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COLUMNS OF FIXED HEIGHT IN BARGRAPHS

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Abstract. We obtain the generating function for the number of columns of fixed height r in a bargraph (classified according to semi-perimeter). As initial case for two distinct methods we first find the generating function for columns of height 1. Then using a first-return-to-level-1 decomposition, we obtain the rational function version of the continued fraction generating function which allows us to derive separate recursions for its numerator and denominator. This then allows us to get the asymptotic average number of columns for each r. We also obtain an equivalent generating function by exploiting a sequential decomposition for bargraphs in terms of columns of height r.

1. Introduction

By definition, a bargraph is a non-intersecting lattice path in $\mathbb{N}_0^2$ with three allowable steps: up $(0,1)$, down $(0,-1)$ and horizontal $(1,0)$. An up step may not immediately follow a down step nor vice versa. A bargraph starts at the origin with an up step and terminates immediately upon return to the x-axis. In the generating function for bargraphs (according to semi-perimeter), x tracks the number of horizontal (right) steps and y tracks the number of up steps.

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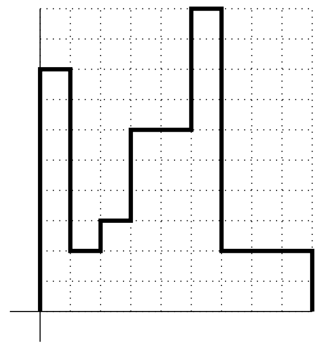

FIGURE 1. A bargraph with 16 up steps (y^{16}) and nine horizontal steps (x^9).

The basic enumeration of bargraphs according to semi-perimeter was a problem defined and solved in the seminal papers [18, 19]. These papers show the influence of this work in a broader setting.

As may already be apparent from the definition of a bargraph and its illustration in Figure 1, the definition provides no direct information about the height of the columns of the bargraphs which are tracked by the generating functions for semi-perimeter (see Equation (1.1) below). Equation (1.1) is simply a consequence obtained by expressing the first-return-to-level-one decomposition illustrated in Figure 2 in algebraic terms. Only two parameters are tracked in this equation. These are x for horizontal semi-perimeter and y for vertical semi-perimeter. The elegant simplicity of this decomposition obscures the fact that it is based implicitly on the height of the various columns of the lattice paths which it describes. Part 1 of this decomposition yields the expression xy in the consequent generating function and the y reflects not only the semi-perimeter but also the height of such columns. Parts 3, 4 and 5 represent the different ways in which this decomposition captures raising a given bargraph by one level. Hence the column heights are increased by the raising operation by one and implicitly the heights of the columns before raising is used in the consequent generating function obtained.

In summary, this decomposition has implicitly used the heights of each column as a starting point to obtain the generating function which tracks the semi-perimeter. The "inverse" of this process is the problem dealt with here. Namely: given a bargraph of fixed semi-perimeter n, how do we obtain the heights of the various columns?

The lacuna of not having the heights of the columns being specified in the generating function given in Equation (1.1) gained in significance when the definition of a bargraph was extended by Mansour et al into a graphical representation in the first quadrant of $\mathbb{N}_0^2$ of any combinatorial structure which can be represented as a sequence of such columns. For examples of the latter, i.e., bargraphs of permutations, see [5]; for bargraphs of Catalan words, see [8]; for bargraphs of compositions, see [14, 15, 16] and for bargraphs of set partitions, see [15, 16]. Lastly, for a general survey on the extended notion of a bargraph, see [13].

Our attempt to remove this lacuna is the fundamental motivation for the current paper. Hence we will enumerate the number of columns of a fixed height in a (semi-perimeter defined) bargraph.

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This is equivalent to counting the number of single horizontal steps at some fixed level (say, r). For example, in Figure 1 there is one column of height $r = 8$ (the first column) and four columns of height $r = 2$ (columns 2, 7, 8, 9).

For previous examples of the methods employed in (site-perimeter defined) bargraph statistics, see [1, 2, 3, 4]. The earliest papers on such bargraphs were in a Physics setting [18, 19] and the first combinatorial paper was [6]. A predecessor of the latter was unpublished, see [11]. Papers straddling the domains of Physics and Mathematics are [7, 12]. More recently, in [9], a bijection with cornerless Motzkin paths was used to enumerate many bargraph statistics.

The generating function that counts all bargraphs is given by

$$B(x,y) = \frac{1-x-y-xy-\sqrt{(1-x-y-xy)^2-4x^2y}}{2x}, \tag{1.1}$$

where x counts the number of horizontal steps and y counts the number of vertical up steps (see, for example, [19]).

The asymptotics of the coefficient of x^n in $B(x,x)$ have been considered in [6], and in order to compute it, the dominant singularity ρ is the positive root of $1-4x+2x^2+x^4=0$. By singularity analysis (for example, see [10]) we have

$$[x^n]B(x,x) \sim \frac{1}{2}\sqrt{\frac{1-\rho-\rho^3}{\pi\rho n^3}}\rho^{-n} \tag{1.2}$$

with

$$\rho = \frac{1}{3}\left(-1-\frac{2^{8/3}}{(13+3\sqrt{3})^{1/3}}+2^{1/3}(13+3\sqrt{33})^{1/3}\right) \approx 0.295598\cdots. \tag{1.3}$$

These asymptotics are used predominantly in the main result in this paper, which is

Theorem 1.1. *The average number of columns of height r in bargraphs of semi-perimeter n is asymptotic (as $n \to \infty$) to*

$$\frac{2(1-\rho)^2}{1+\rho^2} + \frac{4(1-\rho)\rho}{1+\rho^2}\, r. \tag{1.4}$$

The structure of the paper is as follows: In Section 2 as a starting case for the arguments in the rest of the paper, we find the generating function counting the number of columns of height 1. In Section 3 we use the decomposition (colloquially known as the wasp-waist, see [6]) to create a continued fraction form of our generating function. In Section 4 we solve this continued fraction recurrence by considering separate recursions for its numerator and denominator. The result is used to compute the asymptotic average in Section 5. Finally, in Section 6 we give a theoretically interesting derivation of an equivalent generating function by using a sequential decomposition for bargraphs in terms of the columns of height r.

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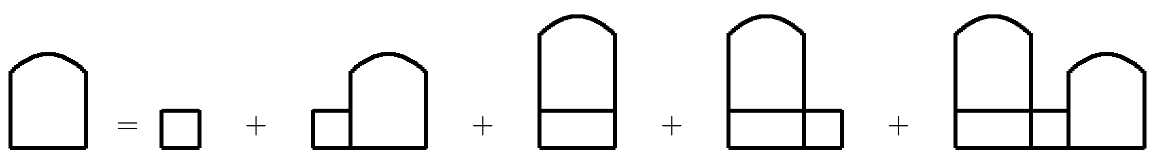
2. Number of columns of height one

We begin by enumerating all columns of height one in bargraphs of semi-perimeter n. The method decomposes a bargraph according to the first time the perimeter of the bargraph returns to level one. This was called the wasp-waist decomposition in [6].

A symbolic representation of the method is reproduced in Figure 2. The five different cases of a (non-empty) bargraph are explained as follows. Case 1 is a single part of height one. Case 2 is a single part of height one to which a further bargraph is attached. The third case is a raised bargraph which means that there is no part of height one. Case 4 is similar to Case 3, but has a final part of height one attached to the raised portion. Finally, Case 5 is the same as Case 4 but with a further bargraph appended to the end. Note that all the cases excluding the third have an explicit occurrence of the first return to level one.



1 2 3 4 5

FIGURE 2. Wasp-waist decomposition of bargraphs

Now define $f_1(x, y, w)$ to be the bargraph generating function which tracks the number of columns of height one with the variable w. Variables y and x track the up and right steps respectively. Using the five possible bargraph decompositions in Figure 2, we set up a recursive equation for f_1 as follows.

$$f_1 = xyw + xwf_1 + yB(1 + xw + \frac{xw}{y} f_1) \tag{2.1}$$

where $B = B(x, y)$ is the generating function of all bargraphs as given in (1.1), which does not depend on the variable w since the raised bargraph does not have any parts of height one.

The first two terms of Equation (2.1) represent the first two cases of Figure 2, each of which starts with a part of height one (marked by w). Note that in the second term, the first up step is absorbed into the bargraph f_1 that follows the single part, hence there is no explicit y factor required. For the remaining three terms, the yB represents the raised bargraph which is either followed by nothing, or by a part of height one, or by a part of height one and a further bargraph which may have parts of height one.

This functional equation (2.1) can be solved by making f_1 the subject of the formula, which yields the following theorem.

Theorem 2.1. *The bargraph generating function which tracks the number of columns of height one with the variable w is*

$$\begin{aligned}(2.2)\qquad & f_1(x,y,w)\\ &= \frac{wxy(1-y+x-xy)+y(1-y-x-xy)-y(1+wx)\sqrt{(y-1)\left(x^2(y-1)+y+2x(y+1)-1\right)}}{2x-wx(1-xy+x-y)+wx\sqrt{(y-1)\left(x^2(y-1)+y+2x(y+1)-1\right)}}.\end{aligned}$$

The variables y and x track the up and right steps respectively. Alternatively, in terms of semi-perimeter (i.e., putting $y = x$), we have

$$f_1(x,x,w) = \frac{1-x^2-wx^3-\sqrt{1-4x+2x^2+x^4}-x\left(2+w\left(\sqrt{1-4x+2x^2+x^4}-1\right)\right)}{2+w\left(x^2+\sqrt{1-4x+2x^2+x^4}-1\right)}. \tag{2.3}$$

Note that Deutsch and Elizalde in [9] find an equivalent generating function to Equation (2.2) using a bijection with Motzkin paths.

To count the total number of columns of height one, we differentiate with respect to w and obtain

$$\frac{\partial}{\partial w}f_1(x,y,w)\Big|_{w=1} = \frac{y\left(x(y-1)+y-1+\sqrt{(y-1)\left(-1+x^2(y-1)+y+2x(y+1)\right)}\right)^2}{x\left(x(y-1)+y+1+\sqrt{(y-1)\left(-1+x^2(y-1)+y+2x(y+1)\right)}\right)^2}$$

or (in terms of semi-perimeter only)

$$df_1(x) := \frac{\partial}{\partial w}f_1(x,x,w)\Big|_{w=1} = \frac{\left(x^2+\sqrt{1-4x+2x^2+x^4}-1\right)^2}{\left(x^2+\sqrt{1-4x+2x^2+x^4}+1\right)^2}. \tag{2.4}$$

As $x \to \rho$ (as given by Equation (1.3)),

$$df_1(x) \sim \frac{4\left(x^2-1\right)\sqrt{Z}}{\left(x^2+1\right)^3} + \frac{x^4-2x^2+1}{\left(x^2+1\right)^2}. \tag{2.5}$$

Note that $f \sim g$ means that $\frac{f}{g} \to 1$ as $x \to \rho$. By singularity analysis,

$$[x^n]df_1(x) \sim \frac{2\left(\frac{1}{n}\right)^{3/2}\left(1-\rho^2\right)\sqrt{-4\rho^3-4\rho+4}}{\sqrt{\pi}\left(\rho^2+1\right)^3}\rho^{\frac{1}{2}-n}.$$

Now dividing by the asymptotic number of bargraphs whose semi-perimeter is n, we find that the average number of columns of height 1 is (asymptotically)

$$\frac{8\rho\left(1-\rho^2\right)}{\left(\rho^2+1\right)^3} \approx 1.67857. \tag{2.6}$$

Here the symbol $\approx$ means "approximately equal to".

3. Number of columns of height r

The recursion given in (2.1) can be extended naturally to the general case. The generating function $f_r = f_r(x, y, w)$ which counts columns of height r in bargraphs according to semi-perimeter satisfies the recursion

$$f_r = xy + xf_r + yf_{r-1}(1 + x + \frac{x}{y}f_r). \tag{3.1}$$

The five cases in the wasp-waist method (see Figure 2) lead to the five terms in Equation (3.1). For example the last term $yf_{r-1}\frac{x}{y}f_r = f_{r-1}xf_r$ represents an up step; a raised bargraph in which we count columns of height $r-1$ (which become columns of height r after being raised); a horizontal step; and a bargraph without its first up step in which we count the columns of height r. Solving for f_r leads to a fractional recursive equation

$$f_r = \frac{xy + y(1-x)f_{r-1}}{1 - x - xf_{r-1}}. \tag{3.2}$$

We will take advantage of this rational form in Section 4 but for the moment we show that the function f_r can be written in a continued fraction form.

First rewrite the function with only one appearance of f_{r-1} on the right:

$$f_r = -y - \frac{y}{x} + \frac{y}{x(1 - x - xf_{r-1})}.$$

This can then be nested until you reach f_1, see below for the case when $r = 5$.

$$\begin{aligned} &f_5(x, y, w) \\ &= -y - \frac{y}{x} + \frac{y}{x\left(1 - x + yx + y - \cfrac{y}{1 - x + yx + y - \cfrac{y}{1 - x + yx + y - \cfrac{y}{1 - x - xf_1}}}\right)}. \end{aligned} \tag{3.3}$$

If we consider only semi-perimeter x (i.e., the case where $y = x$), this simplifies further as can be seen later in (4.13).

4. Simultaneous recursions

We treat the denominator and numerator recurrences separately (a technique also used in [1]). Let

$$f_r = f_r(x, y, w) := \frac{p(r)}{q(r)} \tag{4.1}$$

be the generating function for bargraphs where the number of columns of height r is tracked by w, and x and y mark the right steps and up steps as usual.

4.1. **Denominator recurrence.** We start with the recurrence (3.2), i.e.,

$$f_r = \frac{p(r)}{q(r)} = \frac{y}{x\left(-\frac{xp(r-1)}{q(r-1)} - x + 1\right)} - \frac{y}{x} - y \tag{4.2}$$

$$= -\frac{y((x+1)p(r-1) + xq(r-1))}{xp(r-1) + (x-1)q(r-1)}. \tag{4.3}$$

This yields simultaneous recursions:

$$p(r) = y((1+x)p(r-1) + xq(r-1)) \tag{4.4}$$

and

$$q(r) = -xp(r-1) + (1-x)q(r-1). \tag{4.5}$$

By substituting

$$p(r-1) = \frac{-q(r) + (1-x)q(r-1)}{x}$$

into the first recursion (4.4), we obtain

$$\frac{-q(r+1) + (1-x)q(r)}{x} = y\left(\frac{(1+x)(-q(r) + (1-x)q(r-1))}{x} + xq(r-1)\right)$$

which gives the second order recursion for the denominator as

$$yq(r-1) - (1 + x(y-1) + y)q(r) + q(1+r) = 0,$$

with initial conditions which for now we will call $q(1) = q_1$ and $q(0) = q_0$. For ease of expression we define $X := -4y + (-1 + x - y - xy)^2$ and then the solution, in terms of X is

$$q(r) = \frac{q_0\left(\left(1+\sqrt{X}+x(y-1)+y\right)\left(1-\sqrt{X}+x(y-1)+y\right)^r - \left(1-\sqrt{X}+x(y-1)+y\right)\left(1+\sqrt{X}+x(y-1)+y\right)^r\right)}{2^{r+1}\sqrt{X}}$$

$$+ \frac{q_1\left(\left(1+\sqrt{X}+x(y-1)+y\right)^r - \left(1-\sqrt{X}+x(y-1)+y\right)^r\right)}{2^r\sqrt{X}}. \tag{4.6}$$

For the initial conditions, $q_1 = q(1)$ is the denominator of f_1 and hence

$$q_1 = x\left(w\left(x(y-1) + \sqrt{X} + y - 1\right) + 2\right)$$

and by setting $q(2)$ as the denominator of f_2, and solving simultaneously with the $q(1)$ equation above,

$$q_0 = \frac{4xy\left(w\left(x(y-1) + \sqrt{X} + y - 1\right) + x(-y) + x - \sqrt{X} - y + 3\right)}{x^2(y-1)^2 + 2x\left(y^2-1\right) - X + (y+1)^2}.$$

We now set $y = x$ (so that x marks semi-perimeter in the bargraph), which yields

$$q_1 = x\left(w\left(x^2 + \sqrt{x^4 + 2x^2 - 4x + 1} - 1\right) + 2\right)$$

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and

$$q_0 = x\left(w\left(x^2+\sqrt{x^4+2x^2-4x+1}-1\right)-x^2-\sqrt{x^4+2x^2-4x+1}+3\right)$$

and thus in total, by setting $Z := X|_{y=x} = (1+x^2)^2-4x$, $\alpha := x^2+\sqrt{Z}+1$ and $\beta := x^2-\sqrt{Z}+1$, we simplify Equation (4.6) to

$$q(r) = \frac{x\left(\alpha^r\left(-\beta^2-(\beta-2)w\left(2x^2-\beta\right)+2\beta\left(x^2-1\right)+4\right)+(w-1)\beta^r\left(\beta-2x^2\right)^2\right)}{2^{r+1}\sqrt{Z}}. \tag{4.7}$$

4.2. **Numerator recurrence.** Again we use the simultaneous recursions in the previous section and substitute

$$\frac{p(r)-y((1+x)p(r-1))}{xy} = q(r-1)$$

into (4.5) to obtain

$$yp(r-1)-(1+x(y-1)+y)p(r)+p(1+r)=0.$$

We let the initial conditions be $p(1)=p_1$ and $p(2)=p_2$ for the time being and recall that $y=x$ and $Z=(1+x^2)^2-4x$. The solution to the recursion in terms of semi-perimeter is

$$\begin{aligned} p(r) =& \frac{p_1\left(\left(Z+2x+\sqrt{Z}\left(x^2+1\right)\right)\left(1+x^2-\sqrt{Z}\right)^r-\left(Z+2x-\sqrt{Z}\left(x^2+1\right)\right)\left(1+x^2+\sqrt{Z}\right)^r\right)}{2^{r+1}x\sqrt{Z}} \\ &- \frac{p_2\left(\left(x^2+1+\sqrt{Z}\right)\left(1+x^2-\sqrt{Z}\right)^r-\left(x^2+1-\sqrt{Z}\right)\left(1+x^2+\sqrt{Z}\right)^r\right)}{2^{r+1}x\sqrt{Z}}. \end{aligned} \tag{4.8}$$

For the initial conditions, since $p(1)=p_1$ is the numerator of f_1, we have

$$p_1 = -x\left(wx^3+x\left(w\left(\sqrt{x^4+2x^2-4x+1}-1\right)+2\right)+x^2+\sqrt{x^4+2x^2-4x+1}-1\right)$$

and, similarly,

$$\begin{aligned} p_2 = -x^2\Bigg(& wx^4+x^2\left(w\left(\sqrt{x^4+2x^2-4x+1}-1\right)+3\right)+x^3 \\ &+\left(\sqrt{x^4+2x^2-4x+1}-1\right)x+\sqrt{x^4+2x^2-4x+1}-1\Bigg). \end{aligned}$$

Altogether, with α and β as defined above,

$$\begin{aligned} p(r) =& \frac{2x^3\left(2(w-1)(\beta-1)\beta^r+\alpha^r((w-1)\beta-2)\right)-\beta\left(\alpha^r\beta+(\beta-2)\beta^r\right)}{\sqrt{Z}2^{r+1}} \\ &+ \frac{x\left(\beta^r\left(2(w-1)\beta-(w-1)\beta^2-4\right)-\alpha^r\beta((w-1)\beta-4)\right)-4(w-1)x^5\beta^r}{\sqrt{Z}2^{r+1}} \\ &+ \frac{4x^4\left(w\left(\beta^r-\alpha^r\right)-\beta^r\right)+2x^2\left(\alpha^r(\beta+w\beta-2)-(w-2)\beta^{r+1}\right)}{\sqrt{Z}2^{r+1}}. \end{aligned} \tag{4.9}$$

4.3. **Solution for f_r.** Since we have a solution for the numerator and denominator of $f_r = \frac{p(r)}{q(r)}$ we use the same substitutions for Z, α and β, as well as the formulae in (4.9) and (4.7) to write

$$\begin{aligned} f_r =& \frac{x\left(\alpha\beta^r - \alpha^r\beta\right)\left(2x^3 + 2wx^4 - \beta - x\beta + x^2(4 - w\beta)\right)}{2x(2 + w(\alpha - 2))\left(\alpha^r - \beta^r\right) + x\left(2 - 2x^2 + w(\alpha - 2) + \beta\right)\left(\alpha\beta^r - \alpha^r\beta\right)} \\ &- \frac{\left(2wx^3 + \alpha + x(2 - w\beta) - 2\right)\left(2x^4\beta^r - \alpha^r\beta - (\beta - 2)\beta^r + 2x\left(\alpha^r - \beta^r\right) - x^2\left(\alpha^r\beta + (\beta - 4)\beta^r\right)\right)}{2x(2 + w(\alpha - 2))\left(\alpha^r - \beta^r\right) + x\left(2 - 2x^2 + w(\alpha - 2) + \beta\right)\left(\alpha\beta^r - \alpha^r\beta\right)} \end{aligned} \tag{4.10}$$

where w tracks the number of columns of height r for a fixed r and x tracks the semi-perimeter of the bargraph. In order to enumerate the total number of columns of height r, we differentiate with respect to w and then set $w = 1$ to obtain

$$df_r(x) := \frac{\partial}{\partial w} f_r\Big|_{w=1} = \frac{\left(1 + x^2 - \sqrt{Z}\right)^r \left(1 - x^2 - \sqrt{Z}\right)^2}{4x\left(1 + x^2 + \sqrt{Z}\right)^r}. \tag{4.11}$$

The first few terms of the series expansion for the case where $r = 5$ are given by

$$x^6 + 10x^7 + 57x^8 + 254x^9 + 1006x^{10} + 3752x^{11} + 13519x^{12} + O\left(x^{13}\right). \tag{4.12}$$

Setting $y = x$ in the continued fraction representation (3.3), we obtain

$$f_5 = -x - 1 + \cfrac{1}{1 + x^2 - \cfrac{x}{1 + x^2 - \cfrac{x}{1 + x^2 - \cfrac{x}{1 - x - xf_1}}}}. \tag{4.13}$$

We note that if we differentiate Equation (4.13) with respect to w and set $w = 1$, the series expansion matches (4.12), as expected.

For the convenience of the reader, using Equation (4.10), we list the first few generating functions for small values of r in Table 1.

Name	Function
f_1	$-\frac{wx^3+x\left(w\left(\sqrt{Z}-1\right)+2\right)+x^2+\sqrt{Z}-1}{w\left(x^2+\sqrt{Z}-1\right)+2}$
f_2	$-\frac{x\left(x^2\left(w\left(\sqrt{Z}-1\right)+3\right)+wx^4+x^3+x\left(\sqrt{Z}-1\right)+\sqrt{Z}-1\right)}{w(x^2-x+1)\left(x^2+\sqrt{Z}-1\right)+x^3+2x^2+x\left(\sqrt{Z}-3\right)+2}$
f_3	$-\frac{x^2\left(w\left(x^3+x-1\right)\left(x^2+\sqrt{Z}-1\right)+x^2\sqrt{Z}+x^4+3x^3+x\sqrt{Z}+x+\sqrt{Z}-3\right)}{w(x^4-x^3+2x^2-2x+1)\left(x^2+\sqrt{Z}-1\right)+x^3\left(\sqrt{Z}-2\right)+x^5+2x^4+4x^2+x\left(\sqrt{Z}-5\right)+2}$
f_4	$-\frac{x^2\left(w\left(x^5+2x^3-2x^2+x-1\right)\left(x^2+\sqrt{Z}-1\right)+x^4\left(\sqrt{Z}+1\right)+x^3\left(\sqrt{Z}+3\right)+2x^2\left(\sqrt{Z}-3\right)+x^6+3x^5+2x-2\right)}{w(x^6-x^5+3x^4-4x^3+4x^2-3x+1)\left(x^2+\sqrt{Z}-1\right)+x^5\left(\sqrt{Z}-1\right)+x^3\left(2\sqrt{Z}-9\right)-x^2\left(\sqrt{Z}-9\right)+x^7+2x^6+5x^4+x\left(\sqrt{Z}-7\right)+2}$

TABLE 1. Generating functions for f_1 to f_4

As per the analysis in Equation (4.11), we list the derivatives of these functions in Table 2.

Name	Derivative
f_1	$\frac{2\left(x^2\sqrt{Z}+x^4-2x-\sqrt{Z}+1\right)}{\left(x^2+\sqrt{Z}+1\right)^2}$
f_2	$\frac{x\left(x^2+\sqrt{Z}-1\right)^2}{\left(x^2\left(\sqrt{Z}+2\right)+x^4-2x+\sqrt{Z}+1\right)^2}$
f_3	$\frac{x^2\left(x^2+\sqrt{Z}-1\right)^2}{\left(x^4\left(\sqrt{Z}+3\right)+x^2\left(2\sqrt{Z}+3\right)+x^6-3x^3-x\left(\sqrt{Z}+3\right)+\sqrt{Z}+1\right)^2}$
f_4	$\frac{x^3\left(x^2+\sqrt{Z}-1\right)^2}{\left(x^6\left(\sqrt{Z}+4\right)+3x^4\left(\sqrt{Z}+2\right)-2x^3\left(\sqrt{Z}+4\right)+3x^2\left(\sqrt{Z}+2\right)+x^8-4x^5-2x\left(\sqrt{Z}+2\right)+\sqrt{Z}+1\right)^2}$

TABLE 2. Generating functions for df_1 to df_4

Using the same approach that was used in Section 2 for the asymptotics of f_1, we compute (see Table 3) the average number of columns of height r for $r = 1, 2, \ldots, 5$ as $n \to \infty$.

r	Average in terms of ρ	Numerical approximation
1	$\frac{8\rho\left(1-\rho^2\right)}{\left(\rho^2+1\right)^3}$	1.67857
2	$\frac{8\rho^2\left(1-\rho^2\right)\left(\rho^2-\rho+1\right)}{\left(\rho^4+2\rho^2-2\rho+1\right)^3}$	2.44453
3	$\frac{8\rho^3\left(1-\rho^2\right)\left(\rho^4-\rho^3+2\rho^2-2\rho+1\right)}{\left(\rho^6+3\rho^4-3\rho^3+3\rho^2-3\rho+1\right)^3}$	3.21048
4	$\frac{2\rho\left(47\rho^3-88\rho^2+49\rho-8\right)}{8\rho^3+6\rho^2-16\rho+4}$	3.97643
5	$\frac{2\rho\left(172\rho^3-233\rho^2+64\rho-3\right)}{15\rho^3+24\rho^2-49\rho+12}$	4.74238

TABLE 3. Average number of columns of height r for $r = 1, 2, \ldots, 5$

5. Generalising the average formula

Now we consider what happens to the average number of columns of height r as a function of r as $n \to \infty$.

First, consider Equation (2.4) which we rewrite here for convenience

$$df_1(x) = \frac{2\left(1-2x+x^4-\sqrt{1-4x+2x^2+x^4}+x^2\sqrt{1-4x+2x^2+x^4}\right)}{\left(1+x^2+\sqrt{1-4x+2x^2+x^4}\right)^2}.$$

We expand this about ρ, the positive root of $Z = 1-4x+2x^2+x^4$ and define

$$p_a(1,x) := \frac{2\left(1-2x+x^4\right)}{\left(1+x^2\right)^2} \tag{5.1}$$

and

$$p_b(1,x) := \frac{2\left(-1+x^2\right)}{\left(1+x^2\right)^2}, \tag{5.2}$$

in order to rewrite the expression asymptotically as $x \to \rho$ as

$$df_1(x) \sim p_a(1,x) + p_b(1,x)\sqrt{Z}. \tag{5.3}$$

For other r values, as $x \to \rho$

$$df_r(x) \sim p_a(r,x) + p_b(r,x)\sqrt{Z}, \tag{5.4}$$

and we show below that $p_a(r,x)$ and $p_b(r,x)$ are rational functions. First we present an argument to establish a recursion. We have

$$df_{r+1}(x) = p_a(r+1,x) + p_b(r+1,x)\sqrt{Z} \tag{5.5}$$

$$= \frac{x df_r(x)}{(1-x-xB(x,x))^2} \tag{5.6}$$

$$= \frac{4x df_r(x)}{\left(1+x^2+\sqrt{Z}\right)^2} \tag{5.7}$$

$$\sim \left(x p_a(r,x) + x p_b(r,x)\sqrt{Z}\right)\left(\frac{4}{(1+x^2)^2} - \frac{8\sqrt{Z}}{(1+x^2)^3}\right) \tag{5.8}$$

since as $x \to \rho$

$$\frac{1}{(1-x-xB(x,x))^2} = \frac{4}{\left(1+x^2+\sqrt{Z}\right)^2} \tag{5.9}$$

$$\sim \frac{4}{(1+x^2)^2} - \frac{8\sqrt{Z}}{(1+x^2)^3}. \tag{5.10}$$

Now expand out the right-hand side to obtain

$$\frac{4x p_a(r,x)}{(1+x^2)^2} - \frac{8xX p_b(r,x)}{(1+x^2)^3} + \sqrt{Z}\left(-\frac{8x p_a(r,x)}{(1+x^2)^3} + \frac{4x p_b(r,x)}{(1+x^2)^2}\right) \tag{5.11}$$

so

$$p_a(r+1,x) + p_b(r+1,x)\sqrt{Z} = \frac{4x p_a(r,x)}{(1+x^2)^2} + \sqrt{Z}\left(-\frac{8x p_a(r,x)}{(1+x^2)^3} + \frac{4x p_b(r,x)}{(1+x^2)^2}\right). \tag{5.12}$$

Now equate the terms with and without $\sqrt{Z}$, to obtain

$$p_a(r,x) = \frac{4x p_a(r-1,x)}{(1+x^2)^2} \tag{5.13}$$

and

$$p_b(r,x) = -\frac{8x p_a(r-1,x)}{(1+x^2)^3} + \frac{4x p_b(r-1,x)}{(1+x^2)^2}. \tag{5.14}$$

It is now clear that by induction on r, $p_a(r,x)$ and $p_b(r,x)$ are indeed rational. Solving recursions (5.13) and (5.14) simultaneously, we obtain

$$p_b(r,x) = \frac{2^{2r-1}x^{r-1}\left(-1+x^4-2r\left(1-2x+x^4\right)\right)}{(1+x^2)^{2r+1}}, \tag{5.15}$$

which is the relevant term in $df_r(x)$ for asymptotics. Note that $\frac{4\rho}{(1+\rho^2)^2}=1$. Thus the asymptotics for the total number of columns of height r are

$$p_b(r,\rho)[x^n]\sqrt{Z}=\frac{2^{2r-1}\rho^{r-1}\left(-1+\rho^4-2r\left(1-2\rho+\rho^4\right)\right)}{\left(1+\rho^2\right)^{2r+1}}[x^n]\sqrt{Z}, \tag{5.16}$$

and to get the average we have

$$\frac{p_b(r,\rho)[x^n]\sqrt{Z}}{[x^n]B(x,x)}=\frac{\frac{2^{2r-1}\rho^{r-1}\left(-1+\rho^4-2r\left(1-2\rho+\rho^4\right)\right)}{\left(1+\rho^2\right)^{2r+1}}[x^n]\sqrt{Z}}{[x^n]\sqrt{Z}/(-2\rho)} \tag{5.17}$$

$$=\frac{-2\rho 2^{2r-1}\rho^{r-1}\left(-1+\rho^4-2r\left(1-2\rho+\rho^4\right)\right)}{\left(1+\rho^2\right)^{2r+1}} \tag{5.18}$$

$$=\frac{-(4\rho)^r\left(-1+\rho^4-2r\left(1-2\rho+\rho^4\right)\right)}{\left(1+\rho^2\right)^{2r}\left(1+\rho^2\right)} \tag{5.19}$$

$$=\frac{-\left(-1+\rho^4-2r\left(1-2\rho+\rho^4\right)\right)}{1+\rho^2}. \tag{5.20}$$

Since ρ is a root of $Z=1-4x+2x^2+x^4$, we have that $1-4\rho+2\rho^2+\rho^4=0$, so the average number of columns of height r in bargraphs is (as $x\to\rho$)

$$-\frac{2(\rho-1)(\rho(2r-1)+1)}{\rho^2+1}, \tag{5.21}$$

which leads us to Theorem 1.1. By replacing ρ in the formula (Equation (1.4))

$$\frac{2(1-\rho)^2}{1+\rho^2}+\frac{4(1-\rho)\rho}{1+\rho^2}\,r$$

of this theorem by its approximate numerical value (see Equation (1.3)), we obtain an average of

$$0.912622+0.765952r \tag{5.22}$$

which matches the numerical values in Table 3.

6. Sequence construction method

The wasp-waist method used in the previous sections is a first return decomposition. Here, we present an alternative approach to count the columns of height r in bargraphs using semi-perimeter which is based instead on a sequence deconstruction where the elements of the deconstruction are columns of height less than r (∇) and columns of height r or more ($\mathcal{R}$).

To avoid double-counting the number of up steps (represented by y in the generating function) we do not include the first or last vertical column for each (∇) (i.e., the height of the first/last column). We will thus assume that these are omitted from the ∇ component unless indicated by a subscript. The component ∇ has either zero, one or two subscripts according to which vertical portions (start or end) are included. For example, ∇_f represents a small component where we do include the first vertical portion and do not include the last vertical portion of the bargraph and ∇_{fl} represents a small component where we do include both the first and last vertical portion of the bargraph. Let $\mathcal{B}$ represent all bargraphs. Then, the decomposition works as follows:

$$\mathcal{B} = \nabla_{fl} + (\epsilon + \nabla_f)(\mathcal{R}\nabla)^*(\mathcal{R} + \mathcal{R}\nabla_l) \tag{6.1}$$

where the Kleene star is the usual notation for a possibly empty sequence. We can represent the right-hand side of Equation (6.1) more visually as in Figure 3.

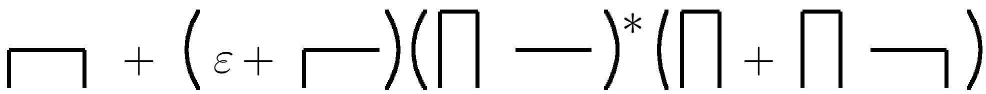

FIGURE 3. 'Staple' decomposition of bargraphs according to columns of height r

Hereafter, the elements in Figure 3 with columns of height at most $r-1$ are referred to as small staples and the elements consisting of columns of height r or more are referred to as heavy-duty staples.

Thus in Figure 3, either we have no column of height r (small staple), or we have a bargraph which has at least one column of height r or higher. For the latter, either we start with a column of height r or more (in which case the ε is chosen from the first bracket) or we don't (in which case we start with a non-empty bargraph of height at most $r-1$ in which we do not consider the final descent). This is then followed by a possibly-empty sequence where we do not consider the up or down steps of the portions which have parts of size less than r and finally we end with a portion which may or may not be followed by some columns of height less than r, which is shown by the last bracket.

Now we translate to generating functions. We count all columns of height 1 (denoted by variable w) in a bargraph according to the generating function given in Equation (2.2), namely

$$f_1(x,y,w) = \frac{wxy(1-y+x-xy)+y(1-y-x-xy)-y(1+wx)\sqrt{(y-1)\left(x^2(y-1)+y+2x(y+1)-1\right)}}{2x-wx(1-xy+x-y)+wx\sqrt{(y-1)\left(x^2(y-1)+y+2x(y+1)-1\right)}}.$$

In our case, however, we wish to enumerate columns of height r in the heavy-duty staples in the deconstruction. This is achieved by constructing a bijection between general bargraphs where columns of height 1 are tracked by w and bargraphs whose columns are all of height r or more, where columns of height r are tracked by w. The forward direction of the bijection then raises every bargraph by $r-1$ thus converting columns of height 1 to columns of height r and this is equivalent to multiplying the generating function $f_1(x,y,w)$ by y^{r-1}.

Next we need the generating function for bargraphs with columns of height at most r. We adopt the convention of $g_{\leq r}$ to indicate that we are considering bargraphs with columns at most r to distinguish it from f_r (from Equation (3.1)), which is the generating function for bargraphs where columns of height r are enumerated. The former was obtained by Blecher et al in [1] and is given by

$$g_{\leq r}(x,y) := \frac{xy\left(a_1^r - a_2^r\right)}{x^2y\left(a_2^{r-1} - a_1^{r-1}\right) + (1-x)\left((a_1 - y - xy)a_1^{r-1} - (a_2 - y - xy)a_2^{r-1}\right)}, \tag{6.2}$$

where

$$a_1 = \frac{1}{2}\left(1 - x + y + xy + \sqrt{-4y + (1 - x + y + xy)^2}\right), \tag{6.3}$$

and

$$a_2 = \frac{1}{2}\left(1 - x + y + xy - \sqrt{-4y + (1 - x + y + xy)^2}\right). \tag{6.4}$$

However, as can be seen in Figure 3, we require the height of the first column in certain situations. By symmetry, the generating function for the height of the first column is the same as that for the last column (across all bargraphs). Using a recursion obtained from the wasp-waist decomposition, the generating function for bargraphs of height at most r where the height of the first column is marked by u is

$$h_{\leq r}(x,y,u) = xyu + xug_{\leq r}(x,y) + yuh_{\leq (r-1)}(x,y,u)\left(1 + x + \frac{xg_{\leq r}(x,y)}{y}\right), \tag{6.5}$$

where

$$h_{\leq 1}(x,y,u) = \frac{xyu}{1-x}. \tag{6.6}$$

Now, to facilitate solving the recursion, we rewrite Equation (6.5) using

$$A(r) := xyu + xug_{\leq r}(x,y)$$

and

$$C(r) := yu\left(1 + x + \frac{xg_{\leq r}(x,y)}{y}\right).$$

Thus Equation (6.5) - and the consequent iterations - become:

$$h_{\leq r}(x,y,u) = A(r) + h_{\leq (r-1)}(x,y,u)C(r), \tag{6.7}$$

$$h_{\leq (r-1)}(x,y,u)C(r) = C(r)\Big(A(r-1) + h_{\leq (r-2)}(x,y,u)C(r-1)\Big), \tag{6.8}$$

$$h_{\leq (r-2)}(x,y,u)C(r-1)C(r) = C(r-1)C(r)\Big(A(r-2) + h_{\leq (r-3)}(x,y,u)C(r-2)\Big) \tag{6.9}$$

$$h_{\leq (r-3)}(x,y,u)C(r-2)C(r-1)C(r) = C(r-2)C(r-1)C(r)\Big(A(r-3) + h_{\leq (r-4)}(x,y,u)C(r-3)\Big) \tag{6.10}$$

$$\dots$$

$$h_{\leq r}(x,y,u) = \prod_{j=3}^{r} C(j)\Big(A(2) + h_{\leq 1}(x,y,u)C(2)\Big). \tag{6.11}$$

Adding these together gives

$$h_{\leq r}(x,y,u) = \sum_{i=2}^{r} A(r-i+2)\prod_{j=3}^{i} C(r-j+3) + \frac{uxy}{1-x}\prod_{j=2}^{r} C(j). \tag{6.12}$$

The last function we need before combining everything is again obtained using the wasp-waist decomposition and is for the small staple (∇), tracking the first and last verticals. The recursion is

$$\begin{aligned} f_{\leq r}(x,y,v,u) &= vxyu + xuf_{\leq r}(x,y,v,1) + vyuf_{\leq (r-1)}(x,y,v,u) + vxyuf_{\leq (r-1)}(x,y,1,u) \\ &\quad + xuf_{\leq r}(x,y,v,1)f_{\leq (r-1)}(x,y,1,u) \\ &= vxyu + xuh_{\leq r}(x,y,v) + vyuf_{\leq (r-1)}(x,y,v,u) + vxyuh_{\leq (r-1)}(x,y,u) \\ &\quad + xuh_{\leq r}(x,y,v)h_{\leq (r-1)}(x,y,u), \end{aligned} \tag{6.13}$$

since tracking the first vertical of a small staple is equivalent to tracking the last vertical, over all of the staples by left-to-right symmetry. Now we solve the recursion. The first step is given in Equation (6.13). Let

$$D(r-1) := vxyu + xuh_{\leq r}(x,y,v) + vxyuh_{\leq (r-1)}(x,y,u) + xuh_{\leq r}(x,y,v)h_{\leq (r-1)}(x,y,u). \tag{6.14}$$

The first few steps of the recursion are

$$f_{\leq r}(x,y,v,u) = vyuf_{\leq (r-1)}(x,y,v,u) + D(r-1), \tag{6.15}$$

$$vyuf_{\leq (r-1)}(x,y,v,u) = (vyu)^2 f_{\leq (r-2)}(x,y,v,u) + vyuD(r-2), \tag{6.16}$$

$$(vyu)^2 f_{\leq (r-2)}(x,y,v,u) = (vyu)^3 f_{\leq (r-3)}(x,y,v,u) + (vyu)^2 D(r-3), \tag{6.17}$$

and so forth until

$$(vyu)^{r-2} f_{\leq 2}(x,y,v,u) = (vyu)^{r-1} f_{\leq 1}(x,y,v,u) + (vyu)^{r-2} D(1). \tag{6.18}$$

So summing these gives:

$$f_{\le r}(x,y,v,u) = (vyu)^{r-1} f_{\le 1}(x,y,v,u) + \sum_{i=1}^{r-1} (vyu)^{i-1} D(r-i), \tag{6.19}$$

or, alternatively

$$\begin{aligned} f_{\le r}(x,y,v,u) =& (vyu)^{r-1} \frac{vxyu}{1-x} \\ &+ \sum_{i=1}^{r-1} (vyu)^{i-1} \Big(vxyu + xuh_{\le(r-i+1)}(x,y,v) + vxyuh_{\le(r-i)}(x,y,u) \\ &\qquad + xuh_{\le(r-i+1)}(x,y,v)h_{\le(r-i)}(x,y,u) \Big). \end{aligned} \tag{6.20}$$

We now insert these components into Equation (6.1) or the staple decomposition in Figure 3 to create a generating function ($B_r(x,y,w)$) for all bargraphs ($\mathcal{B}$). Note that we replace u and v with $1/y$ to account for excess up steps.

(6.21)

$$B_r(x,y,w) := g_{<r}(x,y) + \frac{\big(1 + h_{<r}(x,y,1/y)\big)\big(y^{r-1} f_1(x,y,w) + y^{r-1} f_1(x,y,w) h_{<r}(x,y,1/y)\big)}{1 - y^{r-1} f_1(x,y,w) f_{<r}(x,y,1/y,1/y)}$$

$$= g_{<r}(x,y) + \frac{y^{r-1} f_1(x,y,w)\big(1 + h_{<r}(x,y,1/y)\big)^2}{1 - y^{r-1} f_1(x,y,w) f_{<r}(x,y,1/y,1/y)}. \tag{6.22}$$

As an example, a series expansion for $r = 4$ is

$$\begin{aligned} B_4(x,x,w) = {} & x^2 + 2x^3 + 5x^4 + (w+12)x^5 + \left(w^2 + 6w + 28\right) x^6 + \left(w^3 + 6w^2 + 23w + 67\right) x^7 \\ & + \left(w^4 + 6w^3 + 25w^2 + 76w + 167\right) x^8 + O\left(x^9\right), \end{aligned} \tag{6.23}$$

and we note that this is the same as $f_4(x,x,w)$ (see Equation (4.10)) as expected.

7. Conclusion

Dealing with the notion of a bargraph as parametrised by semi-perimeter has always presented the challenge that the height of the various columns is not part of this parameterisation but is nevertheless an important component of the geometrical structure of these bargraphs. In this paper we have attempted to address this issue.

Employing two different methods, we have shown that one can count the number of columns of height r in bargraphs of fixed semi-perimeter. More specifically, we used the wasp-waist decomposition directly to establish a generating function, starting with the case where $r = 1$. The general rth case of this generating function was expressed as a rational function recursion or a continued fraction recursion, and both versions are useful as detailed in the text. To solve the recursion, we split it into two recursions using the numerator and denominator of the rational function. This solution was used to calculate the asymptotic average for the number of columns of height r, for a fixed r. Finally, we used a different ('staple') decomposition (based initially on the wasp-waist decomposition) to obtain an equivalent generating function enumerating columns of fixed height.

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