1}{\lambda}), & for\ t_i=\frac{1}{\lambda}\\ 4m(\lambda), & for\ t_i=-\frac{1}{\lambda}\end{cases} & m(-\frac{1}{t_i}\lambda)=\begin{cases}2m(\lambda), & for\ \frac{1}{t_i}\neq\frac{1}{\lambda}\\ 4m(\lambda), & for\ \frac{1}{t_i}=\frac{1}{\lambda}\\ 4m(\frac{1}{\lambda}), & for\ \frac{1}{t_i}=-\frac{1}{\lambda}\end{cases}\\ t_0\lambda & -\frac{1}{t_0}\lambda\\ m(\lambda) & m(\lambda)\end{pmatrix}.$$

Let $\omega\in spec(\mathtt{Z}_p(\mathtt{G}))$ where

$$\omega=t_i\lambda=\frac{2cos\frac{2\pi i}{p/2}+\sqrt{(2cos\frac{2\pi i}{p/2})^2+4}}{2}\lambda.$$

Then

$$-\frac{1}{\omega}=-\frac{1}{t_i\lambda}=-\frac{2cos\frac{2\pi i}{p/2}-\sqrt{(2cos\frac{2\pi i}{p/2})^2+4}}{2}(\frac{1}{\lambda})\in spec(\mathtt{Z}_p(\mathtt{G})).$$

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Moreover,

$$m(t_i\lambda) = \begin{cases} 2m(\lambda), & for\ t_i \neq \pm\frac{1}{\lambda} \\ 4m(\frac{1}{\lambda}), & for\ t = \frac{1}{\lambda} \\ 4m(\lambda), & for\ t = -\frac{1}{\lambda} \end{cases}$$

and

$$m(-\frac{1}{t_i\lambda}) = \begin{cases} 2m(\frac{1}{\lambda}), & for\ \frac{1}{t} \neq \pm\lambda \\ 4m(\frac{1}{\lambda}), & for\ \frac{1}{t} = -\lambda \\ 4m(\lambda), & for\ \frac{1}{t} = \lambda, \end{cases}$$

since the graph $\mathtt{G}$ strictly satisfies property ($\mathtt{R}$) then there must exist at least one eigenvalue λ in the spectrum of $\mathtt{G}$ such that $m(\lambda) \neq m(\frac{1}{\lambda})$ which implies that at least, for one eigenvalue λ,

$$m(\omega) \neq m(\frac{1}{\omega}).$$

Similarly, for $\nu = t_0\lambda \in spec(\mathtt{Z}_p(\mathtt{G}))$ there exist $-\frac{1}{\nu} = -\frac{1}{t_0\lambda} \in spec(\mathtt{Z}_p(\mathtt{G}))$ as $\frac{1}{\lambda} \in spec(\mathtt{G})$ with different multiplicities. Thus, we conclude that $\mathtt{Z}_p(\mathtt{G})$ satisfies strictly satisfies property ($-\mathtt{R}$). □

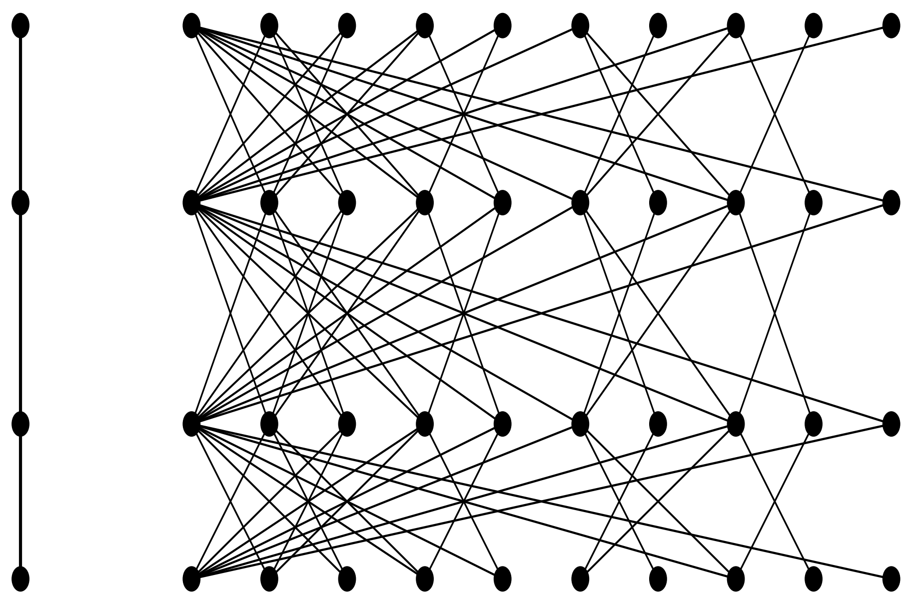

FIGURE 3. $\mathtt{P}_4$ and $\mathtt{A}_{10}(\mathtt{P}_4)$.

Now we construct a family of graphs which strictly satisfies property ($-\mathtt{R}$) under some conditions.

Definition 2.3. *Consider p copies of a graph $\mathtt{H}$ say $\mathtt{H}_1, \mathtt{H}_2, \ldots, \mathtt{H}_p$. Then $\mathtt{D}_p(\mathtt{H})$, $p \geq 10$ be the p-shadow graph obtained from p copies of $\mathtt{H}$ according to the Definition 1.3. Let*

$$\mathtt{A} = \{\mathtt{A}_p(\mathtt{H}) : p \in \mathbb{N}, p \geq 10\}$$

be the family of non-bipartite graphs which can be obtained from $\mathtt{D}_p(\mathtt{H})$ *by deleting all edges except the joining edges of the following copies*

$$\begin{cases} \mathtt{H}_1 \text{ and } \mathtt{H}_{2i+1},, & \text{for } i=1,2; \\ \mathtt{H}_1 \text{ and } \mathtt{H}_i, & \text{for even values of } i \leq p; \\ \mathtt{H}_i \text{ and } \mathtt{H}_{i+1}, & \text{for even values of } i < p; \\ \mathtt{H}_{2i} \text{ and } \mathtt{H}_{2i+2}, & \text{for odd values of } i < \frac{p}{2}. \end{cases}$$

For example the graph $\mathtt{A}_{10}(\mathtt{H})$ *is shown in Figure 2, obtained from* 10 *copies of* $\mathtt{P}_4$ *and deleting all the edges in the* 10*-shadow graph* $\mathtt{D}_{10}(\mathtt{P}_4)$ *except the joining edges of the following copies*

$$\begin{cases} \mathtt{H}_1 \text{ and } \mathtt{H}_{2i+1},, & \text{for } i=1,2; \\ \mathtt{H}_1 \text{ and } \mathtt{H}_i, & \text{for even values of } i \leq 10; \\ \mathtt{H}_i \text{ and } \mathtt{H}_{i+1}, & \text{for even values of } i < 10; \\ \mathtt{H}_{2i} \text{ and } \mathtt{H}_{2i+2}, & \text{for odd values of } i < 5. \end{cases}$$

The following result tells us that each graph in $\mathtt{A}$ strictly satisfies property $(-\mathtt{R})$ for $p \cong 2 \; mod \; 4$.

Theorem 2.4. *Let* $\mathtt{H}$ *be any graph with property* $(-\mathtt{SR})$ *but not* $(\mathtt{SR})$*. Then the graph* $\mathtt{A}_p(\mathtt{H}) \in \mathtt{A}$ *as defined in Definition 2.3 strictly satisfies property* $(-\mathtt{R})$ *for* $p \cong 2 \; mod \; 4$*.*

Proof. Given that $\mathtt{H}$ be any graph with property $(-\mathtt{SR})$ but not $(\mathtt{SR})$, p be the number of copies of $\mathtt{H}$ such that $p \cong 2 \; mod \; 4$ and λ be any eigenvalue in the spectrum of $\mathtt{H}$. Let

$$\mu = \frac{-1+\sqrt{5}}{2},$$

$$\kappa = \frac{1+\sqrt{2p+21}+\sqrt{2p+6+2\sqrt{2p+21}}}{4},$$

and

$$\delta = \frac{1-\sqrt{2p+21}+\sqrt{2p+6-2\sqrt{2p+21}}}{4}.$$

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Then the spectrum of the graph $\mathtt{A}_p(\mathtt{H})$ can be written as,

$$\left(\begin{array}{cc}
\kappa\lambda = \begin{cases} \kappa\lambda, & \text{for } \lambda \neq \pm\frac{1}{\kappa}; \\ \pm 1, & \text{for } \lambda = \pm\frac{1}{\kappa}. \end{cases} & \frac{1}{\kappa}\lambda = \begin{cases} \frac{1}{\kappa}\lambda, & \text{for } \lambda \neq \pm\kappa; \\ \pm 1, & \text{for } \lambda = \pm\kappa. \end{cases} \\
m(\kappa\lambda) = \begin{cases} m(\lambda), & \text{if } \lambda \neq \pm\frac{1}{\kappa}; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \frac{1}{\kappa}; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\frac{1}{\kappa}. \end{cases} & m(\frac{1}{\kappa}\lambda) = \begin{cases} m(\lambda), & \text{if } \lambda \neq \pm\kappa; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \kappa; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\kappa. \end{cases} \\
\delta\lambda = \begin{cases} \delta\lambda, & \text{for } \lambda \neq \pm\frac{1}{\delta}; \\ \pm 1, & \text{for } \lambda = \pm\frac{1}{\delta}. \end{cases} & \frac{1}{\delta}\lambda = \begin{cases} \frac{1}{\delta}\lambda, & \text{for } \lambda \neq \pm\delta; \\ \pm 1, & \text{for } \lambda = \pm\delta. \end{cases} \\
m(\delta\lambda) = \begin{cases} m(\lambda), & \text{if } \lambda \neq \pm\frac{1}{\delta}; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \frac{1}{\delta}; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\frac{1}{\delta}. \end{cases} & m(\frac{1}{\delta}\lambda) = \begin{cases} m(\lambda), & \text{if } \lambda \neq \pm\delta; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \delta; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\delta. \end{cases} \\
\mu\lambda = \begin{cases} \mu\lambda, & \text{for } \lambda \neq \pm\frac{1}{\mu}; \\ \pm 1, & \text{for } \lambda = \pm\frac{1}{\mu}. \end{cases} & \frac{1}{\mu}\lambda = \begin{cases} \frac{1}{\mu}\lambda, & \text{for } \lambda \neq \pm\mu; \\ \pm 1, & \text{for } \lambda = \pm\mu. \end{cases} \\
m(\mu\lambda) = \begin{cases} (\frac{p-2}{4})m(\lambda), & \text{if } \lambda \neq \pm\frac{1}{\mu}; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \frac{1}{\mu}; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\frac{1}{\mu}. \end{cases} & m(\frac{1}{\mu}\lambda) = \begin{cases} (\frac{p-6}{4})m(\lambda), & \text{if } \lambda \neq \pm\mu; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \mu; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\mu. \end{cases} \\
-\mu\lambda = \begin{cases} -\mu\lambda, & \text{for } \lambda \neq \pm\frac{1}{\mu}; \\ \pm 1, & \text{for } \lambda = \pm\frac{1}{\mu}. \end{cases} & -\frac{1}{\mu}\lambda = \begin{cases} -\frac{1}{\mu}\lambda, & \text{for } \lambda \neq \pm\mu; \\ \pm 1, & \text{for } \lambda = \pm\mu. \end{cases} \\
m(-\mu\lambda) = \begin{cases} (\frac{p-6}{4})m(\lambda), & \text{if } \lambda \neq \pm\frac{1}{\mu}; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = \frac{1}{\mu}; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = -\frac{1}{\mu}. \end{cases} & m(-\frac{1}{\mu}\lambda) = \begin{cases} (\frac{p-2}{4})m(\lambda), & \text{if } \lambda \neq \pm\mu; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = \mu; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = -\mu. \end{cases}
\end{array} \right).$$

Since, the graph $\mathtt{H}$ satisfies property $(-\mathtt{SR})$, therefore for each eigenvalue $\lambda \in spec(\mathtt{H})$ there exist $-\frac{1}{\lambda} \in spec(\mathtt{H})$ such that $m(\lambda) = m(-\frac{1}{\lambda})$. Let $\alpha \in spec(\mathtt{A}_p(\mathtt{H}))$ where

$$\alpha = \kappa\lambda = \frac{1 + \sqrt{2p+21} + \sqrt{2p+6+2\sqrt{2p+21}}}{4}\lambda,$$

then

$$-\frac{1}{\alpha} = -\frac{1}{\kappa\lambda} = \frac{1}{\kappa}(-\frac{1}{\lambda}) \in spec(\mathtt{A}_p(\mathtt{H}))$$

as $\mathtt{H}$ satisfies property $(-\mathtt{SR})$. Moreover, from $spec(\mathtt{A}_p(\mathtt{H}))$ we can see that

$$m(\kappa\lambda) = \begin{cases} m(\lambda), & \text{if } \lambda \neq \pm\frac{1}{\kappa}; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \frac{1}{\kappa}; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\frac{1}{\kappa}. \end{cases}$$

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and

$$m(-\frac{1}{\kappa\lambda}) = \begin{cases} m(\lambda), & \text{if } \frac{1}{\lambda} \neq \pm\kappa; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \frac{1}{\lambda} = -\kappa; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \frac{1}{\lambda} = \kappa. \end{cases}$$

which implies that there exist atleast one eigenvalue $\alpha \in spec(\mathtt{A}_p(\mathtt{H}))$ such that $m(\alpha) \neq m(-\frac{1}{\alpha})$. Now, let $\beta \in spec(\mathtt{A}_p(\mathtt{H}))$ and

$$\beta = \delta\lambda = \frac{1 - \sqrt{2p+21} + \sqrt{2p+6-2\sqrt{2p+21}}}{4},$$

then

$$-\frac{1}{\beta} = -\frac{1}{\delta\lambda} = \frac{1}{\delta}(-\frac{1}{\lambda}) \in spec(\mathtt{A}_p(\mathtt{H}))$$

as $\mathtt{H}$ satisfies property $(-\mathtt{SR})$. Moreover, from $spec(\mathtt{A}_p(\mathtt{H}))$ we can see that

$$m(\delta\lambda) = \begin{cases} m(\lambda), & \text{if } \lambda \neq \pm\frac{1}{\delta}; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \lambda = \frac{1}{\delta}; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \lambda = -\frac{1}{\delta}. \end{cases}$$

and

$$m(-\frac{1}{\delta\lambda}) = \begin{cases} m(\lambda), & \text{if } \frac{1}{\lambda} \neq \pm\delta; \\ (\frac{p-6}{2})m(\lambda), & \text{if } \frac{1}{\lambda} = -\delta; \\ (\frac{p-2}{2})m(\lambda), & \text{if } \frac{1}{\lambda} = \delta. \end{cases}$$

which implies that $m(\delta\lambda) \neq m(-\frac{1}{\delta\lambda})$. Similarly for $\pm\gamma = \pm\mu\lambda \in spec(\mathtt{A}_p(\mathtt{H}))$ there exist $\pm\frac{1}{\gamma} = \pm\frac{1}{\mu\lambda} = \frac{1}{\mu}(\pm\frac{1}{\lambda}) \in spec(\mathtt{A}_p(\mathtt{H}))$ with different multiplicities. Hence, we conclude that $\mathtt{A}_p(\mathtt{H})$ strictly satisfies property $(-\mathtt{R})$. □

Example 2.1. *Consider the graph* $\mathtt{G}$ *which strictly satisfies property* $(\mathtt{R})$, *shown in Figure 4. Then the spec*$(\mathtt{G})$ *is:*

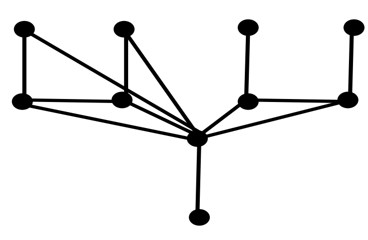

FIGURE 4. Graph $\mathtt{G}$ satisfying property $(\mathtt{R})$

$$\begin{pmatrix} \frac{1\pm\sqrt{5}}{2} & \frac{-1\pm\sqrt{5}}{2} & \frac{1+\sqrt{41}+\sqrt{26+2\sqrt{41}}}{4} & \frac{1+\sqrt{41}-\sqrt{26+2\sqrt{41}}}{4} & \frac{1-\sqrt{41}+\sqrt{26-2\sqrt{41}}}{4} & \frac{1-\sqrt{41}-\sqrt{26-2\sqrt{41}}}{4} \\ 1 & 2 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Case 1: *Consider* 12 *copies of* $\mathtt{G}$*, join them according to the Definition 2.1. Then the spectrum of resulting graph* $\mathtt{Z}_{12}(\mathtt{G})$ *is:*

$$\begin{pmatrix} \pm 8.22799615 & \pm 5.51449865 & \pm -5.45337325 & \pm 3.9062796 & \pm 3.65491413 & \pm 2.61803399 \\ 1 & 2 & 1 & 3 & 2 & 6 \\ \pm 2.10635105 & \pm 1.49206604 & \pm 1.41170096 & \pm 1.39605297 & \pm 1.06877466 & \pm 1 \\ 2 & 3 & 1 & 2 & 1 & 12 \\ \pm 0.93565093 & \pm 0.7163052 & \pm 0.70836532 & \pm 0.67021162 & \pm 0.47475467 & \pm 0.38196601 \\ 1 & 2 & 1 & 3 & 2 & 6 \\ \pm 0.27360424 & \pm 0.25599806 & \pm 0.18337274 & \pm 0.18134015 & \pm 0.12153627 & \\ 2 & 3 & 1 & 2 & 1 & \end{pmatrix}$$

Hence the graph $\mathtt{Z}_{12}(\mathtt{G})$ *satisfies property* $(-\mathtt{SR})$*.*

Case 2: *Now, consider* 6 *copies of* $\mathtt{G}$*, join them according to the Definition 2.1. Then the spectrum of resulting graph* $\mathtt{Z}_{6}(\mathtt{G})$ *is:*

$$\begin{pmatrix} -5.51449865 & -5.45337325 & -3.9062796 & -2.61803399 & -1.49206604 & -1.41170096 \\ 2 & 1 & 2 & 2 & 1 & 1 \\ -1.39605297 & -1.06877466 & -1 & -0.67021162 & -0.47475467 & -0.38196601 \\ 2 & 1 & 8 & 1 & 2 & 2 \\ -0.27360424 & -0.25599806 & -0.12153627 & 0.18134015 & 0.18337274 & 0.25599806 \\ 2 & 2 & 1 & 2 & 1 & 1 \\ 0.38196601 & 0.67021162 & 0.70836532 & 0.7163052 & 0.93565093 & +1 \\ 4 & 2 & 1 & 2 & 1 & 4 \\ 1.49206604 & 2.10635105 & 2.61803399 & 3.65491413 & 3.9062796 & 8.22799615 \\ 2 & 2 & 4 & 2 & 1 & 1 \end{pmatrix}$$

Hence the graph $\mathtt{Z}_{6}(\mathtt{G})$ *strictly satisfies property* $(-\mathtt{R})$*.*

Example 2.2. *Consider the graph* $\mathtt{H}$*, shown in Figure 5.*

$$spec(\mathtt{H}) = \begin{pmatrix} -2.09529399 & -0.73764031 & -0.41421356 & 0.47726 & 1.35567429 & 2.41421356 \\ 2 & 2 & 1 & 2 & 2 & 1 \end{pmatrix},$$

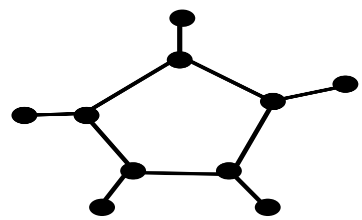

FIGURE 5. Graph H satisfying property $(-\mathtt{SR})$

here the graph $\mathtt{H}$ *satisfies property* $(-\mathtt{SR})$ *but not* $(\mathtt{SR})$. *Now, consider* 14 *copies of* $\mathtt{H}$, *join them according to the Definition 2.3. Then the spectrum of resulting graph* $\mathtt{A}_{14}(\mathtt{H})$ *is:*

$$\begin{pmatrix} -7.81974 & -6.32049 & -3.90627 & -3.5492 & -3.39025 & -2.75291 & -0.25599 \\ 2 & 1 & 3 & 2 & 4 & 2 & 3 \\ -2.19352 & -1.54586 & -1.49206 & -1.29496 & -1.24948 & -1.19352 & -0.45588 \\ 6 & 1 & 2 & 6 & 2 & 4 & 6 \\ -0.92214 & -0.83785 & -0.77222 & -0.67021 & -0.56143 & -0.51782 & -0.29496 \\ 1 & 4 & 6 & 2 & 2 & 2 & 4 \\ -0.19765 & -0.18229 & -0.11098 & 1.08442 & 1.19352 & 1.29496 & 1.49206 \\ 2 & 2 & 1 & 1 & 6 & 4 & 3 \\ 0.12788 & 0.15821 & 0.25599 & 0.281753 & 0.29496 & 0.36325 & 1.78115 \\ 2 & 1 & 2 & 2 & 6 & 2 & 2 \\ 0.45588 & 0.64688 & 0.67021 & 0.77222 & 0.80033 & 0.83785 & 1.93116 \\ 4 & 1 & 3 & 4 & 2 & 6 & 2 \\ 2.19352 & 3.39025 & 3.90627 & 5.05944 & 5.48555 & 9.00996 & \\ 4 & 6 & 2 & 2 & 2 & 1 & \end{pmatrix}$$

Hence the graph $\mathtt{A}_{14}(\mathtt{H})$ *strictly satisfies property* $(-\mathtt{R})$.

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