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THE GENEROUS ROMAN DOMINATION NUMBER

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ABSTRACT. Let $G = (V, E)$ be a simple graph and $f : V \to \{0, 1, 2, 3\}$ be a function. A vertex u with $f(u) = 0$ is called an undefended vertex with respect to f if it is not adjacent to a vertex v with $f(v) \geq 2$. We call the function f a generous Roman dominating function (GRDF) if for every vertex with $f(u) = 0$ there exists at least a vertex v with $f(v) \geq 2$ adjacent to u such that the function $f' : V \to \{0, 1, 2, 3\}$, defined by $f'(u) = \alpha$, $f'(v) = f(v) - \alpha$ where $\alpha = 1$ or 2, and $f'(w) = f(w)$ if $w \in V - \{u, v\}$ has no undefended vertex. The weight of a generous Roman dominating function f is the value $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a generous Roman dominating function on a graph G is called the generous Roman domination number of G, denoted by $\gamma_{gR}(G)$. In this paper, we initiate the study of generous Roman domination and show its relationships. Also, we give the exact values for paths and cycles. Moreover, we present an upper bound on the generous Roman domination number, and we characterize cubic graphs G of order n with $\gamma_{gR}(G) = n-1$, and a Nordhaus-Gaddum type inequality for the parameter is also given. Finally, we study the complexity of this parameter.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. The size $|E|$ of G is denoted by $m = m(G)$. For every vertex $v \in V$, the open neighborhood $N_G(v)$ of v is the set $\{u \in V(G) : uv \in E(G)\}$ and the closed neighborhood $N_G[v]$ of v is the set $N_G(v) \cup \{v\}$. For any $S \subseteq V$, we denote by $G[S]$ the subgraph of G induced

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by S. The degree of a vertex $v \in V$ is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A vertex of degree one is called a leaf or a pendant vertex and its neighbor is called a support vertex. If $\Delta(G) = 0$ then G is called empty graph, and if $\delta(G) = \Delta(G) = r \geq 1$ then G is called r-regular graph. A $(n-1)$-regular graph is called complete graph of order n, denoted by K_n, and $K_{p,q}(p \geq q)$ denotes the complete bipartite graph with two partite sets of sizes p, q. A star denoted by $K_{1,n-1}$ is a graph in which $n-1$ vertices have degree 1 and a single vertex of degree $n-1$. We write $P_n = v_1v_2 \cdots v_n$ for the path of order $n \geq 2$, and $C_n = v_1v_2 \cdots v_nv_1 = P_n + \{v_nv_1\}$ for the cycle of order $n \geq 3$. The subpath of P_n from v_i to v_j is denoted by v_iPv_j. A cubic graph (also called a 3-regular graph) is a graph in which every vertex has degree 3, for example, Peterson graph P and prism graph $C_3\square K_2$ are cubic.

If G and H are two vertex-disjoint graphs, the union of G and H is the graph $G\bigoplus H$ whose vertex-set is $V(G) \cup V(H)$ and edge-set is $E(G) \cup E(H)$. For an integer $p \geq 2$, the union of p copies of a graph G is denoted by pG.

A set $S \subseteq V$ is called a *dominating set* if every vertex in $V - S$ is adjacent to some vertex in S. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. Domination and its variations in graphs are now well studied. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [11, 12].

Cockayne et al. in [8] defined the *Roman domination*. A *Roman dominating function* (*RDF*) is a function $f = (V_0, V_1, V_2)$ such that for every vertex $v \in V_0$ there exists a vertex $u \in V_2$ which is adjacent to v. The *Roman domination number,* denoted by $\gamma_R(G)$, is the minimum weight among all RDFs on G. Since 2004, a plus hundred papers have been published on this topic, where several new variations were introduced modifying the way in which the vertices are dominated or to adding an extra property to the Roman domination property itself [1, 2, 7, 9, 14, 15].

Henning et al. [13] defined the *weak Roman dominating.* Let $f : V \to \{0, 1, 2\}$ be a function. A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex u with $f(u) \in \{1, 2\}$. A function $f : V \to \{0, 1, 2\}$ is a *weak Roman dominating function* (*WRDF*) on a graph G if every vertex u with $f(u) = 0$ has a neighbor u with $f(v) \in \{1, 2\}$ and the function $f' : V \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. The weight $w(f)$ of f is $w(f) = f(V) = \sum_{u \in V} f(u)$. The *weak Roman domination number*, denoted by $\gamma_r(G)$, is the minimum weight of a *WRDF* in G, that is, $\gamma_r(G) = min\{w(f) : f$ is a *WRDF* in $G\}$ (see that [5, 6, 8]).

Beeler et al. [4] have defined *double Roman dominating.* A *double Roman dominating function* (*DRDF*) on a graph G is a function $f = (V_0, V_1, V_2, V_3)$ such that the following conditions are hold:

(a) if $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3,

(b) if $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$.

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The double Roman domination number, denoted by $\gamma_{dR}(G)$, is the minimum weight among all DRDFs on G, see also [3, 5, 6]. We consider double Roman dominating functions with no vertex assigned 1. We use the notation $f = (V_0, V_2, V_3)$ for a *DRDF*.

In this paper, we introduce and study a variant of Roman dominating functions, namely, generous Roman dominating functions. Let $f = (V_0, V_1, V_2, V_3)$ be a function. A vertex u with $u \in V_0$ is called an *undefended vertex* with respect to f if it is not adjacent to a vertex v with $v \in V_2 \cup V_3$. We call the function f a *generous Roman dominating function* (*GRDF*) if for every vertex with $u \in V_0$ there exists at least a vertex v with $v \in V_2 \cup V_3$ adjacent to u such that the function f', defined by $f'(u) = \alpha$, $f'(v) = f(v) - \alpha$ where $\alpha = 1$ or 2, and $f'(w) = f(w)$ if $w \in V - \{u, v\}$ has no *undefended* vertex (that is we will apply the following rule: If u and v are adjacent with $u \in V_0$ and $v \in V_2 \cup V_3$ then v sends 1 legion or 2 legions from v to u depending on the situation such that the function $f' = (V_0, V_1, V_2, V_3)$ has no undefended vertex). The weight of a generous Roman dominating function f is the value $w(f) = f(V) = \sum_{u \in V} f(u) = V_1 + 2V_2 + 3V_3$. The minimum weight of a *GRDF* on a graph G is called the *generous Roman domination number* of G, denoted by $\gamma_{gR}(G)$. A *GRDF* of G with weight $\gamma_{gR}(G)$ is called a $\gamma_{gR}(G)$-function (or γ_{gR}-function of G).

The strategy adopted in the *GRDF* is based on the same style of strategy as the *WRDF*, except that an undefended city is required to be adjacent to a city with at least two legions (in order to save the Roman Empire from a possible double attack). A city is undefended if it has no legions and each adjacent city to it is not secure (i.e. each have at most one legion stationed there). As an undefended city is vulnerable to attack or a double attack, we require that each undefended city be adjacent to a secured city (with at least two legions), in such a way that the movement of a legion or two from the defended city to an undefended city does not lead to the creation undefended cities, which means that this strategy leads to respecting the condition that each city is adjacent to at least one city with at least two legions. So that the movement of legions, does not lead to the creation of a undefended city, and in this way any city can still be defended in the Roman Empire and protected from any second attack.

Such a placement of legions corresponds to a *GRDF* and such a minimum placement of legions corresponds to a minimum *GRDF*. Since the potential exists to save the Emperor from substantial legion maintenance costs while defending the Roman Empire (from a possible double attack), this concept of generous Roman domination is an attractive alternative to the Emperor Constantine to the notion of Roman.

For instance, for the tree T shown in Figure 1, we have three different $\gamma_{gR}(G)$-functions. For this example a vertex of weight 2 can sends 1 or 2 legions to a vertex with weight 0. But a vertex of weight 3 can sends at most 1 legion to a vertex with weight 0. For this example we have,

$$\gamma(T) = 2 < 3 = \gamma_r(T) < \gamma_R(T) = 4 < 5 = \gamma_{gR}(T) < \gamma_{dR}(T) = 6.$$

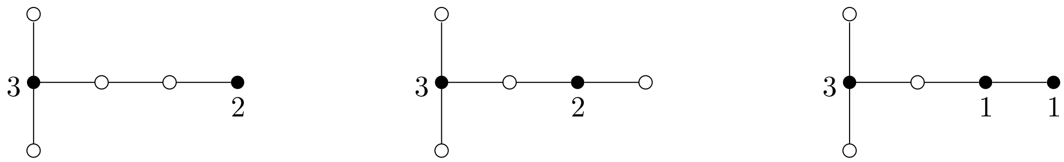

FIGURE 1. Three placements of legions which correspond to three different generous Roman dominating functions on the same tree T.

In the first section, we initiate the study of generous Roman domination and show its relationship to both Roman domination and double Roman domination. In the second section, we give the exact values of the generous Roman domination number for paths and cycles. In the third section, various bounds on $\gamma_{gR}(G)$ of a graph G are presented, we characterize cubic graphs G of order n with $\gamma_{gR}(G) = n-1$, and a Nordhaus-Gaddum type inequality for the parameter is also given. Finally, in the last section, we study the complexity of this parameter, we prove that the decision problem associated with $\gamma_{gR}(G)$ is $\mathcal{NP}$-complete for bipartite graphs and the problem of deciding whether $\gamma_{gR}(G) = n - \Delta + 2$ is co-$\mathcal{NP}$-complete for graphs.

2. Preliminary results

The Roman domination, weak Roman domination and double Roman domination number of a path P_n and a cycle C_n on n vertices is established in [8], [13] and [3] respectively.

Proposition 2.1. [8] *For $n \geq 3$, $\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.*

Proposition 2.2. [13] *For $n \geq 3$, $\gamma_r(P_n) = \gamma_r(C_n) = \left\lceil \frac{3n}{7} \right\rceil$.*

Proposition 2.3. [3] *For $n \geq 1$, $\gamma_{dR}(P_n) = \begin{cases} n & \text{if} \quad n \equiv 0 \pmod 3, \\ n+1 & \text{if} \quad n \equiv 1,2 \pmod 3. \end{cases}$*

Proposition 2.4. [3] *For $n \geq 3$, $\gamma_{dR}(C_n) = \begin{cases} n & \text{if} \quad n \equiv 0,2,3,4 \pmod 6, \\ n+1 & \text{if} \quad n \equiv 1,5 \pmod 6. \end{cases}$*

In [4, 13] the authors gave the following inequalities of the relationships between $\gamma(G)$, $\gamma_R(G)$, $\gamma_r(G)$ and $\gamma_{dR}(G)$.

Proposition 2.5. [4, 13] *For any graph G,*

$$\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq 2\gamma(G) \leq \gamma_{dR}(G) \leq 3\gamma(G).$$

Now, we begin with an inequality chain relating the Roman domination number, the generous Roman domination number and the double Roman domination number.

Observation 2.6. $\gamma_{gR}(K_n) = 2$ *and* $\gamma_{gR}(K_{1,n}) = 3$.

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Proposition 2.7. *For any graph G,*

$$\gamma_{gR}(G) \leq \gamma_{dR}(G),$$

and this bound is sharp for $G = K_{1,n}$.

Note that the inequality given in Proposition 2.7 is strict for complete graphs K_n and equality for stars $K_{1,n}$. We have $\gamma_{gR}(K_n) = 2 < 3 = \gamma_{dR}(K_n)$ and $\gamma_{gR}(K_{1,n}) = 3 = \gamma_{dR}(K_{1,n})$.

Proposition 2.8. *For any graph G,*

$$\gamma_R(G) \leq \gamma_{gR}(G),$$

and this bound is sharp for $G = K_n$.

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a γ_{gR}-function of G. We define a function g on G by $g = (V_0', V_1', V_2') = (V_0, V_1, V_2 \cup V_3)$. Clearly g is a Roman dominating function on G of weight $w(g) = |V_1'| + 2|V_2'| \leq |V_1| + 2|V_2| + 2|V_3| \leq |V_1| + 2|V_2| + 3|V_3| = \gamma_{dr}(G)$. Hence $\gamma_R(G) \leq \gamma_{gR}(G)$. □

By Proposition 2.5, 2.7, and 2.8, we have the following inequalities of the relationships between $\gamma(G)$, $\gamma_r(G)$, $\gamma_R(G)$, $\gamma_{gR}(G)$ and $\gamma_{dR}(G)$.

Corollary 2.9. *For any graph G without isolated vertices,*

$$\gamma(G) \leq \gamma_r(G) \leq \gamma_R(G) \leq \gamma_{gR}(G) \leq \gamma_{dR}(G) \leq 3\gamma(G).$$

Note that if $G = P_5$, then $\gamma(G) = 2$, $\gamma_r(G) = 3$, $\gamma_R(G) = 4$, $\gamma_{gR}(G) = 5$ and $\gamma_{dR}(G) = 6$. Hence there exists a connected graphs G with,

$$\gamma(G) < \gamma_r(G) < \gamma_R(G) < \gamma_{gR}(G) < \gamma_{dR}(G).$$

Also, there are graphs for which equality is obtained for the parameters of Corollary 2.9. For the stars $K_{1,n-1}$ of order $n \geq 1$, we have $\gamma(K_{1,n-1}) = 1$ and $\gamma_{gR}(K_{1,n-1}) = \gamma_{dR}(K_{1,n-1}) = 3$ which gives $\gamma_{gR}(G) = \gamma_{dR}(G) = 3\gamma(G)$. For the complete graphs K_n, $n \geq 2$, we have $\gamma_R(K_n) = \gamma_{gR}(K_n) = 2$.

Proposition 2.10. [8] *For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K}_n$.*

The following corollary follows immediately from previous Corollary 2.9 and Proposition 2.10.

Corollary 2.11. *For any graph G of order n, $\gamma(G) = \gamma_{gR}(G)$ if and only if $G = \overline{K}_n$.*

3. Paths and cycles

In this section we determine the generous Roman domination number of paths and cycles. We start with this useful result for the next step.

Observation 3.1.

(1) *Let P_n be a path of order $n = 1, 2, \ldots, 6$. Then $f(V(P_n)) \geq n$ for every* GRDF *f of P_n. Indeed, $\gamma_{gR}(P_n) = n$ for $n = 1, 2, \ldots, 6$.*

(2) *Let P_7 be a path of order 7. Then $f(V(P_7)) \geq 6$ for every* GRDF *f of P_7. Indeed, $\gamma_{gR}(P_7) = 6$.*

Proof. **1.** For a path P_n of order $n = 1, 2, \ldots, 6$, we consider only the case $n = 6$, with a similar reasoning, we can verify the other cases. Let $P_6 = v_1v_2v_3v_4v_5v_6$ be a path of order 6 and f a *GRDF* of P_6. It is clear that $f(N_G[v_2]) \geq 2$ and $f(N_G[v_5]) \geq 2$. If $f(N_G[v_2]) \geq 3$ and $f(N_G[v_5]) \geq 3$, then $f(V(P_6)) \geq 6$, and since the function f with $f(v_1) = f(v_3) = f(v_5) = 0$ and $f(v_2) = f(v_4) = f(v_6) = 2$, is a GRDF of P_6, and so $\gamma_{gR}(P_6) = 6$. If $f(N_G[v_2]) = 2$, then without loss of the generality we can suppose that $f(v_2) = 2$ and so, $f(v_4) \geq 2$ otherwise f would not be a *GRDF* of P_6. We distinguish two cases.

Case 1. $f(v_4) = 2$. If $f(v_5) = 0$, then $f(v_6) \geq 2$, otherwise f would not be a *GRDF* of P_6. And if $f(v_5) = 1$, then $f(v_6) \geq 1$ otherwise f would not be a *GRDF* of P_6. In all cases we have $f(V(P_6)) \geq 6$.

Case 2. $f(v_4) = 3$. If $f(v_5) = 0$, then $f(v_6) \geq 1$, otherwise f would not be a *GRDF* of P_6. In all cases we obtain $f(V(P_6)) \geq 6$.

By symmetry and with the same argument like above, it is easy to check that if $f(N_G[v_5]) = 2$, then we have $f(V(P_6)) \geq 6$.

2. Let $P_7 = v_1v_2v_3v_4v_5v_6v_7$ be a path of order 7 and f is a *GRDF* of P_7. It is clear that $f(N_G[v_2]) \geq 2$ and $f(N_G[v_6]) \geq 2$. If $f(N_G[v_2]) \geq 3$ and $f(N_G[v_6]) \geq 3$. Then $f(V(P_7)) \geq 6$, and since the function f with $f(v_1) = f(v_3) = f(v_5) = f(v_7) = 0$ and $f(v_2) = f(v_4) = f(v_6) = 2$, is a *GRDF* of P_7, and so $\gamma_{gR}(P_7) = 6$. If $f(N_G[v_2]) = 2$, then without loss of the generality we can suppose that $f(v_2) = 2$ and so, $f(v_4) \geq 2$ otherwise f would not be a *GRDF* of P_6. We distinguish two cases relatively of $f(v_4)$.

Case 1. $f(v_4) = 2$. If $f(v_5) = 0$, then $f(v_6) \geq 2$, otherwise f would not be a *GRDF* of P_7. And if $f(v_5) = 1$, then $f(v_6) + f(v_7) \geq 2$ otherwise f would not be a *GRDF* of P_7. In all cases we have $f(V(P_7)) \geq 6$.

Case 2. $f(v_4) = 3$. If $f(v_5) = 0$, then $f(v_6) + f(v_7) \geq 2$, otherwise f would not be a *GRDF* of P_7. In all cases we obtain $f(V(P_7)) \geq 6$.

By symmetry and with the same argument like above, it is easy to check that if $f(N_G[v_6]) = 2$, then we have $f(V(P_7)) \geq 6$. □

Lemma 3.2. *If G is a graph obtained from H and the path $P_7 = v_1v_2v_3v_4v_5v_6v_7$ by joining a vertex v of H to a vertex v_7 of P_7, then $f(V(P_7)) \geq 6$ for any GRDF f of G and $\gamma_{gR}(G) = \gamma_{gR}(H) + 6$.*

Proof. Let G be a graph obtained from H and the path $P_7 = v_1v_2v_3v_4v_5v_6v_7$ by joining a vertex v of G to a vertex v_7 of P_7. Then $\deg_G(v_i) = 2$ for $i = 2, \ldots, 6, 7$ and $\deg_G(v_1) = 1$. Let $f = (V_0, V_1, V_2, V_3)$

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be a γ_{gR}-function of G. If $f(v_7) \geq 1$, then the restriction of f to P_7 is a *GRDF* of P_7. Hence, by Item 2 of Observation 3.1 $f(V(P_7)) \geq 6$. If $f(v_7) = 0$, then the restriction of f to $P_6 = v_1v_2v_3v_4v_5v_6$ is a *GRDF* of P_6. Thus, by Item 1 of Observation 3.1, $f(V(P_6)) \geq 6$.

To show the equality $\gamma_{gR}(G) = \gamma_{gR}(H) + 6$, first any γ_{gR}-function of H can be extended to a *GRDF* of G by assigning the value 2 to v_2, v_4 and v_6 and the value 0 to v_1, v_3, v_5 and v_7. Hence, $\gamma_{gR}(G) \leq \gamma_{gR}(H) + 6$.

On the other hand, suppose $f = (V_0, V_1, V_2, V_3)$ be a γ_{gR}-function of G. If $f(v) = 1$ or $f(v) = 0$ and $f(v_7) = 0$ or 1, then the restriction g of f to H is a *GRDF* of H. Hence, $\gamma_{gR}(H) \leq w(g) = w(f) - f(V(P_7)) \leq w(f) - 6 = \gamma_{gR}(G) - 6$. So, $\gamma_{gR}(G) \geq \gamma_{gR}(H) + 6$. If $f(v) = 0$ and $f(v_7) \geq 2$, then the restriction g of f to H with $g(v) = 1$ is a *GRDF* of H. Hence, $\gamma_{gR}(H) \leq w(g) = w(f) - f(V(P_7)) + 1$. If $f(v_6) \geq 1$, then the restriction of f to $P_6 = v_1v_2v_3v_4v_5v_6$ is a *GRDF* of P_6, so $f(V(P_7)) \geq f(V(P_6)) + 2$ which by Item 1 of Observation 3.1 implies $f(V(P_7)) \geq 8$. If $f(v_6) = 0$, then the restriction of f to $P_5 = v_1v_2v_3v_4v_5$ is a *GRDF* of P_5, so $f(V(P_7)) \geq f(V(P_5)) + 2$ which by Item 1 of Observation 3.1 implies $f(V(P_7)) \geq 7$. In all cases $f(V(P_7)) \geq 7$. Thus, $\gamma_{gR}(H) \leq w(g) = w(f) - f(V(P_7)) + 1 \leq w(f) - 7 + 1 = \gamma_{gR}(G) - 6$. Hence, $\gamma_{gR}(G) \geq \gamma_{gR}(H) + 6$. Consequently, $\gamma_{gR}(G) = \gamma_{gR}(H) + 6$. □

Proposition 3.3. *For $n \geq 1$,*

$$\gamma_{gR}(P_n) = \left\lceil \frac{6n}{7} \right\rceil$$

Proof. We proceed by induction on n. By Observation 3.1, it is straightforward to verify the result for small n, $1 \leq n \leq 7$. Assume that the result holds for all paths of order less than n, where $n \geq 8$. Let $P_n = v_1v_2 \cdots v_n$ be a path of order n. Let $P_{n'} = v_1Pv_{n-7}$ denote the subpath from v_1 to v_{n-7} of order $n-7$ and $v_{n-6}Pv_n$ the subpath from v_{n-6} to v_n of order 7. Applying the inductive hypothesis to $P_{n'}$, $\gamma_{gR}(P_{n-7}) = \left\lceil \frac{6(n-7)}{7} \right\rceil$. By Lemma 3.2, $\gamma_{gR}(P_n) = \gamma_{gR}(P_{n-7}) + 6$, and so, $\gamma_{gR}(P_n) = \left\lceil \frac{6(n-7)}{7} \right\rceil + 6 = \left\lceil \frac{6n}{7} \right\rceil$. The result now follows by mathematical induction. □

Note that $\gamma_{gR}(C_3) = 2$, so for the next we consider the generous Roman domination number of a cycle C_n of order $n \geq 4$. We begin by the simple following observation.

Observation 3.4. *For any spanning subgraph H of a graph G, $\gamma_{gR}(G) \leq \gamma_{gR}(H)$.*

Proposition 3.5. *For $n \geq 4$,*

$$\gamma_{gR}(C_n) = \left\lceil \frac{6n}{7} \right\rceil.$$

Proof. It is clear that $\left\lceil \frac{6n}{7} \right\rceil = \frac{6n+r}{7}$, if $n \equiv r(\text{mod}\, 7)$. It is easy to check that if $4 \leq n \leq 6$ then $\gamma_{gR}(C_n) = \frac{6n+r}{7}$, where $n \equiv r(\text{mod } 7)$. For the next, assume that $n \geq 7$.

By Proposition 3.3 and Observation 3.4 we have

$$\gamma_{gR}(C_n) \leq \gamma_{gR}(C_n - \{uv\}) = \gamma_{gR}(P_n) = \frac{6n+r}{7}, \text{ if } n \equiv r(mod\ 7).$$

To prove the inverse inequality, let f be a γ_{gR}-function of C_n. If there are two consecutive vertices v_i and v_{i+1} for some i such that $f(v_i) = f(v_{i+1}) = 0$ or $f(v_i) = 0$, $f(v_{i+1}) = 1$ or $f(v_i) > 0$, $f(v_{i+1}) > 0$, then the function f, restricted to $C_n - \{v_i v_{i+1}\} = P_n$ is a *GRDF* of P_n and we deduce from Observation 3.4 that $\gamma_{gR}(C_n) \geq \gamma_{gR}(P_n)$ which leads to the desired equality. Hence we may assume that vertices of C_n are assigned alternately the values 0 and k, with $k \in \{2, 3\}$. Clearly, in this case n is even. Assume that there exists an edge uv of C_n such that $f(u) + f(v) = 3$ and without loss of generality assume that $(f(u), f(v)) = (0, 3)$. Let w be the neighbor of v, so, $f(w) = 0$. Define $g : V(C_n) \to \{0, 1, 2, 3\}$ by $g(x) = f(x)$ for $x \in V - \{v\}$ and $g(v) = 2$. Clearly, g is a *GRDF* on C_n of weight $w(f) - 1$, a contradiction. So, for every edge uv of C_n verifies $f(u) + f(v) = 2$ and $\gamma_{gR}(C_n) = 2|V_2| = 2(\frac{n}{2}) = n$. But in this case, we have

$$n = \gamma_{gR}(C_n) \leq \gamma_{gR}(C_n - \{uv\}) = \gamma_{gR}(P_n) = \frac{6n + r}{7} \leq \frac{6n + 6}{7}.$$

This implies that $n \leq 6$ a contradiction with the fact that $n \geq 7$, and the proof is complete. $\square$

If a graph G of order n has a spanning cycle C_n containing every vertex once, then G is called a Hamiltonian graph and C_n a Hamiltonian cycle. Since a Hamiltonian graph contains C_n, the following result can be deduce easily by Observation 3.4 and Proposition 3.5.

Proposition 3.6. *For any Hamiltonian graph G of order $n \geq 4$, $\gamma_{gR}(G) \leq \left\lceil \frac{6n}{7} \right\rceil$.*

4. Bounds on γ_{gR} and Nordhaus-Gaddum inequalities

First we fix the notation to be used in the rest of the paper. Let $G = (V, E)$ be a graph, we denote by X_Δ the set of all vertices of G with maximum degree $\Delta(G) \geq 1$, i.e., $X_{\Delta(G)} = \{x \in V : \deg_G(x) = \Delta(G)\}$.

Proposition 4.1. *Let G be a graph of order $n \geq 1$ and maximum degree $\Delta(G) \geq 1$. Then,*

$$\gamma_{gR}(G) \leq n - \Delta(G) + 2,$$

and this bound is sharp for $G = K_{1,n-1}$, $n \geq 3$.

Proof. Let x be a vertex with $\deg_G(x) = \Delta(G)$. Define $f = \left(N_G(x), \overline{N_G[x]}, \emptyset, \{x\}\right)$. Clearly, f is a *GRDF* of G of weight $w(f) = n - \Delta(G) + 2$, and this leads to upper bound. Since $\gamma_{gR}(K_{1,n-1}) = 3$ for $n \geq 3$, this bound is sharp. $\square$

It is clear that for every graph of order n, $\gamma_{gR}(G) \leq n$, so the natural question is for which graphs we have $\gamma_{gR}(G) = n$.

Theorem 4.2. *Let G be a connected graph of order $n \geq 1$. Then $\gamma_{gR}(G) = n$ if and only if G is a path P_n with $1 \leq n \leq 6$ or a cycle C_n with $4 \leq n \leq 6$.*

Proof. The sufficiency condition came from Proposition 3.3 and Proposition 3.5. For the necessity condition, let G be a graph of order n with $\gamma_{gR}(G) = n$. By Proposition 4.1, $n = \gamma_{gR}(G) \leq n - \Delta(G) + 2$, so $\Delta(G) \leq 2$. Hence, G is a path or a cycle. Now by Proposition 3.3 and Proposition 3.5, if G is a path P_n of order $n \geq 1$, then $n = \left\lceil \frac{6n}{7} \right\rceil = \frac{6n+r}{7}$, if $n \equiv r (\bmod 7)$ and so, $n = r$ with $1 \leq n \leq 6$. If G is a cycle C_n of order $n \geq 4$, then $n = \left\lceil \frac{6n}{7} \right\rceil = \frac{6n+r}{7}$, if $n \equiv r (\bmod 7)$ and so, $n = r$ with $4 \leq n \leq 6$. Consequently, G is a path P_n with $1 \leq n \leq 6$ or a cycle C_n with $4 \leq n \leq 6$. □

Observation 4.3. *Let G be a graph of maximum degree $\Delta(G) \geq 1$, and let $x \in X_{\Delta(G)}$. If $\gamma_{gR}(G) = n - \Delta(G) + 2$, then $G[N_G(x)]$ is not a complete graph, $G\left[\overline{N_G[x]}\right]$ does no contain C_3, every vertex in $N_G(x)$ has at most three neighbors in $\overline{N_G[x]}$, and every vertex in $\overline{N_G[x]}$, has at most two neighbors in $\overline{N_G[x]}$.*

We note that the necessary conditions given in Observation 4.3 are not sufficient for an arbitrary graph and in particular for a graph H, Peterson graph P and prism graph $H' = C_3 \square K_2$ shown in Figure 2. Assign a value 2 to every white vertex and 0 to every black vertex. Clearly that $\gamma_{gR}(G) < n - \Delta(G) + 2$. However, for the graph $G = K_{3,3}$ we have $\gamma_{gR}(G) = n - \Delta(G) + 2 = 5$.

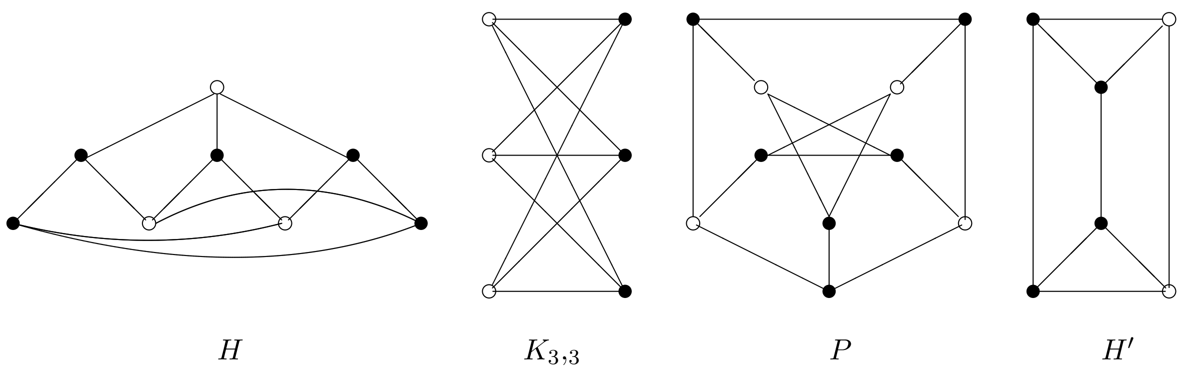

FIGURE 2. $\gamma_{gR}(G) < n - \Delta(G) + 2$ and $\gamma_{gR}(K_{3,3}) = n - \Delta(K_{3,3}) + 2$.

For the next results, we denote for every vertex $x \in X_{\Delta(G)}$, $Y = N_G(x)$ and $Z = \overline{N_G[x]} = V - N_G[x]$. Let $A, B \subseteq V$, we denote by $m' = m'(A, B)$ the number of edges joining A to B. We begin this section by given a new upper bound for $\gamma_{gR}(G)$.

Theorem 4.4. *Let G be a cubic graph of order n. Then,*

$$\gamma_{gR}(G) = n - 1 \textit{ if and only if } G = K_{3,3}.$$

Proof. For the sufficiency condition, it is easy to verify that $G = K_{3,3}$ satisfies $\gamma_{gR}(G) = n - 1$. For the necessity condition, let x be a vertex of G and $Y = \{y_1, y_2, y_3\}$. It follows from Observation 4.3 and the fact that G is 3-regular, $Z = N_G(Y) - \{x\}$, $|N_G(z) \cap Y| \geq 1$, for every vertex $z \in Z$, and $1 \leq |Z| \leq 6$. We consider six cases relatively to the values of $|Z|$.

Case 1. $|Z| = 1$. Let $Z = \{z\}$, since G is 3-regular, $N_G(z) = Y$. This implies that there is one vertex, say y_i in Y with $\deg_G(y_i) = 2$, a contradiction with the fact that G is a cubic graph.

Case 2. $|Z| = 2$. Let $Z = \{z_1, z_2\}$. If $z_1z_2 \notin E(G)$, then $N_G(z_1) = N_G(z_2) = Y$. Therefore, $G = K_{3,3}$. If $z_1z_2 \in E(G)$, then $1 \leq |N_G(z_1) \cap N_G(z_2)| \leq 2$. If $|N_G(z_1) \cap N_G(z_2)| = 2$, without loss of generality, let $N_G(z_1) \cap N_G(z_2) = \{y_1, y_2\}$, then $\deg_G(y_3) = 1$, a contradiction. So, $|N_G(z_1) \cap N_G(z_2)| = 1$, and without loss of generality, let $N_G(z_1) \cap N_G(z_2) = \{y_2\}$, then $y_1y_3 \in E(G)$. Therefore, $G = C_3 \Box K_2$. Define the function $f : V \to \{0, 1, 2, 3\}$ on G by: $f(y_1) = f(z_1) = 2$ and $f(v) = 0$ otherwise. Clearly f is a *GRDF* of G. Thus, $\gamma_{gR}(G) \leq w(f) = 4 < 5 = n - 1 = \gamma_{gR}(G)$, a contradiction.

Case 3. $|Z| = 3$. In this case we have $m'(Y, Z) \in \{2, 4, 6\}$ and $m'(Z, Y) \in \{3, 5, 7, 9\}$. So, in this case no cubic graph exists.

Case 4. $|Z| = 4$. Let $Z = \{z_1, z_2, z_3, z_4\}$. Since $m'(Y, Z) \in \{2, 4, 6\}$ and $m'(Z, Y) \geq 4$, we obtain $m'(Y, Z) = 4$ or $m'(Y, Z) = 6$. If $m'(Y, Z) = 4$, without loss of generality, we may assume that $y_1z_1, y_1z_2 \in E(G)$. Thus, $y_2z_3, y_2z_4 \in E(G)$ or $y_2z_3, y_3z_4 \in E(G)$. If $y_2z_3, y_2z_4 \in E(G)$ then $\deg_G(y_3) = 1$, a contradiction. And if $y_2z_3, y_3z_4 \in E(G)$ then $y_2y_3 \in E(G)$. Hence $G\left[\overline{N_G[z_1]}\right]$ contains $C_3 = xy_2y_3x$, a contradiction with Observation 4.3. Now, if $m'(Y, Z) = 6$, then by Observation 4.3 and the fact that G is cubic graph, we may assume, without loss of generality that $y_1z_1, y_1z_2, y_2z_2, y_2z_3, y_3z_3$ and $y_3z_4 \in E(G)$. Now, if $z_2z_3 \in E(G)$, then $z_1z_4 \in E(G)$ and $\deg_G(z_1) = \deg_G(z_4) = 2$, a contradiction. So, z_1z_3, z_1z_4 and $z_2z_4 \in E(G)$. Therefore, $G = H$ (see Figure 2). Define the function $f : V \to \{0, 1, 2, 3\}$ on G by: $f(x) = f(z_2) = f(z_3) = 2$ and $f(v) = 0$ otherwise. Clearly f is a *GRDF* of G. Thus, $\gamma_{gR}(G) \leq w(f) = 5 < 6 = n - 1 = \gamma_{gR}(G)$, a contradiction.

Case 5. $|Z| = 5$. Since $m'(Y, Z) \in \{2, 4, 6\}$ and $m'(Z, Y) \geq 5$, we obtain $m'(Y, Z) = 6$ and in this case, it is easy to check that there is one vertex z_i in Z with $\deg_G(z_i) = 2$, a contradiction.

Case 6. $|Z| = 6$. Clearly that $m'(Y, Z) = 6$. Let $Z = \{z_1, z_2, z_3, z_4, z_5, z_6\}$. Without loss of generality we may assume that $N_G(y_i) \cap Z = \{z_{2i-1}, z_{2i}\}$, $1 \leq i \leq 3$. Since every vertex z in $G[Z]$ has $\deg_G(z) = 2$, we have $G[Z] = C_6$. Now, if there exists $i \in \{1, 2, 3\}$ such that $z_{2i-1}z_{2i} \in E(G)$, then $G\left[\overline{N_G[y_j]}\right]$ contains $C_3 = y_iz_{2i-1}z_{2i}y_i$ where $i \neq j \in \{1, 2, 3\}$, a contradiction with Observation 4.3. Moreover, if there exists $i \in \{1, 2, 3\}$ such that $z_{2i-1}z_{2j-1}, z_{2i-1}z_{2j} \in E(G)$, then $N_G(z_{2i-1}) = \{y_i, z_{2j-1}, z_{2j}\} \subset \overline{N_G[y_k]}$ where $i, j \neq k \in \{1, 2, 3\}$ and $i \neq j$, a contradiction with Observation 4.3. So, without loss of generality we can assume that $z_1z_3, z_1z_5 \in E(G)$. Then $z_2z_4, z_2z_6 \in E(G)$ and this implies that $z_3z_6, z_4z_5 \in E(G)$. Therefore, $G[Z] = C_6 = z_1z_3z_6z_2z_4z_5z_1$ and $G = P$ where P is the Peterson graph. Define the function $f : V \to \{0, 1, 2, 3\}$ on G by: $f(y_1) = f(y_2) = f(z_5) = f(z_6) = 2$ and $f(v) = 0$ otherwise. Clearly f is a *GRDF* of G. Thus, $\gamma_{gR}(G) \leq w(f) = 7 < 8 < n - 1 = \gamma_{gR}(G)$, a contradiction.

Hence, the only graph G which satisfies $\gamma_{gR}(G) = n - 1$ is $G = K_{3,3}$. □

We are now ready to state the result on the Nordhaus-Gaddum type inequalities of a graph.

Theorem 4.5. *If G is a graph of order $n \geq 3$, then*

$$5 \leq \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) \leq n + 5.$$

Moreover,

(1) $\gamma_{gR}(G) + \gamma_{gR}(\overline{G}) = 5$ *if and only if* $G \in \{K_3, 3K_1\}$,

(2) $\gamma_{gR}(G) + \gamma_{gR}(\overline{G}) = n + 5$ *if and only if* $G = C_5$.

Proof. When G has at least three vertices, $\gamma_{gR}(G) \geq 2$, with equality if and only if $G = K_n$. If $G = K_n$ and $n \geq 3$, then $\gamma_{gR}(G) + \gamma_{gR}(\overline{G}) = 2 + n \geq 5$. If $G \neq K_n$, then $\gamma_{gR}(G) + \gamma_{gR}(\overline{G}) \geq 5$. Hence, $\gamma_{gR}(G) + \gamma_{gR}(\overline{G}) = 5$ if and only if $\gamma_{gR}(G) = 2$ and $\gamma_{gR}(\overline{G}) = 3$ or $\gamma_{gR}(G) = 3$ and $\gamma_{gR}(\overline{G}) = 2$, which equivalent to $G = K_3$. Consequently, equality holds if and only if $G \in \{K_3, 3K_1\}$.

For the upper bound, since $\Delta(\overline{G}) + \delta(G) = n - 1$, Proposition 4.1 yields

$$\begin{aligned} \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) &\leq (n - \Delta(G) + 2) + (n - \Delta(\overline{G}) + 2) \\ &= n - (\Delta(G) - \delta(G)) + 5 \leq n + 5. \end{aligned} \tag{4.1}$$

We begin with the necessary condition. Let G be a graph of order $n \geq 3$, and suppose that

$$\gamma_{gR}(G) + \gamma_{gR}(\overline{G}) = n + 5. \tag{4.2}$$

It follows from (4.1 and 4.2), we deduce that $\Delta(G) = \delta(G) = k$. Thus G is a k-regular graph and $\overline{G}$ is also k'-regular graph with $k + k' = n - 1$. We may assume that $k \leq (n-1)/2$, since the argument is symmetric in G and $\overline{G}$, we can deduce from (4.1) that

$$\gamma_{gR}(G) = n - k + 2 \text{ and } \gamma_{gR}(\overline{G}) = k + 3 \tag{4.3}$$

Since $\gamma_{gR}(G) \leq n$, we can deduce from (4.3) that $k \geq 2$. Now, we distinguish between two cases.

Case 1. $k = 2$ or $n-3$. If $k = 2$, Then $G = \bigoplus_{i=1}^{l} C_{n_i}$ with $n_i \geq 3, i = 1, \dots, l$, and $\overline{G}$ is $(n-3)$-regular graph. Then $\gamma_{gR}(\overline{G}) \leq n - \Delta(\overline{G}) + 2 = 5$. First, suppose that G has $l \geq 2$ cycles. So,

$$\gamma_{gR}(G) = \gamma_{gR}(\bigoplus_{i=1}^{l} C_{n_i}) = \sum_{i=1}^{t} \gamma_{gR}(C_{n_i}) \leq \sum_{i=1}^{l} n_i = n.$$

Now, if there exists a copy C_3 in G, then $\gamma_{gR}(G) \leq n - 1$ (since $\gamma_{gR}(C_3) = 2$), and

$$n + 5 = \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) \leq n + 4,$$

a contradiction. So, every cycle C_{n_i} of G has order $n_i \geq 4$. Let C_{n_1} and C_{n_2} be two cycles of G and x a vertex of C_{n_1} and y a vertex of C_{n_2}. Define the function $f: V \to \{0, 1, 2, 3\}$ on $\overline{G}$ by: $f(x) = 2$, $f(y) = 2$ and $f(v) = 0$ otherwise. Clearly f is a *GRDF* on $\overline{G}$ of weight $w(f) = 4$. Indeed, without loss of generality suppose to the contrary that there exists a vertex z of $\overline{C_{n_1}}$ with $f(z) = 0$ and $xz \notin E(\overline{G})$ (otherwise if $xz \in E(\overline{G})$ then x sends 2 legions from x to z) such that the function defined by $f'(z) = 2$, $f'(y) = 0$ and $f'(v) = f(v)$ for every vertex $v \neq z, y$, has an undefended vertex, say t in $V\left(\overline{G}\right)$. Since every vertex in $V\left(\overline{G}\right) - V(\overline{C_{n_1}})$ is defended by x with respect to f', thus the vertex t must be in $V(\overline{C_{n_1}})$ moreover, $tz \notin E(\overline{C_{n_1}})$ and $tx \notin E(\overline{C_{n_1}})$. So, $xztx$ is a cycle C_3 of G, a contradiction. Thus, $\gamma_{gR}(\overline{G}) \leq 4$ and

$$n + 5 = \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) \leq n + 4,$$

again a contradiction. Hence, $l = 1$, and $G = C_n$.

It follows from (4.2) and Proposition 3.5, that

$$n + 5 = \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) \leq \left\lceil \frac{6n}{7} \right\rceil + 5 \leq \frac{6n+6}{7} + 5.$$

This simplifies to $n \leq 6$. And in this case it is easy to check that the only cycle of order $n \leq 6$ satisfies (4.2) is C_5.

For $k = n - 3$, by symmetry and with the same argument like above, we obtain $G = C_5$.

Case 2. $3 \leq k \leq n - 4$. It follows from (4.3) and Observation 4.3 that, every vertex y in Y has at most three neighbors in Z and every vertex z in Z has at least $k - 2$ neighbors in Y and let $m' = m'\left(N_G\left[x\right], \overline{N_G\left[x\right]}\right)$, so

$$(k-2)(n-k-1) \leq m' \leq 3k, \tag{4.4}$$

and since $k \leq (n-1)/2$, we have

$$2k + 1 \leq n \leq k + 1 + \frac{3k}{k-2}. \tag{4.5}$$

On the one hand, since $n \geq 2k + 1$, we have $k \leq \frac{3k}{k-2}$, which leads to $k \leq 5$, and on the other hand we have $kn = \sum_{v \in V} \deg_G(v) = 2m$ which implies that if k is odd, then n must be even. We consider three subcases.

SubCase 2.1. $k = 3$. By (4.5) and the fact n must be even, $n \in \{8, 10, 12\}$. Since $\gamma_{gR}(G) = n - k + 2 = n - 1$, by Theorem 4.4, $G = K_{3,3}$. It is easy to check that $K_{3,3}$ doesn't satisfy (4.2), because $\gamma_{gR}(K_{3,3}) = 5$ and $\gamma_{gR}(\overline{K_{3,3}}) = 4$.

SubCase 2.2. $k = 4$. By (4.5), we have $n \in \{9, 10, 11\}$ which leads to $|Z| \in \{4, 5, 6\}$. By (4.4), we have $m' \in \{8, 9, 10, 11, 12\}$. If $|Z| = 6$, then $m' = 12$, $G\left[Y\right]$ is independent and $G\left[Z\right] = C_6$. Since $\gamma_{gR}(\overline{G}) = n - k' + 2$, we have in $\overline{G}$, $N_{\overline{G}}(x) = Z$ and $\overline{N_{\overline{G}}\left[x\right]} = Y$, thus $G[Y]$ contains C_3, a contradiction with Observation 4.3. If $\left|\overline{N_G\left[x\right]}\right| = 5$, then by Observation 4.3, we obtain $10 \leq m' \leq 12$ and $G\left[Y\right]$ has at least one isolated vertex y, otherwise we have $m' \leq 8$, a contradiction. Define on $\overline{G}$ the function g by: $g\left(x\right) = 3$ and $g\left(y\right) = 3$, where y is an isolated vertex in $G\left[Y\right]$ in G, and $g\left(v\right) = 0$ otherwise. Clearly, $g : V \to \{0, 1, 2, 3\}$ is a *GRDF* on $\overline{G}$ of weight $w\left(g\right) = 6$, which gives $n + 5 = \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) < n + 5$, a contradiction. Finally, if $|Z| = 4$, then by Observation 4.3, we obtain $8 \leq m' \leq 12$.

If $9 \leq m' \leq 12$, then $G\left[Y\right]$ has at least one isolated vertex y, otherwise we have $m' \leq 8$, a contradiction. Define on $\overline{G}$ the function $g : V \to \{0, 1, 2, 3\}$ by: $g\left(x\right) = 3$ and $g\left(y\right) = 3$, where y is an isolated vertex in $G[N_G(x)]$ in G, and $g(v) = 0$ otherwise. Clearly, g is a *GRDF* on $\overline{G}$ of weight $w(g) = 6$, which gives $n + 5 = \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) < n + 5$, a contradiction.

If $m' = 8$, then every vertex y in Y has exactly two neighboors in Z and every vertex z in Z has exactly two neighboors in Y, by Observation 4.3, $G[Y] = 2K_2$ and $G[Z] = C_4$. Without loss of generality, assume that y_1y_2, $y_3y_4 \in E(G)$ and $C_4 = z_1z_2z_3z_4z_1$. If there exists a vertex z_i such that $z_iy_1, z_iy_2 \in E(G)$ (or $z_iy_3, z_iy_4 \in E(G)$, respectively), then $G\left[\overline{N_G[z_i]}\right]$ contains $C_3 = xy_3y_4x$, (or $G\left[\overline{N_G[z_i]}\right]$ contains $C_3 = xy_1y_2x$, respectively) a contradiction with Observation 4.3. So, $|N_G(z_i) \cap \{y_1, y_2\}| = 1$ and $|N_G(z_i) \cap \{y_3, y_4\}| = 1$ for every $i \in \{1, .., 4\}$, and since G is 4-regular graph it is easy to check that there exists a pair of distinct vertices, say z_i, z_j $1 \leq i, j \leq 4$ and $i \neq j$ such that $N_G(z_i) \cap N_G(z_j) \cap N_G(x) = \emptyset$ and $(N_G(z_i) \cup N_G(z_j)) \cap N_G(x) = N_G(x)$ (otherwise if for every two distinct vertices z_i and z_j satisfy $N(z_i) \cap N(z_j) \cap N(x) \neq \emptyset$, then $\deg_G(z_i) = 5$, a contradiction). Define the function $g : V \to \{0, 1, 2, 3\}$ on G by: $g(z_i) = g(z_j) = g(x) = 2$ and $g(v) = 0$ otherwise. Clearly, g is a *GRDF* on $\overline{G}$ of weight $w(g) = 6$, which gives $14 = n + 5 = \gamma_{gR}(G) + \gamma_{gR}(\overline{G}) \leq w(g) + \gamma_{gR}(\overline{G}) \leq 6 + 7 = 13$, a contradiction.

SubCase 2.3. $k = 5$. Then $11 = 2k + 1 \leq n \leq k + 1 + \frac{3k}{k-2} = 11$, so $n = 11$ and since k and n are odds, we deduce that there is no 5-regular graph with 11 vertices.

Consequently, $G = C_5$.

For the sufficiency condition, since $\overline{C_5} = C_5$ and $\gamma_{gR}(C_5) = \lceil \frac{6n}{7} \rceil = 5$, $\gamma_{gR}(C_5) + \gamma_{gR}(\overline{C_5}) = 5 + 5 = n + 5$. And the proof is complete. □

5. Complexity

For the next subsection we show that the decision problem associated with generous Roman dominating functions is $\mathcal{NP}$-complete for bipartite graphs. Consider the following decision problem.

5.1. $\mathcal{NP}$-Completeness of the generous Roman Domination number. .

GENEROUS ROMAN DOMINATING FUNCTION (*GRDF*)

Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question: Does G have a generous Roman dominating function of weight at most k?

We show that this problem is $\mathcal{NP}$-complete by reducing the well-known $\mathcal{NP}$-complete problem(see [10]) , Exact-3-Cover (***X3C***), to ***GRDF***.

EXACT 3-COVER (*X3C*)

Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

Theorem 5.1. *Problem **GRDF** is $\mathcal{NP}$-Complete for bipartite graphs.*

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Proof. ***GRDF*** is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f: V \rightarrow \{0,1,2,3\}$ has weight at most k and is a generous Roman dominating function. Now let us show how to transform any instance of ***X3C*** into an instance G of ***GRDF*** so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $Y = \{Y_1, Y_2, \ldots, Y_t\}$ be an arbitrary instance of ***X3C***, where Y is a collection of 3-element subsets of X.

For each $x_i \in X$, we build a graph H_i obtained from a path $P_3 = \left(x_i^1, x_i^2, x_i^3\right)$. For each $Y_j \in Y$, we build a star $K_{1,4}$ centered at z_j for which one leaf is labeled y_j. Let $Z = \{z_1, z_2, \ldots, z_t\}$. Now to obtain a graph G, we add edges $y_j x_i^1$ if $x_i \in Y_j$. Clearly G is bipartite. Set $k = 3t + 8q$.

Suppose that the instance Y of ***X3C*** has a solution Y'. We construct a generous Roman dominating function f on G of weight k. For every vertex of Y_j, assign the value 2 to y_j if $Y_j \in Y'$ and 0 if $Y_{j} \notin Y'$. Also, assign the value 3 to every vertex of Z, the value 2 to every vertex of x_i^2, and 0 otherwise. Since Y' exists, with $|Y'| = q$, the number of y_j's with weight 2 is q. Clearly, every vertex x_i^1 has two neighbors assigned 2. Hence, it is easy to check that f is a generous Roman dominating function with weight $f(V) = 3t + 8q = k$.

Conversely, suppose that G has a *GRDF*, g of weight at most $k = 3t + 8q$, and we prove that Y has a solution Y'. Clearly, each star needs a weight of at least 3, and so we may assume, without loss of generality, that $g(z_j) = 3$ and all its leaves are assigned 0. Since $z_j y_j \in E(G)$, it follows that each vertex y_j may be assigned the value 0. Let H be the subgraph of G induced by the union of the $V(H_i)$, thus $V(H) = \underset{1 \leq i \leq 3q}{\cup} V(H_i)$. Since $|V(H_i)| = 3$. We consider two cases:

Case 1. $g(H_i) = 2$. Then, $g(x_i^1) = 0$, otherwise, if $g(x_i^1) \geq 1$, then x_i^2 or x_i^3 is not generous Roman dominated by g. So, without loss of generality, we can assume that $g(x_i^2) = 2$.

Case 2. $g(H_i) = 3$. Then, $g(x_i^1) \leq 2$, and the same reasoning like case 1, without loss of generality, we can assume that $g(x_i^2) = 3$.

Let p be the number of H_i's having weight 3. Hence $g(V(H)) = 3p + 2(3q - p) = 6q + p$.

Now, note that for every H_i such that $g(H_i) = 2$, there exists at least one Y_j with $g(y_j) = 2$ (if not g would not be a *GRDF* of G). Let s be the number of y_j's assigned 2. Then $3t + 2s + 6q + p \leq k = 3t + 8q$, which gives $2s + p \leq 2q$. On the other hand, since each y_j has exactly three neighbors in $\{x_1^1, x_2^1, \ldots, x_{3q}^1\}$, we must have $3s \geq 3q - p$. Combining these two inequalities, we obtain $s = q$ and $p = 0$. Consequently, $Y' = \{Y_j : g(y_j) = 2\}$ is an exact cover for X. □

5.2. co-$\mathcal{NP}$-Completeness of the generous Roman Domination Problem with $\gamma_{gR}(G) = n - \Delta(G) + 2$. .

In this subsection we consider the complexity of the problem of deciding whether a graph G has $\gamma_{gR}(G) = n - \Delta(G) + 2$, to which we shall refer as ***MGRDF***$(n - \Delta(G) + 2)$.

MINIMUM GENEROUS ROMAN DOMINATING FUNCTION (MGRDF $(n - \Delta(G) + 2)$**)**

INSTANCE: An graph $G = (V, E)$ with a maximum degree $\Delta(G)$.

QUESTION: Does G have a minimum *GRDF* f with $f(V) = n - \Delta(G) + 2$?

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We show that this problem is *co*-$\mathcal{NP}$-complete by reducing the 3-satisfiability problem (***3-SAT***) to the problem of deciding whether $\gamma_{gR}(G) \leq n - \Delta(G) + 1$ is $\mathcal{NP}$-complete. Recall that the ***3-SAT*** problem is a well known NP-complete problem [10].

3-SATISFIABILITY (*3-SAT*)

INSTANCE: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ of clauses over a finite set X of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, q$.

QUESTION: Is there a truth assignment for X that satisfies all the clauses in $\mathcal{C}$?

GENEROUS DOMINATING FUNCTION (*GRDF*($n - \Delta(G) + 1$))

INSTANCE: An graph $G = (V, E)$ with a maximum degree $\Delta(G)$.

QUESTION: Does G have a *GRDF* f with $f(V) \leq n - \Delta(G) + 1$?

We denote by H_i the graph of order 4 describe in Fig. 3.

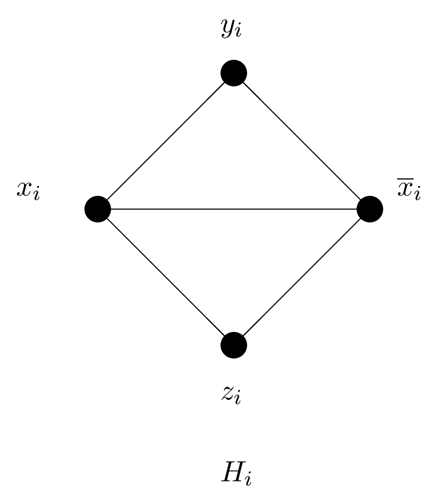

FIGURE 3. A literal gadget H_i.

Theorem 5.2. *Problem **GRDF**($n - \Delta(G) + 1$) is $\mathcal{NP}$-complete for graphs.*

Proof. First, ***GRDF*** ($n - \Delta(G) + 1$) is a member of $\mathcal{NP}$, since we can check in polynomial time that a function $f : V \to \{0, 1, 2, 3\}$ has weight at most $n - \Delta(G) + 1$ and is a generous Roman dominating function. Second, let us show how to transform any instance of ***3-SAT*** into an instance G of ***GRDF***($n - \Delta(G) + 1$) so that one of them has a solution if and only if the other one has a solution.

Let I be an arbitrary instance of ***3-SAT*** for the set of clauses $\mathcal{C} = \{C_1, C_2, \ldots, C_q\}$ on the set of variables $X = \{x_1, x_2, \ldots, x_p\}$. We construct a graph $G(I)$ such that I has a satisfying truth assignment if and only if $G(I)$ has a *GRDF* $f = (V_0, V_1, V_2, V_3)$ with $f(V) \leq n - \Delta(G) + 1$.

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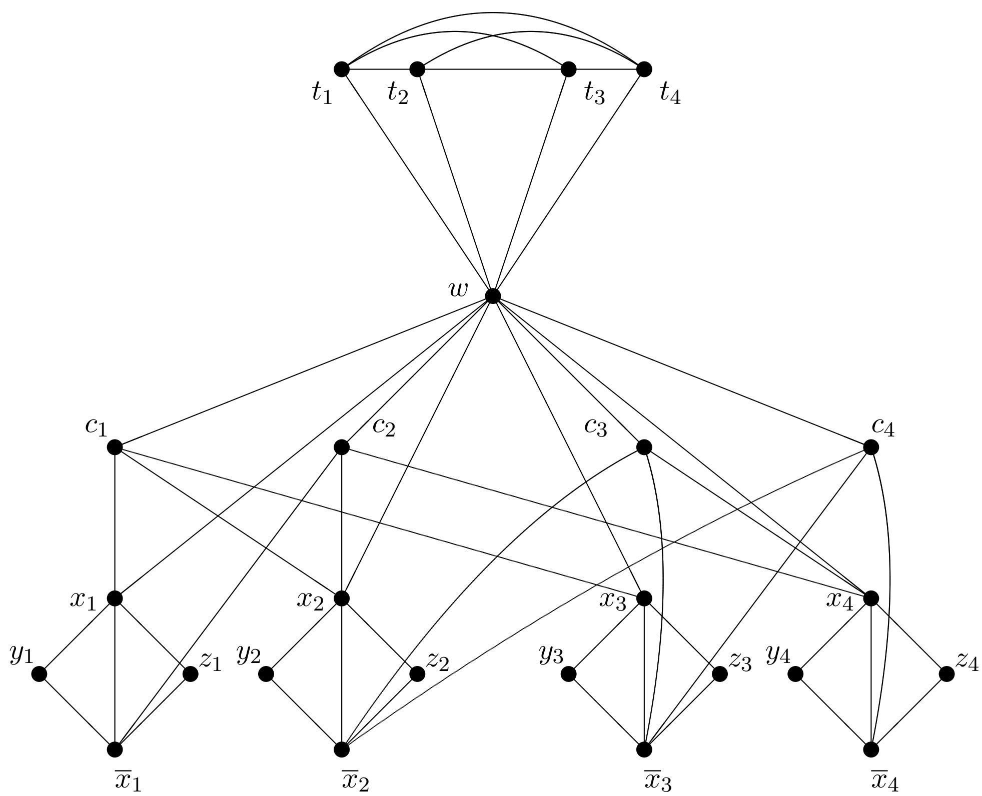

Figure 4. Instance $I = C_1 \wedge C_2 \wedge C_3 \wedge C_4 = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee x_4) \wedge (\overline{x}_2 \vee \overline{x}_3 \vee x_4) \wedge (\overline{x}_2 \vee \overline{x}_3 \vee \overline{x}_4)$.

Corresponding to each clause C_j, we create a vertex labeled c_j, and for each variable $x_i \in X$, $i = 1$ to $p \geq 3$, we build a graph labeled literal gadget H_i described in Figure 3. Moreover, we add a complete graph K_{q+1}, and join it from a vertex w of K_{q+1} on the one hand by edges from w to vertex x_i of H_i, for $i = 1$ to p, and on the other hand, we join it by edges from w to c_j for $j = 1$ to q. Now, for each clause C_j, we add three edge from three literals in clause C_j to corresponding clause vertex c_j, for $j = 1$ to q (i.e., we add edges $c_j x_i$ if $x_i \in C_j$ or $c_j \overline{x}_i$ if $\overline{x}_i \in C_j$, for $j = 1$ to q) (see Figure 4). Clearly $G(I)$ can be constructed from an instance of ***3-SAT*** in polynomial time. Also, $G(I)$ has order $4p + 2q + 1$ and w is a vertex of maximum degree $\Delta(G(I)) = p + 2q$. We shall show that the given instance of ***3-SAT*** has a satisfying truth assignment if and only if the graph $G(I)$ has a *GRDF* $f = (V_0, V_1, V_2, V_3)$ with $f(V) \leq n - \Delta(G) + 1$.

Set $k = 3p+2 = n - \Delta(G) + 1$. Let the given instance of ***3-SAT*** have a satisfying truth assignment A. We first prove that $G(I)$ has a *GRDF* $f = (V_0, V_1, V_2, V_3)$ with $f(V) \leq k$. Define the function $f : V \to \{0, 1, 2, 3\}$ as follows: $f(w) = 2$, and for $1 \leq i \leq p$, we set $f(y_i) = f(z_i) = 0$, and

$(f(x_i), f(\overline{x}_i)) = (3,0)$ if $I(x_i) = True$, and $(f(x_i), f(\overline{x}_i)) = (0,3)$ if $I(x_i) = False$, and all the remaining vertices are in $V(G)$ are in V_0. It is easy to see that the function f is a $GRDF$ on $G(I)$ of weight $f(V) = 3p + 2$.

Conversely, we prove that I has a satisfying truth assignment. To accomplish this, we suppose that $G(I)$ has a $GRDF$ $f = (V_0, V_1, V_2, V_3)$ of weight $f(V) \leq 3p+2$. We must show that I has a satisfying truth assignment. Clearly that $f(V(K_{q+1})) \geq 2$. Also, it is easy to check that for each subgraph H_i, we must have $f(V(H_i)) \geq 3$. Therefore, $f(V) \geq 3p + 2$. Equality is obtained from the fact that $f(V) \leq 3p + 2$, which implies that $f(V(H_i)) = 3$ for each $i \in \{1, 2, \ldots, p\}$ and $f(V(K_{q+1})) = 2$, and $\sum_{j=1}^{q} f(c_j) = 0$. Since $f(V(H_i)) = 3$ and $|V(H_i)| = 4$, for each H_i, we have two possibilities, either,

$$(f(x_i), f(\overline{x}_i)) = (3,0) \text{ and } f(y_i) = f(z_i) = 0$$

or

$$(f(x_i), f(\overline{x}_i)) = (0,3) \text{ and } f(y_i) = f(z_i) = 0$$

Since $(V_1 \cup V_2) \cap V(H_i) = \emptyset$, each clause vertex c_j must be dominated by at least one vertex of V_3. Now, we can create a satisfying truth assignment for I as follows: for each variable $v_i \in \{x_i, \overline{x}_i\}$, if $x_i \in V_3$ then, assign x_i the value $True$ (Instance $I(x_i) = True$) , and if $x_i \in V_0$ then, assign the value $False$ (Instance $I(x_i) = False$). It is easy to see that this is a satisfying truth assignment for I. □

Theorem 5.2 implies the next result immediately.

Corollary 5.3. ***MGRDF***$(n - \Delta(G) + 2)$ *is co-$\mathcal{NP}$-complete for graphs.*

6. Open problems

The idea of generous Roman dominating functions in graphs provides numerous interesting theoretical and computational questions. Many problems have yet to be settled. We close this paper with a partial listing of some of these problems and open questions.

1. For an integer $n \geq 2$. Characterize the trees T satisfying $\gamma_{gR}(T) = \gamma_R(T)$ or $\gamma_{gR}(T) = \gamma_{dR}(T)$.
2. Does there exist a 1inear algorithm for computing $\gamma_{gR}(T)$ for any tree T?
3. Characterize all graphs G satisfying $\gamma_{gR}(G) = n - 1$.
4. Characterize all trees T satisfying $\gamma_{gR}(T) = n - \Delta + 2$.
5. For an integer $r \geq 4$, give a characterization of r-regular graphs G satisfying $\gamma_{gR}(G) = n - \Delta + 2$.

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