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NOTE ON SKEW-EIGENVALUES OF DIGRAPHS

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ABSTRACT. Let G^σ be an oriented graph with underlying simple graph G. The skew-adjacency matrix of G^σ is the $\{0, 1, -1\}$-matrix $S = S(G^\sigma) = [s_{ij}]$, such that $s_{ij} = 1$ if (v_i, v_j) is an arc in G^σ, $s_{ij} = -1$ if (v_j, v_i) is an arc in G^σ and $s_{ij} = 0$, otherwise. In this paper, all connected oriented graphs with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$ are characterized.

1. Introduction

In this section, we recall some definitions that will be used in the paper. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(G)$ and G^σ be an orientation of G, which assigns to each edge of G a direction. The graph G is called the underlying graph of G^σ. The skew-adjacency matrix of G^σ is defined as follow:

$$S = S(G^\sigma) = [s_{ij}] = \begin{cases} 1 & (v_i, v_j) \ is\ an\ arc\ in\ G^\sigma \\ -1 & (v_j, v_i) \ is\ an\ arc\ in\ G^\sigma \\ 0 & Otherwise, \end{cases}$$

see [3, 5, 7, 11] for more details. The characteristic polynomial of G^σ, denoted by $\phi(G^\sigma, x)$, is defined as follow:

$$\phi(G^\sigma, x) = \det(xI_n - S) = x^n + s_1 x^{n-1} + \cdots + s_{n-1}x + s_n.$$

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The roots of the polynomial $\phi(G^\sigma, x)$ are called the eigenvalues of G^σ. Since $S = S(G^\sigma)$ is a skew-symmetric matrix, these eigenvalues are either 0 or pure imaginary number. Thus the eigenvalues of the $n \times n$ matrix S are the form $\mathbf{i}\lambda_1$, $\mathbf{i}\lambda_2, \ldots, \mathbf{i}\lambda_n$ where each of the λ_k, $1 \leq k \leq n$, are real and $\mathbf{i}^2 = -1$. These eigenvalues form the spectrum of $S(G^\sigma)$ and is said to be the skew-spectrum of G^σ, denote $Sp(G^\sigma)$. Our notation is standard and can be taken from the [2, 9, 10, 12, 13, 15].

Throughout this paper we denote by $deg(v_i)$, for simplicity d_i, the degree of the vertex $v_i \in V(G)$. Also we denote by P_n and C_n, a path and a cycle with n vertices, and $K_{1,n}$ a star graph with $n+1$ vertices, respectively. In [1] Adiga et al. defined the concept of the skew energy of an oriented graph G^σ as the sum of the absolute values of all the eigenvalues of $S(G^\sigma)$, denoted by $\mathcal{E}_s(G^\sigma)$. They proved that for any oriented graph G^σ with order n and maximum degree Δ, $\mathcal{E}_s(G^\sigma) \leq n\sqrt{\Delta}$ and the equality holds if and only if $S(G^\sigma)^t S(G^\sigma) = \Delta I_n$, which implies that G^σ is $\Delta-$regular.

Xu in [6] determined the $3-$regular oriented graphs with optimum skew energy. Later the authors in [4] characterized $4-$regular oriented graphs with optimum skew energy. Lastly the authors in [16] found the conditions for oriented graphs to have two and three skew-eigenvalues. They determined all oriented graphs with mentioned skew-eigenvalues. Consider the skew adjacency matrix of an oriented graph G^σ, $S(G^\sigma) = [s_{ij}]$, and let $W = v_1e_1v_2e_2 \cdots v_ke_kv_{k+1}$ be a walk with length k. Then the sign of W is defined as:

$$sgn(W^\sigma) = s_{1,2}s_{2,3} \cdots s_{k,k+1}.$$

Let $\overline{W} = v_{k+1}e_kv_ke_{k-1}v_{k-1} \cdots e_1v_1$. Then $sgn(\overline{W}) = sgn(W)$ if k is even, and $sgn(\overline{W}) = -sgn(W)$ if k is odd. Denote by $w_k^+(ij)$ and $w_k^-(ij)$ the number of all positive and negative walks starting v_i and terminating v_j with length k, respectively.

Theorem 1.1. [6] *Let S be the skew-adjacency matrix of an oriented graph G^σ and v_i and v_j be two arbitrary vertices of G^σ. Then*

$$[S^k]_{ij} = w_k^+(ij) - w_k^-(ij).$$

It is easy to see that $[S^k]_{ii} = 0$, for each odd k. In [14] the authors computed the skew spectral moments of an oriented graph.

Definition 1.2. [8] *An even cycle C is called evenly oriented if for either choice of direction of traversing around C the number of edges of C which are directed in the direction of traversal is even. Otherwise C is oddly oriented.*

Throughout this paper denoted by C_n^+ an evenly oriented even cycle and C_n^- an oddly oriented even cycle with n vertices. Also an oriented square is an oriented cycle with four vertices.

Theorem 1.3. [16] *Suppose G^σ is an oriented graph with the skew adjacency matrix S. Then G^σ has two distinct skew-eigenvalues if and only if the matrix S satisfies $S^2 = -\lambda^2 I_n$, for some λ.*

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Theorem 1.4. [16] *Let $G = (V, E)$ be a simple graph and G^σ be an orientation of G. Then the following are equivalent.*

i. *The oriented graph G^σ has exactly two skew-eigenvalues.*
ii. *The graph G is regular and for any vertices v_i and v_j, $i \neq j$, $w_2^+(ij) = w_2^-(ij)$.*

Theorem 1.5. [16] *Let G^σ be an oriented graph. Then G^σ has exactly two distinct skew-eigenvalues if and only if G^σ has optimum skew energy.*

Theorem 1.6. [16] *Suppose G^σ is an oriented graph with skew adjacency matrix S such that either G is non-regular or $w_2^+(ij) \neq w_2^-(ij)$ for some i and j. Then G^σ has three distinct skew-eigenvalues if and only if the matrix S satisfies $S^3 = -\lambda^2 S$.*

Lemma 1.7. [16] *Let G^σ be an oriented graph and S be the skew-adjacency matrix associated to σ. If (v_i, v_j) is an arc of G^σ, then $[S^3]_{ij} = -d_i - d_j - q_{ij}^+ + q_{ij}^- + 1$, where q_{ij}^+ and q_{ij}^- are the number of evenly oriented and oddly oriented squares that are contain the edge v_iv_j, respectively. Also if G^σ has three distinct skew-eigenvalues, then $[S^3]_{ij} = 0$, for any v_i and v_j such that $v_iv_j \notin E(G)$.*

Theorem 1.8. [16] *Suppose G^σ is an oriented graph such that either G is non-regular or $w_2^+(ij) \neq w_2^-(ij)$ for some i and j. Then G^σ has exactly three distinct skew-eigenvalues if and only if the following statemetnts hold.*

(a) *If (v_i, v_j) is an arc in G^σ, then $d_i + d_j + q_{ij}^+ - q_{ij}^- - 1$ is a constant where q_{ij}^+ and q_{ij}^- are the number of evenly oriented and oddly oriented squares that are contain the edge v_iv_j, respectively.*
(b) *If (v_i, v_j) is not an arc in G^σ, then $w_3^+(ij) = w_3^-(ij)$.*

Theorem 1.9. [16] *Let G^σ be an oriented graph with underlying graph G where has three skew-eigenvalues 0, $\mathbf{i}\lambda$ and $-\mathbf{i}\lambda$. Then $\Delta \leq \lambda^2$, which Δ is the maximum degree of the vertices of G.*

The authors in [16] determined all oriented graphs with three distinct skew-eigenvalues 0, $\pm\mathbf{i}\sqrt{2}$ and 0, $\pm\mathbf{i}\sqrt{3}$. In this paper all oriented graphs with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$ are characterized.

2. Main Results

The aim of this section is to characterize all connected oriented graphs with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. Assume that G^σ has three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. By Theorem 1.8, we have $\lambda^2 = d_i + d_j + q_{ij}^+ - q_{ij}^+ - 1 = 4$. On the other hand, by Theorem 1.9, for any vertex v_i,

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we have $d_i \leq \Delta \leq \lambda^2$. Therefore for any arc (v_i, v_j) we have

$$\begin{cases} d_i + d_j + q_{ij}^+ - q_{ij}^- = 5 \\ d_i + d_j \geq 3 \\ d_i + d_j \leq 2\Delta \leq 2\lambda^2 = 8, \end{cases}$$

and for any two non-adjacent vertices v_i and v_j, we have $w_3^+(ij) = w_3^-(ij)$. Thus for an arbitrary arc (v_i, v_j) the followings hold:

$$\begin{cases} d_i + d_j = 3 \textit{ and } q_{ij}^+ - q_{ij}^- = 2, \\ d_i + d_j = 4 \textit{ and } q_{ij}^+ - q_{ij}^- = 1, \\ d_i + d_j = 5 \textit{ and } q_{ij}^+ - q_{ij}^- = 0, \\ d_i + d_j = 6 \textit{ and } q_{ij}^+ - q_{ij}^- = -1, \\ d_i + d_j = 7 \textit{ and } q_{ij}^+ - q_{ij}^- = -2, \\ d_i + d_j = 8 \textit{ and } q_{ij}^+ - q_{ij}^- = -3. \end{cases}$$

Since $\Delta \leq 4$, we consider the following lemmas.

Lemma 2.1. *Let G^σ be an oriented graph has with $\Delta = 4$. Then G^σ has three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$ if and only if $G^\sigma \cong K_{1,4}^\sigma$ and $G^\sigma \cong G_i^\sigma$, for $i = 1, 2, \ldots, 11$, which are depicted in Figure 1.*

Proof. If G^σ is one the mentioned oriented graph, then by calculation with MAPLE one can see that G^σ has three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. Conversely, assume that G^σ has three mentioned skew-eigenvalues. Suppose v_1 is a vertex of degree four and v_2, v_3, v_4 and v_5 are its neighbors. Then the degree of these vertices is either one, two, three or four. First assume that the degree of at least one of these vertices is one, for instance $deg(v_2) = 1$. Then the edge v_1v_2 is not on any square and so the condition $q_{1,2}^+ - q_{1,2}^- = 0$ holds for any orientation. Now we discuss the following cases.

case 1. Assume that $deg(v_3) = 1$. In this case, we check the following situations.

a) If $deg(v_4) = 1$, then the edges v_1v_3 and v_1v_4 are not on any square and so the obtained graph is the star graph $K_{1,4}$. It is easy to see that for every orientation σ of this graph, the obtained oriented graph has all desired conditions. So in this case $G^\sigma \cong K_{1,4}^\sigma$, see Figure 1.

b) If $deg(v_4) = 2$, then the edge v_1v_4 must be on at least one square and so there is a new vertex v_6 that is adjacent to v_4 and v_5. Now we discuss about the degree of v_5. If $deg(v_5) = 4$, then two conditions $d_1 + d_5 = 8$ and $q_{1,5}^+ - q_{1,5}^- = -3$ must be hold for the edge v_1v_5. These conditions require that the edge v_1v_5 lies on at least three squares, which is impossible. If $deg(v_5) = 3$, then two conditions $d_1 + d_5 = 7$ and $q_{1,5}^+ - q_{1,5}^- = -2$ must be hold for the edge v_1v_5. These conditions require that the edge v_1v_5 lies on at least two squares, which is impossible. So $deg(v_5) = 2$ and the edge v_1v_5 is on one square which this square must be oddly oriented. In this case $deg(v_6)$ is either two, three or four. If $deg(v_6) = 4$, then the edges v_4v_6 and v_5v_6 lie on

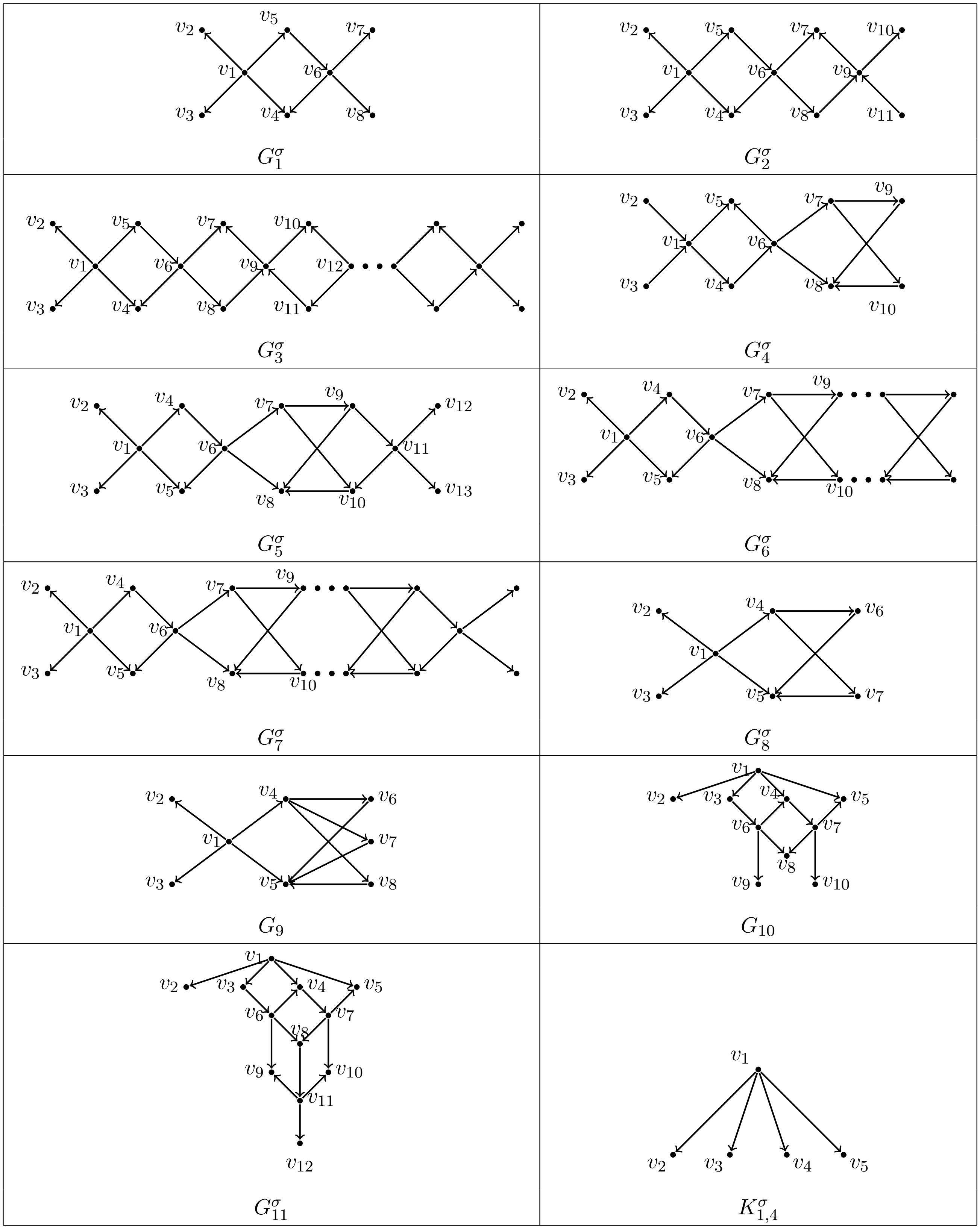

FIGURE 1. Oriented graphs with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$ with maximum degree four.

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one square and so this square should be oddly oriented. Hence there are two new vertices v_7 and v_8 that are adjacent to v_6. If $deg(v_7) = deg(v_8) = 1$, then in the obtained graph, we should consider an orientation to have all desired conditions for three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. This implies that $G^\sigma \cong G_1^\sigma$ which is shown in Figure 1. If $deg(v_7) = 2$, because the degree of v_6 is equal to 4, then we must have condition $q^+_{6,7} - q^-_{6,7} = -1$ for the edge v_6v_7. So, there is a new vertex v_9 that is adjacent to two vertices v_8 and v_7 and the square $v_6v_7v_9v_8v_6$ should be oddly oriented. Like what we said about the vertex v_6, it is also true for vertex v_9, that is, the degree of vertex v_9 must be equal to 4. So there are two new vertices v_{10} and v_{11} which are adjacent to vertex v_9. In this graph we should consider an orientation to have all desired conditions for three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. This implies that $G^\sigma \cong G_2^\sigma$ which is depicted in Figure rm1. By repeating the above process, a new vertex is created in each step, which must be adjacent to two vertices of degree two of the previous step. If we continue this process, at the end we will reach a graph, in the last step, the vertex that we add, must be of degree 4 and two pendant vertices must be connected to it. In the obtained graph we should consider an orientation to have all desired conditions for three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. This implies that we have a family of oriented graph with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$, which is depicted in Figure 1, G_3^σ. Now if $deg(v_7) = 3$, then the edge v_6v_7 must be on at least two squares. This implies that there are two new vertices v_9 and v_{10} such that $\{v_7v_9, v_7v_{10}\} \subseteq E(G)$. By checking the degree of the vertices v_9 and v_{10}, it can be concluded that the degree of these two vertices must be equal to either two or four. If $deg(v_9) = 2$, then in the obtained graph, we must direct the edges in such a way that it has the conditions to have three desired skew-eigenvalues. Thus in this case we obtain the oriented graph G_4^σ, which is depicted in Figure 1. If $deg(v_9) = 4$, then there must be two new vertices v_{11} and v_{12} so that are adjacent to two vertices v_9 and v_{10}. Like what we said about vertex v_9, it is also true for vertex v_{11}, that is, the degree of vertex v_{11} must be equal to four. By repeating the above process, two new vertex are created in each step, which must be adjacent to two vertices of the previous step. If we continue this process, at the end we will reach a graph, in the last step, the vertex that we add, must be of degree 4. Thus we have two items. In the first item, this new vertex must be connected to two pendant vertices and so we obtain the oriented graph G_7^σ which is depicted in Figure 1. In the second item, this new vertex must be connected to two vertices of degree four and so we obtain the oriented graph G_6^σ which is depicted in Figure 1. By a simple check one can see that if $deg(v_6) = 3$, no oriented graph will be obtained with three skew-eigenvalues 0 and $\pm 2\mathbf{i}$.

c) If $deg(v_4) = 3$, then the edge v_1v_4 must be on at least two squares. So, there are two new vertices v_6 and v_7 such that $\{v_4v_6, v_4v_7, v_5v_7, v_5v_6\} \subseteq E(G)$. In this case by a simple check we have the oriented graph G_8^σ which is depicted in Figure 1.

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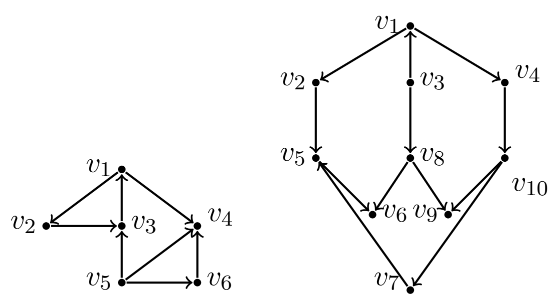
d) If $deg(v_4) = 4$, then the edge v_1v_4 must be on at least three squares. So, there are three new vertices v_6, v_7 and v_8 such that $\{v_4v_6, v_4v_7, v_4v_8, v_5v_6, v_5v_7, v_5v_8\} \subseteq E(G)$. Thus by considering an orientation for this graph, we have the oriented graph G_9^σ which is depicted in Figure 1.

case 2. Assume that $deg(v_3) = 2$. Then the edge v_1v_3 must be on at least one square. So there is a new vertex v_6 that is adjacent to v_3 and at least one of the vertices v_4 and v_5. By checking the degree of vertices, one can see that if $deg(v_3) = deg(v_4) = 2$, then there is no oriented graph with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. Assume that $deg(v_4) = 3$. In this case by checking the degree of the vertices and adding new vertices, we obtain oriented graphs with three skew-eigenvalues 0 and $\pm 2\mathbf{i}$, which are shown in Figure 1, G_{10}^σ and G_{11}^σ.

case 3. Assume that $deg(v_3) = 3$. Then the edge v_1v_3 must be on at least two squares. A simple check proves that there is no new oriented graph with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$.

case 4. Assume that $deg(v_3) = 4$. Then the edge v_1v_3 must be on at least three squares. So there are three vertices v_6, v_7 and v_8 that are adjacent to v_3. With a similar argument one can see that there is no new oriented graph with three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$. □

Finally, if the degree of vertices v_2, v_3, v_4 and v_5 is at least two, then by a similar argument we obtain oriented graphs that are the same graphs as in Figure 1.



H_1^σ H_2^σ

FIGURE 2. The oreinted graphs H_1^σ and H_2^σ.

Lemma 2.2. *Let G^σ be an oriented graph with $\Delta = 3$. Then G^σ has three distinct skew-eigenvalues 0 and $\pm 2\mathbf{i}$ if and only if $G^\sigma \cong H_i^\sigma$, $i = 1, 2$, which are depicted in Figure 2.*

Proof. Let v_1 be a vertex of degree three and v_2, v_3 and v_4 be its neighbors. Then the degree of these vertices is either one, two or three. It is easy to see that the degree of any of these vertices can not be equal to one. Thus we discuss the following cases.

case 1. Assume that $deg(v_2) = 2$. Then we must have $q_{1,2}^+ - q_{1,2}^- = 0$. Consider two items. First assume that $v_2v_3 \in E(G)$, so $deg(v_3) \neq 2$. Let $deg(v_3) = 3$. Then the edge v_1v_3 must be on at least one square. This implies that $v_3v_4 \notin E(G)$. So there is a new vertex v_5 such that $v_3v_5 \in E(G)$ and

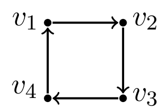


FIGURE 3. The oreinted graph C_4^+.

$v_4v_5 \in E(G)$. Also since the edge v_1v_4 lies on one square, the degree of v_4 is not two. So $deg(v_4) = 3$ and there is a new vertex v_6 that $v_4v_6 \in E(G)$. Since the edge v_3v_5 lies on one square, $deg(v_5) \neq 2$. So $deg(v_5) = 3$ and this implies that $v_5v_6 \in E(G)$. In the obtained graph, the edges v_4v_6 and v_5v_6 are not on any square. Thus we consider an orientation for this graph that: $q^+_{i,j} - q^-_{i,j} = 0$, for the edges v_4v_6 and v_5v_6, and $q^+_{i,j} - q^-_{i,j} = -1$, for the edges v_1v_3, v_1v_4, v_4v_5 and v_3v_5. This orientation is shown in Figure 2. Then in this item $G^\sigma \cong H_1^\sigma$. In the next item assume that $v_2v_3 \notin E(G)$. So there is a new vetrex v_5 that $v_2v_5 \in E(G)$. It is easy to see that $deg(v_5) \neq 1, 2$ and so $deg(v_5) = 3$. The vertex v_5 is not adjacent to both v_3 and v_4. So there are two new vertices v_6 and v_7 that are adjacent to v_5. It is clear that $deg(v_6) \neq 1, 3$ and $deg(v_7) \neq 1, 3$, so $deg(v_6) = deg(v_7) = 2$. Thus there is a new vertex v_8 that $v_6v_8 \in E(G)$. Since $deg(v_6) = 2$ and the edge v_6v_8 is not on any square, $deg(v_8) = 3$. This implies that $v_3v_8 \in E(G)$. By a similar argument there is a new vertex v_9 that $v_8v_9 \in E(G)$ and $deg(v_9) = 2$. So there is a new vertex v_{10} that is adjacent to v_9. One can see that the degree of v_{10} is equal to three, so we assume that $v_7v_{10} \in E(G)$. In the obtained graph, none of the edges are on any square. Thus for the edges we have $q^+_{i,j} - q^-_{i,j} = 0$. So we consider an orientation such that $w_3^+(i, j) = w_3^-(i, j)$. This orientation is shown in Figure 2. Therefore in this item $G^\sigma \cong H_2^\sigma$.

case 2. $deg(v_2) = 3$. Then the edge v_1v_2 must be on at least a square. So there are two new vertices v_5 and v_6 that $\{v_3v_5, v_3v_6\} \subseteq E(G)$. By discussion on the degree of these vertices, one can see that in this case there is no oriented graph with three skew-eigenvalues 0 and $\pm 2\mathbf{i}$. □

Lemma 2.3. *Let G^σ be an oriented graph with $\Delta = 2$. Then G^σ has three skew-eigenvalues 0 and $\pm 2\mathbf{i}$ if and only if $G^\sigma \cong C_4^+$.*

Proof. Suppose that v_1 is a vertex of degree two and v_2 and v_3 are its neighbors. It is easy to see that $deg(v_i) \neq 1$, for $i = 2, 3$. Let $deg(v_2) = 2$, then the edge v_1v_2 must be on at least one square and $q^+_{1,2} - q^-_{1,2} = 1$ must be hold. Suppose that there is a new vertex v_4 such that $v_2v_4 \in E(G)$. If $deg(v_3) = 2$, then the edge v_1v_3 should be on at least a square. So we have $v_3v_4 \in E(G)$. and we have a cycle C_4. In this graph we consider an orientation such that has mentioned three skew-eigenvalues. Thus in this case $G^\sigma \cong C_4^+$. □

By above lemmas all oriented graphs with three skew-eigenvalues 0 and $\pm 2\mathbf{i}$ are determine as following.

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Theorem 2.4. *Let G^σ be an oriented graph. Then G^σ has three skew-eigenvalues* 0 *and* $\pm 2\mathbf{i}$ *if and only if G^σ is isomorphic to each of the digraphs shown in Figures 1, 2 and 3.*

Proof. If G^σ is isomorphic to each of digraphs of Figures 1, 2 and 3, then by calculation with MAPLE one can see that these oriented graphs have said three skew-eigenvalues. Conversely, if G^σ has three skew-eigenvalues 0 and $\pm 2\mathbf{i}$, then according to above lemmas the result is proved. □

Corollary 2.5. *By Theorem 1.5, all oriented graphs specified in the previous theorem, have the optimum skew energy.*

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