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THE MINIMUM ε-SPECTRAL RADIUS OF t-CLIQUE TREES WITH GIVEN DIAMETER

ZHENGPING QIU, HANYUAN DENG AND ZIKAI TANG*

ABSTRACT. The eccentricity matrix $\varepsilon(G)$ of a graph G is defined as

$$\varepsilon(G)_{uv} = \begin{cases} d_{uv} & d_{uv} = min\{e(u), e(v)\}, \\ 0 & d_{uv} < min\{e(u), e(v)\}. \end{cases}$$

Let T_t be a t-clique tree corresponding to the tree T(underlying graph of T_t) with order $n' = (n-1)t+1$ and diameter d. In this paper, we identify the extremal t-clique trees with given diameter having the minimum ε-spectral radius. Simultaneously, we calculate the lower bound of ε-spectral radius of t-clique trees when $n-d$ is odd.

1. Introduction

The general notation and terminology from graph theory used, the reader is referred to [1, 3, 4, 5, 6, 19, 20]. The graphs considered here are ordinary connected simple graphs. We denote by graph $G = \big(V(G), E(G)\big)$ with vertex set $V(G)$ and edge set $E(G)$, $n = |V(G)|$ and $m = |E(G)|$. If $u, v \in V(G)$, the graph $G + uv$ is obtained from G by adding an edge $uv \notin E(G)$. Similarly, $G - uv$ is constructed from the graph G by deleting an edge $uv \in E(G)$. For $X \subset E(G)$, $G - X$ denotes the subgraph of G obtained by deleting the edges of X.

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Let $N_G(u)$ be the set of vertices adjacent to u in G and $N_G[u] = N_G(u) \cup \{u\}$. Denoted by $d_G(u)$(or simply $d(u)$) the *degree* of vertex u and $d_G(u) = |N_G(u)|$. The distance $d_G(u, v)$ between two vertices u and v is the minimum length of the paths joining them. Let $D(G) = (d_{uv})$ be the *distance matrix* of G, where $d_{uv} = d_G(u, v)$. The *eccentricity* $e_G(u)$ of the vertex $u \in V(G)$ is defined as $e_G(u) = max\{d(u, v)|v \in V(G)\}$, or simply by $e(u)$ if there is no confusion about the graph under consideration. The *diameter* of graph G, denoted by $diam(G)$(or d), is defined as $diam(G) = max\{e(v)|v \in V(G)\}$.

D_{MAX} matrix was firstly appeared in [17, 18]. In order to better study of the novel graph matrix. In 2018, Wang [11] redefined D_{MAX}-matrix and renamed it as *eccentricity matrix* of G, denoted by $\varepsilon(G)$, which is constructed from the distance matrix $D(G)$ so that in each row and each column it only ratain the eccentricities, which other elements of the distance matrix are set to be 0. To be more precise, the elements of *eccentricity matrix* $\varepsilon(G)$ of graph G are defined as follows

$$\varepsilon(G)_{uv} = \begin{cases} d_{uv} & d_{uv} = min\{e(u), e(v)\}, \\ 0 & d_{uv} < min\{e(u), e(v)\}. \end{cases}$$

Proverbially, the ordinary graph spectrum is formed by the eigenvalues of the *adjacency matrix* $A(G)$. Correspondingly, the ε-spectrum of a graph consists of the ε-eigenvalues of its *eccentricity matrix* $\varepsilon(G)$. Since $\varepsilon(G)$ is symmetric, the ε-eigenvalues of G are real. Let $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_k$ be all the distinct ε-eigenvalues. Therefore, the ε-spectrum of G can be written as

$$Spec_\varepsilon(G) = \begin{Bmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_k \\ n_1 & n_2 & \cdots & n_k \end{Bmatrix},$$

where n_i is the algebraic multiplicity of the eigenvalue ε_i $(1 \leq i \leq k)$. Denoted by $\rho(G) = \varepsilon_1(G)$ the largest eigenvalue of $\varepsilon(G)$, and call it the ε-spectral radius of G. By the *Perron-Frobenius Theorem*, we know that $\varepsilon_1(G) > 0$ and there exists a unique positive unit vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, called *perron eigenvector* such that $\varepsilon(G)\mathbf{x} = \rho(G)\mathbf{x}$, where x_i is the coordinate for the vectex v_i. Balaban et al.[2] proposed the use of $\rho(G)$ as a molecular descriptor, and it was used to infer the extent of branching and model boiling points of alkanes in [8].

Studying the eccentricity spectra of grahs is a popular topic in spectral graph theory. In 2018, Wang et al.[11] characterized the relationships between the A-spectrum and ε-spectrum of some graphs, and they proved that the eccentricity matrix $\varepsilon(G)$ of tree G is irreducible. Mahato et al.[7] proved that the conjecture about the least eigenvalue of eccentricity matrices of trees which are presented by Wang[11]. Recently, Wang and Yang et al.[12, 26] determined the exact values of ε-energies of paths, cycles, double stars and digraph. Wang et al.[13, 15] researched some spectral properties of the eccentricity matrix of graphs and characterized a class of graphs whose eccentricity matrices are irreducible. Wei et al.[22] determined the tree with minimum ε-spectral radius and identified all trees with given order and diameter having minimum ε-spectral. Equivalently, Lei et al.[24, 27] characterized the graphs whose second least ε-eigenvalue $\gamma(G) > -\sqrt{15 - \sqrt{193}}$ and given an upper bound for the ε-energy and discuss the corresponding extreme graphs which involve the graphs with three distinct ε-eigenvalues. In 2022, Qiu and Tang[29] showed some properties of the ε-eigenvalues of threshold graphs and determined distinct ε-eigenvalues in

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$[-2, 0]$. For more details about the applications of eccentricity matrix in terms of molecular descriptors, we refer to [14, 16, 21, 23].

Clique tree is a graph whose blocks are cliques[10]. Denote by $\Phi(n, d)$ the set of trees with order n and diameter d. The graph resulting from substituting each edge of tree T with a clique $K_{t+1}(t \geq 1)$ is called *t-clique tree* T_t corresponding to T, and T is called *underlying graph* of T_t. Denote by $\Phi_t(n', d)$ the set of t-clique trees with order $n' = (n-1)t + 1$ and diameter d, where $n \geq 2$ and $t \geq 1$. In general, we call $\mathbb{P}_{n_1,n_2,\ldots,n_k}$ a *clique path* if each edge of path $P_k = v_1 \ldots v_{k+1}$ is replace by a clique K_{n_i} such that $V(K_{n_i}) \cap V(K_{n_{i+1}}) = v_i$ for $1 \leq i \leq k-1$ and $V(K_{n_i}) \cap V(K_{n_j}) = \varnothing$ for $j \neq i-1$, $i+1$ and $1 \leq i \leq k-1$. We call $\mathbb{K}_{u,n_1,n_2,\ldots,n_k}$ as a *clique star* if we replace each edge of the star S_{k+1} with a clique K_{n_i} such that $V(K_{n_i}) \cap V(K_{n_j}) = \{u\}$ for $i \neq j$ and $1 \leq i, j \leq k$, see Figure 1. Qiu et al.[30] have showed the ε-spectrum of *clique star*.

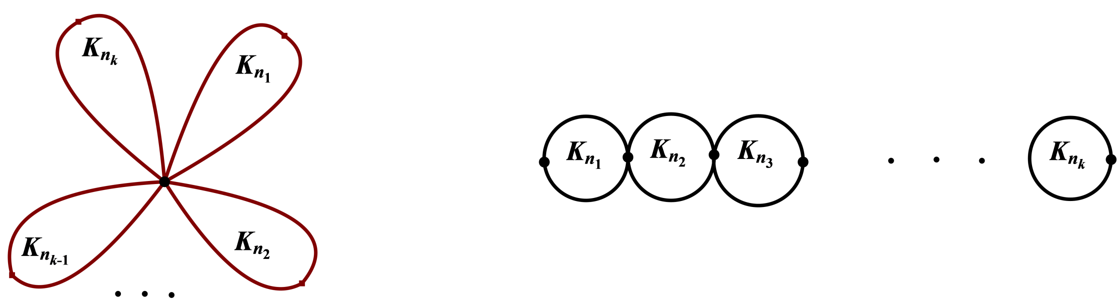

FIGURE 1. A clique path and a clique star

Zhang et al.[25] determined the graphs with maximum and minimum distance Laplacian spectral radius among all clique trees with n vertices and k cliques. Jin et al.[28] deduced the conjecture of distance energy of clique trees proposed by Lin[5]. Qiu et al.[30] determine the upper bound for ε-spectral radius of clique tree. In this paper, we identify all t-clique trees with given diameter d having the minimum ε-spectral radius, and show the corrresponding extremal t-clique trees.

If $d \geq 3$ is odd, let $P_{d+1} = v_0, v_1, v_2, \ldots, v_d$ be a diametrical path of tree $T(n, d, a, b) \in \Phi(n, d)$ such that $a = |N(v_{\frac{d-1}{2}})| - 2$ and $b = |N(v_{\frac{d+1}{2}})| - 2$, where $a + b = n - d - 1$ and $b \geq a \geq 0$. The clique tree corresponding to $T(n, d, a, b)$ is denoted by $T_t(n', d, a, b)$.

If $d \geq 4$ is even, let $P_{d+1} = v_0, v_1, v_2, \ldots, v_d$ be a diametrical path of tree $T(n, d, a, b, c) \in \Phi(n, d)$ such that $a = |N(v_{\frac{d-2}{2}})| - 2$ and $b = |N(v_{\frac{d}{2}})| - 2$, $c = |N(v_{\frac{d+2}{2}})| - 2$, where $a + b + c = n - d - 1$ and $c \geq a \geq 0$, $b \geq 0$. The clique tree corresponding to $T(n, d, a, b, c)$ is denoted by $T_t(n', d, a, b, c)$(see Figure 2).

2. The minimum ε-spectral radius of clique trees with diameter $d \leq 4$

Here, we will consider the clique tree with $d \leq 4$ having the minimum ε-spectral radius. The main tool to identify extremal graphs will be edge shift transformation. Firstly, we give a Lemma which is related to t-clique tree with $d = 3$.

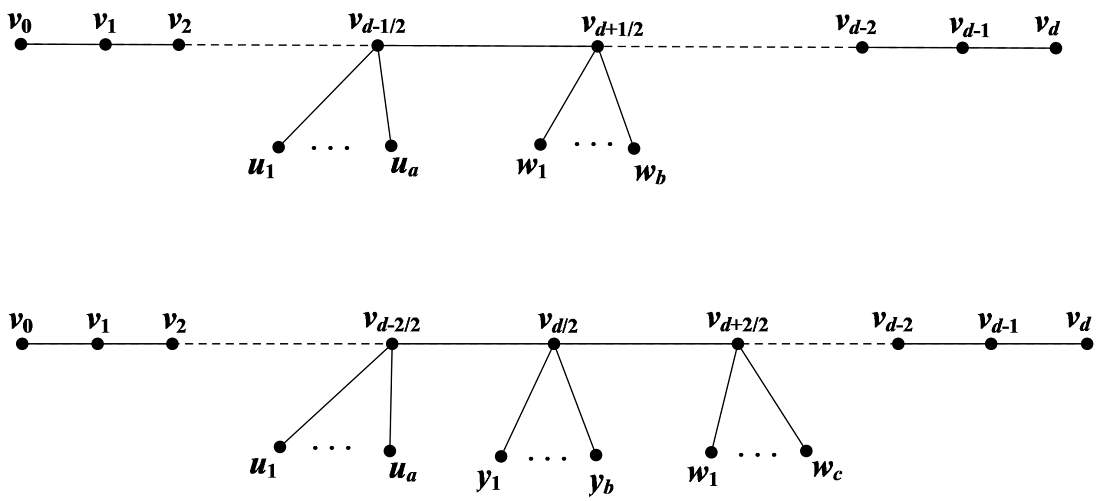

FIGURE 2. Trees $T(n,d,a,b)$ and $T(n,d,a,b,c)$

Lemma 2.1. [30] *Let $T_t(n',3,a,b) \in \Phi_t(n',3)$, where $a+b=n-4$ and $b \geq a \geq 1$, $n'=(n-1)t+1$. Then*

$$\varepsilon_1\big(T_t(n',3,a,b)\big) > \varepsilon_1\big(T_t(n',3,a-1,b+1)\big).$$

In view of the above lemma, the follwing result easily follows:

Theorem 2.2. *The mimimum ε-spectral radius is achieved uniquely by clique tree $T_t(n',3,0,n-4)$ among $\Phi_t(n',3)$.*

Lemma 2.3. [22] *Let G be an n vertex connected graph with diameter d. Then*

$$\varepsilon_1(G) \geq d \text{ and } \varepsilon_n(G) \leq -d.$$

Theorem 2.4. *The mimimum ε-spectral radius is achieved uniquely by clique tree $T_t(n',4,0,n-5,0)$ among $\Phi_t(n',4)$.*

Proof. Let $P_5 = v_0v_1v_2v_3v_4$ be a diametrical path of $T_t(n',4,a,b,c)$. The underlying graph corresponding to $T_t(n',4,a,b,c)$ is $T(n,4,a,b,c) \in \Phi(n,4)$, and P_5 is also the diametrical path of $T(n,4,a,b,c)$, $V\big(T(n,4,a,b,c)\big) \subseteq V\big(T_t(n',4,a,b,c)\big)$.

Let

$$\begin{aligned} T_i &= N_{T_t(n',d,a,b,c)}(v_i) \cap N_{T_t(n',d,a,b,c)}(v_{i+1});\\ U &= N_{T_t(n',d,a,b,c)}(v_1) \backslash \big(T_1 \cup \{v_2\}\big);\\ Y &= N_{T_t(n',d,a,b,c)}(v_2) \backslash \big(T_1 \cup T_2 \cup \{v_1,v_3\}\big);\\ W &= N_{T_t(n',d,a,b,c)}(v_3) \backslash \big(T_2 \cup \{v_2\}\big). \end{aligned}$$

Then it's easy to see that $|U|=(a+1)t$, $|Y|=bt$, $|W|=(c+1)t$ and $|T_i|=t-1 (i=1,2)$.

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By the definition of eccentricity matrix, $\varepsilon\big(T_t(n',d,a,b,c)\big)$ is equal to

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ U \\ Y \\ W \\ T_1 \\ T_2 \end{array}\begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & U & Y & W & T_1 & T_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{3J} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{2J} & \mathbf{0} & \mathbf{2J} & \mathbf{0} & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{3J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2J} & \mathbf{3J} & \mathbf{0} & \mathbf{3J} & \mathbf{4J} & \mathbf{0} & \mathbf{3J} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3J} & \mathbf{0} & \mathbf{3J} & \mathbf{0} & \mathbf{0} \\ \mathbf{3J} & \mathbf{2J} & \mathbf{0} & \mathbf{4J} & \mathbf{3J} & \mathbf{0} & \mathbf{3J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{3J} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}.$$

Let $\mathbf{x}$ be a Perron eigenvector corresponding to $\rho := \varepsilon_1\big(T_t(n',4,a,b,c)\big)$, whose coordinate with respect to vertex v is x_v. For each $t_i, t_i' \in T_i (i = 1,2)$, $u,u' \in U$, $y,y' \in Y$, $w,w' \in W$. Thus,

$$\begin{cases} \rho x_{v_1} &= \rho x_{t_1} = 3\sum_{w\in W} x_w; \\ \rho x_{v_2} &= 2\sum_{u\in U} x_u + 2\sum_{w\in W} x_w; \\ \rho x_{v_3} &= \rho x_{t_2} = 3\sum_{u\in U} x_u; \\ \rho x_u &= 2x_{v_2} + 3x_{v_3} + 3\sum_{y\in Y} x_y + 4\sum_{w\in W} x_w + 3\sum_{t_2\in T_2} x_{t_2}, \quad x_u = x_{u'}; \\ \rho x_y &= 3\sum_{u\in U} x_u + 3\sum_{w\in W} x_w, \quad x_y = x_{y'}; \\ \rho x_w &= 3x_{v_1} + 2x_{v_2} + 4\sum_{u\in U} x_u + 3\sum_{y\in Y} x_y + 3\sum_{t_1\in T_1} x_{t_1}, \quad x_w = x_{w'}. \end{cases}$$

We can simplify it to get

$$\begin{cases} \rho x_{v_1} &= \rho x_{t_1} = 3(c+1)t x_w; \\ \rho x_{v_2} &= 2(a+1)t x_u + 2(c+1)t x_w; \\ \rho x_{v_3} &= \rho x_{t_2} = 3(a+1)t x_u; \\ \rho x_u &= 2x_{v_2} + 3t x_{v_3} + 3bt x_y + 4(c+1)t x_w; \\ \rho x_y &= 3(a+1)t x_u + 3(c+1)t x_w; \\ \rho x_w &= 3t x_{v_1} + 2x_{v_2} + 4(a+1)t x_u + 3bt x_y. \end{cases} \tag{2.1}$$

Then ρ is the largest eigenvalue of

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 3(c+1)t \\ 0 & 0 & 0 & 2(a+1)t & 0 & 2(c+1)t \\ 0 & 0 & 0 & 3(a+1)t & 0 & 0 \\ 0 & 2 & 3t & 0 & 3bt & 4(c+1)t \\ 0 & 0 & 0 & 3(a+1)t & 0 & 3(c+1)t \\ 3t & 2 & 0 & 4(a+1)t & 3bt & 0 \end{pmatrix}.$$

Based on equation (2.1). Firstly we can speculate

$$\begin{aligned}\rho x_u &= 2x_{v_2} + 3tx_{v_3} + 3btx_y + 4(c+1)tx_w\\ &= 4t\big[(a+1)x_u + (c+1)x_w\big] + 9(a+1)t^2x_u + 9bt^2\big[(a+1)x_u + (c+1)x_w\big] + 4(c+1)tx_w\\ &= (a+1)\big(4+9t+9bt\big)tx_u + (c+1)\big(8+9bt\big)tx_w,\end{aligned}$$

and

$$\begin{aligned}\rho x_w &= 3tx_{v_1} + 2x_{v_2} + 4(a+1)tx_u + 3btx_y\\ &= 9(c+1)t^2x_w + 4t\big[(a+1)x_u + (c+1)x_w\big] + 4(a+1)tx_u + 9bt^2\big[(a+1)x_u + (c+1)x_w\big]\\ &= (c+1)\big(9bt+9t+4\big)tx_w + (a+1)\big(8+9bt\big)tx_u.\end{aligned}$$

Since $x_u > 0$ and $x_w > 0$ by Perron Theorem, one has

$$\frac{x_u}{x_w} = \frac{(c+1)\big(8+9bt\big)t}{\rho - (a+1)\big(4+9t+9bt\big)t} > 0,$$

and

$$\frac{x_u}{x_w} = \frac{\rho - (c+1)\big(4+9t+9bt\big)t}{(a+1)\big(8+9bt\big)t} > 0.$$

Thus we get $\rho > (c+1)\big(4+9t+9bt\big)t$ and $4\rho - 9t > 0$.

Let

$$\begin{aligned}f_{a,b,c}(\lambda) &= \begin{vmatrix} -\lambda & 0 & 0 & 0 & 0 & 3(c+1)t\\ 0 & -\lambda & 0 & 2(a+1)t & 0 & 2(c+1)t\\ 0 & 0 & -\lambda & 3(a+1)t & 0 & 0\\ 0 & 2 & 3t & -\lambda & 3bt & 4(c+1)t\\ 0 & 0 & 0 & 3(a+1)t & -\lambda & 3(c+1)t\\ 3t & 2 & 0 & 4(a+1)t & 3bt & -\lambda \end{vmatrix}\\ &= \lambda^2\Big\{\lambda^4 - [8 + (a+c)(9bt+25t+4) + 34t + 18bt + 16act]t\lambda^2\\ &\quad - [32 + (a+c)(72bt+32) + 32ac + 72bt + 72abct]t^2\lambda\\ &\quad + [72 + (a+c)(162bt+81t+72) + 72ac + 81t + 162bt + 81act + 162abct]t^3\Big\}.\end{aligned}$$

Claim 2.1. $a = 0$.

Suppose $a \geq 1$, let $\widetilde{\rho}$ be the largest root of $f_{a-1,b,c+1}(\lambda) = 0$. Note that $a - c - 1 < 0$ and $4\widetilde{\rho} - 9t > 0$, $\widetilde{\rho} = \varepsilon_1\big(T_t(n', 4, a-1, b, c+1)\big) \geq 4$ from Lemma 2.3. Hence, we have

$$\begin{aligned} f_{a,b,c}(\widetilde{\rho}) &= f_{a,b,c}(\widetilde{\rho}) - f_{a-1,b,c+1}(\widetilde{\rho}) \\ &= (a-c-1)\widetilde{\rho}^2 t^2\big[16\widetilde{\rho}^2 + 72bt\widetilde{\rho} - (72 + 81t + 162bt)t\big] \\ &= (a-c-1)\widetilde{\rho}^2 t^2\big(4\widetilde{\rho} - 9t\big)\big(4\widetilde{\rho} + 18bt + 9t + 8\big) \\ &< 0. \end{aligned}$$

Which implies $\widetilde{\rho} = \varepsilon_1\big(T_t(n', 4, a-1, b, c+1)\big) < \varepsilon_1\big(T_t(n', 4, a, b, c)\big)$, a contradiction.

Claim 2.2. $c = 0$.

Suppose $c \geq 1$, let $\widetilde{\rho'}$ be the largest root of $f_{0,b+c,0}(\lambda) = 0$. Then $\widetilde{\rho'} = \varepsilon_1\big(T_t(n', 4, 0, b+c, 0)\big) \geq 4$ from Lemma 2.3.

Thus,

$$\begin{aligned} f_{0,b,c}(\widetilde{\rho'}) &= f_{0,b,c}(\widetilde{\rho'}) - f_{0,b+c,0}(\widetilde{\rho'}) \\ &= -ct\widetilde{\rho'}^2\Big[\big(25t + 9b - 14\big)\widetilde{\rho'}^2 + \big(72bt - 72t + 32\big)t\widetilde{\rho'} - (72 - 81t + 162bt)t^2\Big] \\ &\leq -4^2 \cdot ct\Big[16\big(25t + 9b - 14\big) + 4\big(72bt - 72t + 32\big)t - (72 - 81t + 162bt)t^2\Big] \\ &= -16ct\Big[81\big(1 - 2b\big)t^3 + 72\big(4b - 5\big)t^2 + 672t - 224\Big] \\ &\leq -16c\Big[81\big(1 - 2b\big) + 72\big(4b - 5\big) + 448\Big] \\ &= -16c\big(270b + 25\big) \\ &< 0. \end{aligned}$$

Which implies $\widetilde{\rho'} = \varepsilon_1\big(T_t(n', 4, 0, b+c, 0)\big) < \varepsilon_1\big(T_t(n', 4, 0, b, c)\big)$, a contradiction..

From the above discussion, we can come to the conclusion that the ε-spectral radius of t-clique tree $\Phi_t(n', 4)$ achieves the minimum value when $b = n - 5$ and $a = c = 0$. □

3. The minimum ε-spectral radius of clique trees with diameter $d \geq 4$

In section, firstly we characterize the extremal graphs with diameter $d \geq 5$ having the minimum ε-spectral radius. Nextly we determine the lower bound of ε-spectral radius of t-clique trees when $n - d$ is odd(see Corollaries 3.6 and 3.8).

Lemma 3.1. [30] *Let T_t be a t-clique tree with order n'. Then $\varepsilon(T_t)$ is irreducible.*

Lemma 3.2. [6] *Let M and N be the nonnegative irreducible matrices with same order. If $(N)_{ij} \leq (M)_{ij}$ for each i, j, then $\rho(N) \leq \rho(M)$ with equality if and only if $M = N$, where $\rho(N)$ and $\rho(M)$ denote the spectral radius of N and M, respectively.*

Lemma 3.3. *Let $P_{d+1} = v_0, v_1, \ldots, v_d$ be a diametrical path of graph $\Gamma \in \Phi(n, d)(d \geq 4)$, and let Γ_j be the connected component of $\Gamma - E(P_{d+1})$ and $v_j \in V(\Gamma_j)(0 \leq j \leq d)$. Assume there exists a vertex u_1 in connected component Γ_i such that $d_{\Gamma_i}(v_i, u_1) = e_{\Gamma_i}(v_i) \geq 2$, obviously $d_\Gamma(u_1) = 1$. Denote the vertex $u \sim u_1$ and $N_\Gamma(u) = \{u_0, u_1, u_2, \ldots, u_s\}$, where $d_\Gamma(u_0) \geq 2$ and $d_\Gamma(u_j) = 1$ for $1 \leq j \leq s$. Let*

$$\widetilde{\Gamma} = \Gamma - \{uu_j | 1 \leq j \leq s\} + \{u_0 u_j | 1 \leq j \leq s\}.$$

Then $\varepsilon_1(\widetilde{\Gamma}') < \varepsilon_1(\Gamma')$, where Γ' and $\widetilde{\Gamma}'$ are the t-clique trees corresponding to Γ and $\widetilde{\Gamma}$, see Figure 3.

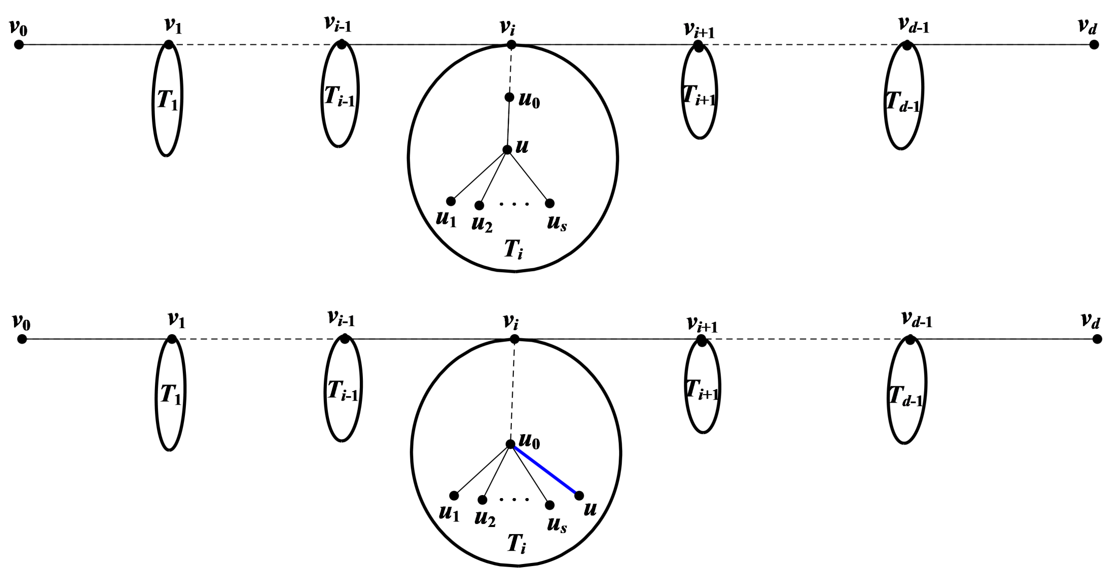

FIGURE 3. Trees Γ and $\widetilde{\Gamma}$

Proof. The t-clique trees obtained by substituting each edge of trees Γ and $\widetilde{\Gamma}$ with a clique $K_{t+1}(t \geq 1)$ are denoted by Γ' and $\widetilde{\Gamma}'$, respectively. Let $U = N_{\Gamma'}[u_0] \cap N_{\Gamma'}[u]$ and $W_i = N_{\Gamma'}[u_i]$, $\{u, u_i\} \subseteq W_i$ and $W_i' = W_i \backslash \{u\}$, $E_1 = \{uw| \ w \in {W_i}'\}$, $E_2 = \{u_0 w| \ w \in {W_i}'\}$, where $1 \leq i \leq s$. Then

$$\widetilde{\Gamma}' = \Gamma' - E_1 + E_2 \text{ and } V(\widetilde{\Gamma}') = V(\Gamma').$$

It's not difficult to learn that $e_{\Gamma'}(w) = e_{\widetilde{\Gamma}'}(w)$ for each vertex $w \in V(\Gamma') \backslash \bigcup_{1 \leq i \leq s} W_i'$, and $d_{\Gamma'}(w, w') = d_{\widetilde{\Gamma}'}(w, w')$ for $\{w, w'\} \subseteq V(\Gamma') \backslash \bigcup_{1 \leq i \leq s} W_i'$. Therefore, $\big(\varepsilon(\Gamma')\big)_{ww'} = \big(\varepsilon(\widetilde{\Gamma}')\big)_{ww'}$ for each $\{w, w'\} \subseteq V(\Gamma') \backslash \bigcup_{1 \leq i \leq s} W_i'$.

Note that

$$d_{\Gamma'}(w, w') = 1 < 4 \leq e_{\Gamma'}(w) = e_{\Gamma'}(w'),$$

$$d_{\widetilde{\Gamma}'}(w, w') = 1 < 3 \leq e_{\widetilde{\Gamma}'}(w) = e_{\widetilde{\Gamma}'}(w').$$

Hence, $\big(\varepsilon(\widetilde{\Gamma}')\big)_{ww'} = \big(\varepsilon(\Gamma')\big)_{ww'} = 0$ for $\{w, w'\} \subseteq W_i'(1 \leq i \leq s)$.

For $w \in W_i'$ and $w' \in W_j'(1 \leq i < j \leq s)$, we have

$$d_{\Gamma'}(w, w') = 2 < 4 \leq e_{\Gamma'}(w) = e_{\Gamma'}(w'),$$

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$$d_{\widetilde{\Gamma}'}(w,w') = 2 < 3 \le e_{\widetilde{\Gamma}'}(w) = e_{\widetilde{\Gamma}'}(w').$$

Therefore, $\big(\varepsilon(\widetilde{\Gamma}')\big)_{ww'} = \big(\varepsilon(\Gamma')\big)_{ww'} = 0$.

On the one hand, for $u' \in U\backslash\{u_0, u\}$ and $w_i \in W_i'(1 \le i \le s)$.

$$d_{\Gamma'}(w_i,u') = 2 < 3 \le e_{\Gamma'}(u') = e_{\Gamma'}(w_i) - 1 < e_{\Gamma'}(w_i),$$

$$d_{\widetilde{\Gamma}'}(w_i,u') = 2 < 3 \le e_{\widetilde{\Gamma}'}(u') = e_{\widetilde{\Gamma}'}(w_i).$$

Hence, $\big(\varepsilon(\widetilde{\Gamma}')\big)_{u'w_i} = \big(\varepsilon(\Gamma')\big)_{u'w_i} = 0$.

On the other hand, for $w_i \in W_i'(1 \le i \le s)$.

$$d_{\Gamma'}(w_i,u_0) = 2 < 3 \le e_{\Gamma'}(u_0) = e_{\Gamma'}(w_i) - 2 < e_{\Gamma'}(w_i),$$

$$d_{\widetilde{\Gamma}'}(w_i,u_0) = 1 < 3 \le e_{\widetilde{\Gamma}'}(u_0) = e_{\widetilde{\Gamma}'}(w_i) - 1.$$

and

$$d_{\Gamma'}(w_i,u) = 1 < 3 \le e_{\Gamma'}(u) = e_{\Gamma'}(w_i) - 1 < e_{\Gamma'}(w_i),$$

$$d_{\widetilde{\Gamma}'}(w_i,u) = 2 < 3 \le e_{\widetilde{\Gamma}'}(u) = e_{\widetilde{\Gamma}'}(w_i).$$

Hence, $\big(\varepsilon(\widetilde{\Gamma}')\big)_{u_0w_i} = \big(\varepsilon(\Gamma')\big)_{u_0w_i} = 0$ and $\big(\varepsilon(\widetilde{\Gamma}')\big)_{uw_i} = \big(\varepsilon(\Gamma')\big)_{uw_i} = 0$.

For $w \in V(\Gamma')\backslash\big(U \cup W_1' \cup W_2' \cup \cdots \cup W_s'\big)$ and $z \in \bigcup_{1\le i\le s} W_i'$. It's not difficult to learn $d_{\widetilde{\Gamma}'}(z,w) = d_{\Gamma'}(z,w) - 1$. If $d_{\Gamma'}(z,w) = min\{e_{\Gamma'}(z), e_{\Gamma'}(w)\}$, then $\big(\varepsilon(\widetilde{\Gamma}')\big)_{zw} < \big(\varepsilon(\Gamma')\big)_{zw} = d_{\Gamma'}(z,w)$, which leads to $\big(\varepsilon(\widetilde{\Gamma}')\big)_{zw} < \big(\varepsilon(\Gamma')\big)_{zw}$. If $d_{\Gamma'}(z,w) < min\{e_{\Gamma'}(z), e_{\Gamma'}(w)\}$, then we have $\big(\varepsilon(\Gamma')\big)_{zw} = 0$. However, In $\widetilde{\Gamma}'$, one has

$$d_{\widetilde{\Gamma}'}(z,w) = d_{\Gamma'}(z,w) - 1 < d_{\Gamma'}(z,w) < e_{\Gamma'}(w) = e_{\widetilde{\Gamma}'}(w)$$

and

$$d_{\widetilde{\Gamma}'}(z,w) = d_{\Gamma'}(z,w) - 1 < d_{\Gamma'}(z,w) < e_{\Gamma'}(z) - 1 = e_{\widetilde{\Gamma}'}(z).$$

Thus, $d_{\widetilde{\Gamma}'}(z,w) < min\{e_{\widetilde{\Gamma}'}(z), e_{\widetilde{\Gamma}'}(w)\}$ and $\big(\varepsilon(\widetilde{\Gamma}')\big)_{zw} = \big(\varepsilon(\Gamma')\big)_{zw} = 0$.

Therefore, we conclude that $\varepsilon(\widetilde{\Gamma}') < \varepsilon(\Gamma')$. Combining with Lemmas 3.1 and 3.2, we have $\varepsilon_1(\widetilde{\Gamma}') < \varepsilon_1(\Gamma')$. □

Consequently, we find that the t-clique tree corresponding to catherpillar tree may have the minimal eccentricity spectral radius.

Lemma 3.4. *Let $P_{d+1} = v_0, v_1, \ldots, v_d (d \ge 5)$ be a diametrical path of catherpillar tree $T \in \Phi(n,d)$. Assume $d_T(v_i) \ge 3$, $u \in N_T(v_i)\backslash\{v_{i-1}, v_{i+1}\}(1 \le i \le \lfloor \frac{d-2}{2} \rfloor)$. Let*

$$\widetilde{\Gamma} = \Gamma - v_i u + v_{\lfloor \frac{d-1}{2} \rfloor} u.$$

Then $\varepsilon_1(\widetilde{\Gamma}') < \varepsilon_1(\Gamma')$, where Γ' and $\widetilde{\Gamma}'$ are the t-clique trees corresponding to Γ and $\widetilde{\Gamma}$.

Proof. It's not hard to get $\widetilde{\Gamma}' = \Gamma' - \left\{v_i w \middle| w \in N_{\Gamma'}[u]\backslash\{v_i\}\right\} + \left\{v_{\lfloor\frac{d-1}{2}\rfloor} w \middle| w \in N_{\Gamma'}[u]\backslash\{v_i\}\right\}$ and $V(\widetilde{\Gamma}') = V(\Gamma')$.

For $\{v, v'\} \subseteq V(\widetilde{\Gamma}')\backslash\big(N_{\widetilde{\Gamma}'}[u]\backslash\{v_i\}\big)$, $d_{\widetilde{\Gamma}'}(v, v') = d_{\Gamma'}(v, v')$ and $e_{\widetilde{\Gamma}'}(v) = e_{\Gamma'}(v)$. Thus, we have $\big(\varepsilon(\widetilde{\Gamma}')\big)_{vv'} = \big(\varepsilon(\Gamma')\big)_{vv'}$ for $\{v, v'\} \subseteq V(\widetilde{\Gamma}')\backslash\big(N_{\widetilde{\Gamma}'}[u]\backslash\{v_i\}\big)$.

For $x \in N_{\widetilde{\Gamma}'}[u]\backslash\{v_i\}$. If $y \in \widetilde{\Gamma}'(v_{d-1})\backslash\Big(\widetilde{\Gamma}'(v_{d-1}) \cap \widetilde{\Gamma}'[v_{d-2}]\Big)$, one has $d_{\Gamma'}(x, y) = e_{\Gamma'}(x) < e_{\Gamma'}(y) = d$ and $d_{\widetilde{\Gamma}'}(x, y) = e_{\widetilde{\Gamma}'}(x) < e_{\widetilde{\Gamma}'}(y) = d$. Thus

$$\big(\varepsilon(\widetilde{\Gamma}')\big)_{xy} = d_{\widetilde{\Gamma}'}(x, y) = \big(d - \lfloor\frac{d-1}{2}\rfloor\big) + 1 < d - i + 1 = d_{\Gamma'}(x, y) = \big(\varepsilon(\Gamma')\big)_{xy}$$

due to $1 \leq i \leq \lfloor\frac{d-2}{2}\rfloor$. If $y \notin \widetilde{\Gamma}'(v_{d-1})\backslash\Big(\widetilde{\Gamma}'(v_{d-1}) \cap \widetilde{\Gamma}'[v_{d-2}]\Big)$, we have $d_{\widetilde{\Gamma}'}(x, y) < min\{e_{\widetilde{\Gamma}'}(x), e_{\widetilde{\Gamma}'}(y)\}$ and $d_{\Gamma'}(x, y) < min\{e_{\Gamma'}(x), e_{\Gamma'}(y)\}$. Hence, we have $\big(\varepsilon(\widetilde{\Gamma}')\big)_{xy} = \big(\varepsilon(\Gamma')\big)_{xy} = 0$.

Therefore, we obtain $\varepsilon_1(\widetilde{\Gamma}') < \varepsilon_1(\Gamma')$ by Lemmas 3.1 and 3.2. □

We will discuss in terms of the parity of diameter d, we can reach the following conclusions by using Lemmas 3.3 and 3.4.

Theorem 3.5. *Among $\Phi_t(n', d)$ with odd $d \geq 5$, the minimum ε-spectral radius is achieved by the t-clique tree $T_t(n', d, \lfloor\frac{n-d-1}{2}\rfloor, \lceil\frac{n-d-1}{2}\rceil)$.*

Proof. Let t-clique tree corresponding to tree $T \in \Phi(n, d)$ have the minimum ε-spectral radius when $d \geq 5$ is odd, it's easy to get $T \cong T(n, d, a, b)$ and $b \geq a \geq 1$.

Let $P_{d+1} = v_0, v_1, \ldots, v_d$ be a diametrical path of tree $T(n, d, a, b)$. All edges of tree $T(n, d, a, b)$ are replaced by clique $K_{t+1} (t \geq 2)$, the t-clique tree corresponding to $T(n, d, a, b)$ is denoted by $T_t(n', d, a, b)$.

Let

$$T_i = N_{T_t(n',d,a,b)}(v_i) \cap N_{T_t(n',d,a,b)}(v_{i+1});$$
$$U = N_{T_t(n',d,a,b)}(v_{\frac{d-1}{2}})\backslash\big(T_{\frac{d-3}{2}} \cup T_{\frac{d-1}{2}} \cup \{v_{\frac{d-3}{2}}, v_{\frac{d+1}{2}}\}\big);$$
$$W = N_{T_t(n',d,a,b)}(v_{\frac{d+1}{2}})\backslash\big(T_{\frac{d-1}{2}} \cup T_{\frac{d+1}{2}} \cup \{v_{\frac{d-1}{2}}, v_{\frac{d+3}{2}}\}\big).$$

Clearly, $|U| = at$, $|W| = bt$, $|T_i| = t - 1 (0 \leq i \leq d - 1)$.

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By the definition of eccentricity matrix, $\varepsilon\big(T_t(n',d,a,b)\big)$ is equal to

$$
\begin{array}{c|ccccccccccccccccccc}
 & v_0 & v_1 & \dots & v_{\frac{d-1}{2}} & v_{\frac{d+1}{2}} & \dots & v_{d-1} & v_d & U & W & T_0 & T_1 & \dots & T_{\frac{d-3}{2}} & T_{\frac{d-1}{2}} & T_{\frac{d+1}{2}} & \dots & T_{d-1} \\
\hline
v_0 & 0 & 0 & \dots & 0 & \frac{d+1}{2} & \dots & d-1 & d & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \frac{d+1}{2}\mathbf{J} & \frac{d+3}{2}\mathbf{J} & \dots & d\mathbf{J} \\
v_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & d-1 & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & (d-1)\mathbf{J} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{\frac{d-1}{2}} & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \frac{d+1}{2} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \frac{d+1}{2}\mathbf{J} \\
v_{\frac{d+1}{2}} & \frac{d+1}{2} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \mathbf{O} & \mathbf{O} & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{d-1} & d-1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \mathbf{O} & \mathbf{O} & (d-1)\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\
v_d & d & d-1 & \dots & \frac{d+1}{2} & 0 & \dots & 0 & 0 & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & d\mathbf{J} & (d-1)\mathbf{J} & \dots & \frac{d+3}{2}\mathbf{J} & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} \\
U & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \frac{d+3}{2}\mathbf{J} \\
W & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\
T_0 & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \frac{d+1}{2}\mathbf{J} & \dots & (d-1)\mathbf{J} & d\mathbf{J} & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \frac{d+1}{2}\mathbf{J} & \frac{d+3}{2}\mathbf{J} & \dots & d\mathbf{J} \\
T_1 & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & (d-1)\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & (d-1)\mathbf{J} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T_{\frac{d-3}{2}} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \frac{d+3}{2}\mathbf{J} \\
T_{\frac{d-1}{2}} & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \frac{d+1}{2}\mathbf{J} \\
T_{\frac{d+1}{2}} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \dots & \mathbf{O} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
T_{d-1} & d\mathbf{J} & (d-1)\mathbf{J} & \dots & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O} & \mathbf{O} & \frac{d+3}{2}\mathbf{J} & \mathbf{O} & d\mathbf{J} & (d-1)\mathbf{J} & \dots & \frac{d+3}{2}\mathbf{J} & \frac{d+1}{2}\mathbf{J} & \mathbf{O} & \dots & \mathbf{O}
\end{array}
$$

Let $\mathbf{x}$ be a Perron eigenvector cornesponding to $\rho := \varepsilon_1\big(T_t(n',d,a,b)\big)$, whose coordinate with respect to vertex v is x_v. For $u,u' \in U$, $w,w' \in W$, $t_j,t'_j \in T_j (0 \le j \le d-1)$, we have $x_u = x_{u'}$, $x_w = x_{w'}$, $x_{t_j} = x_{t'_j}$. Then we can obtain

$$
\begin{cases}
\rho x_{v_0} &= \rho x_{t_0} = \frac{d+1}{2} x_{v_{\frac{d+1}{2}}} + \cdots + d x_{v_d} + \frac{d+3}{2}\sum_{w\in W} x_w + \sum_{\frac{d-1}{2}\le j\le d-1}(j+1)\sum_{t_j\in T_j} x_{t_j};\\
\rho x_{v_1} &= \rho x_{t_1} = (d-1)x_{v_d} + (d-1)\sum_{t\in T_{d-1}} x_t;\\
 & \vdots\\
\rho x_{v_{\frac{d-3}{2}}} &= \rho x_{t_{\frac{d-3}{2}}} = \frac{d+3}{2}x_{v_d} + \frac{d+3}{2}\sum_{t\in T_{d-1}} x_t;\\
\rho x_{v_{\frac{d-1}{2}}} &= \frac{d+1}{2}x_{v_d} + \frac{d+1}{2}\sum_{t\in T_{d-1}} x_t;\\
\rho x_{v_{\frac{d+1}{2}}} &= \frac{d+1}{2}x_{v_0} + \frac{d+1}{2}\sum_{t\in T_0} x_t;\\
\rho x_{t_{\frac{d-1}{2}}} &= \frac{d+1}{2}(x_{v_d} + x_{v_0}) + \frac{d+1}{2}\sum_{t\in T_0\cup T_{d-1}} x_t;\\
\rho x_{v_{\frac{d+3}{2}}} &= \rho x_{t_{\frac{d+1}{2}}} = \frac{d+3}{2}x_{v_0} + \frac{d+3}{2}\sum_{t\in T_0} x_t;\\
 & \vdots\\
\rho x_{v_{d-1}} &= \rho x_{t_{d-2}} = (d-1)v_0 + (d-1)\sum_{t\in T_0} x_t;\\
\rho x_{v_d} &= \rho x_{t_{d-1}} = d x_{v_0} + \cdots + \frac{d+1}{2} x_{v_{\frac{d-1}{2}}} + \frac{d+3}{2}\sum_{u\in U} x_u + \sum_{0\le j\le \frac{d-1}{2}}(d-j)\sum_{t_j\in T_j} x_{t_j};\\
\rho x_u &= \frac{d+3}{2}x_{v_d} + \frac{d+3}{2}\sum_{t\in T_{d-1}} x_t;\\
\rho x_w &= \frac{d+3}{2}x_{v_0} + \frac{d+3}{2}\sum_{t\in T_0} x_t.
\end{cases}
$$

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Simplify, we have

$$\begin{cases}
\rho x_{v_0} &= \rho x_{t_0} = \frac{d+1}{2}x_{v_{\frac{d+1}{2}}} + \cdots + dx_{v_d} + \frac{d+3}{2}btx_w + \sum_{\frac{d-1}{2}\leq j\leq d-1}(j+1)\sum_{t_j\in T_j}x_{t_j};\\
\rho x_{v_1} &= \rho x_{t_1} = (d-1)tx_{v_d};\\
 &\vdots\\
\rho x_{v_{\frac{d-3}{2}}} &= \rho x_{t_{\frac{d-3}{2}}} = \frac{d+3}{2}tx_{v_d};\\
\rho x_{v_{\frac{d-1}{2}}} &= \frac{d+1}{2}tx_{v_d};\\
\rho x_{v_{\frac{d+1}{2}}} &= \frac{d+1}{2}tx_{v_0};\\
\rho x_{t_{\frac{d-1}{2}}} &= \frac{d+1}{2}t(x_{v_d}+x_{v_0});\\
\rho x_{v_{\frac{d+3}{2}}} &= \rho x_{t_{\frac{d+1}{2}}} = \frac{d+3}{2}tx_{v_0};\\
 &\vdots\\
\rho x_{v_{d-1}} &= \rho x_{t_{d-2}} = (d-1)tx_{v_0};\\
\rho x_{v_d} &= \rho x_{t_{d-1}} = dx_{v_0} + \cdots + \frac{d+1}{2}x_{v_{\frac{d-1}{2}}} + \frac{d+3}{2}atx_u + \sum_{0\leq j\leq\frac{d-1}{2}}(d-j)\sum_{t_j\in T_j}x_{t_j};\\
\rho x_u &= \frac{d+3}{2}tx_{v_d};\\
\rho x_w &= \frac{d+3}{2}tx_{v_0}.
\end{cases}$$

Let $\Delta(d) := (\frac{d+1}{2})^2 + (\frac{d+3}{2})^2 + \cdots + (d-1)^2 = \frac{d(d-1)(7d-5)}{24}$. Hence,

$$\begin{aligned}
\rho^2 x_{v_0} &= \rho\Big[\frac{d+1}{2}x_{v_{\frac{d+1}{2}}} + \cdots + (d-1)x_{v_{d-1}} + dx_{v_d} + \frac{d+3}{2}btx_w + \sum_{\frac{d-1}{2}\leq j\leq d-1}(j+1)\sum_{t_j\in T_j}x_{t_j}\Big]\\
&= \Big\{\Big[(\frac{d+1}{2})^2+\cdots+(d-1)^2\Big]tx_{v_0} + d\rho x_{v_d}\Big\} + (\frac{d+3}{2})^2bt^2x_{v_0}\\
&\quad + (t-1)\Big\{(\frac{d+1}{2})^2tx_{v_d} + \Big[(\frac{d+1}{2})^2+\cdots+(d-1)^2\Big]tx_{v_0} + d\rho x_{v_d}\Big\}\\
&= \big[\Delta(d) + (\frac{d+3}{2})^2b\big]t^2x_{v_0} + \big[\rho d + (\frac{d+1}{2})^2(t-1)\big]tx_d,
\end{aligned}$$

and

$$\begin{aligned}
\rho^2 x_{v_d} &= \rho\Big[dx_{v_0} + (d-1)x_{v_1} + \cdots + \frac{d+1}{2}x_{v_{\frac{d-1}{2}}} + \frac{d+3}{2}atx_u + \sum_{0\leq j\leq\frac{d-1}{2}}(d-j)\sum_{t_j\in T_j}x_{t_j}\Big]\\
&= \Big\{d\rho x_{v_0} + \Big[(d-1)^2+\cdots+(\frac{d+1}{2})^2\Big]tx_{v_d}\Big\} + (\frac{d+3}{2})^2at^2x_{v_d}\\
&\quad + (t-1)\Big\{d\rho x_{v_0} + \Big[(d-1)^2+\cdots+(\frac{d+1}{2})^2\Big]tx_{v_d} + (\frac{d+1}{2})^2tx_{v_0}\Big\}\\
&= \big[\Delta(d) + (\frac{d+3}{2})^2a\big]t^2x_{v_d} + \big[\rho d + (\frac{d+1}{2})^2(t-1)\big]tx_{v_0}.
\end{aligned}$$

That is,

$$(3.1)\qquad \begin{cases} \rho^2 x_{v_0} - \big[\Delta(d) + (\frac{d+3}{2})^2 b\big] t^2 x_{v_0} - \big[\rho d + (\frac{d+1}{2})^2 (t-1)\big] t x_{v_d} = 0; \\ \\ \rho^2 x_{v_d} - \big[\Delta(d) + (\frac{d+3}{2})^2 a\big] t^2 x_{v_d} - \big[\rho d + (\frac{d+1}{2})^2 (t-1)\big] t x_{v_0} = 0. \end{cases}$$

Since $x_{v_0} \neq 0$ and $x_{v_d} \neq 0$, ρ is the largest root of $f_a(\lambda) = 0$. Let

$$(3.2)\qquad f_a(\lambda) = \begin{vmatrix} \lambda^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 b\big] t^2 & -\big[d\lambda + (\frac{d+1}{2})^2 (t-1)\big] t \\ \\ -\big[d\lambda + (\frac{d+1}{2})^2 (t-1)\big] t & \lambda^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 a\big] t^2 \end{vmatrix}.$$

Similarly, let ρ' be the largest root of $f_{a-1}(\lambda) = 0$. By calculation and $a + b = n - d - 1$, we obtain

$$\begin{aligned} f_{a-1}(\rho) &= f_{a-1}(\rho) - f_a(\rho) \\ &= \Big[\rho^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 (b+1)\big] t^2\Big]\Big[\rho^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 (a-1)\big] t^2\Big] \\ &\quad - \Big[\rho^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 b\big] t^2\Big]\Big[\rho^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 a\big] t^2\Big] \\ &= (a-b-1)(\frac{d+3}{2})^4 t^4 \\ &< 0. \end{aligned}$$

Therefore, $\rho' > \rho$. This means that t-clique tree $T_t(n', d, \lfloor\frac{n-d-1}{2}\rfloor, \lceil\frac{n-d-1}{2}\rceil)$ with odd $d \geq 5$ achieves the minimal ε-spectral radius.

□

The next result immediately follows from Theorem 3.5.

Corollary 3.6. *Let n be even and T_t be an n'-vectex t-clique tree with odd $d \geq 5$. Let*

$$\Delta(d) := (\frac{d+1}{2})^2 + (\frac{d+3}{2})^2 + \cdots + (d-1)^2 = \frac{d(d-1)(7d-5)}{24} \text{ and } n' = (n-1)t + 1.$$

Then

$$\varepsilon_1(T_t) \geq \frac{1}{2}\Big(dt + \sqrt{d^2t^2 + \big[4\Delta(d)t + (d+3)^2 at + (d+1)^2(t-1)\big]t}\Big).$$

with equality if and only if $T_t \cong T_t(n', d, a, a)$ and $a = \frac{n-d-1}{2}$.

Proof. If $n - d$ is odd and $a = \frac{n-d-1}{2}$. By equation (3.2), we have

$$\begin{aligned} f_a(\lambda) &= \Big[\lambda^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 a\big] t^2\Big]^2 - \Big[\big[d\lambda + (\frac{d+1}{2})^2 (t-1)\big] t\Big]^2 \\ &= \Big\{\lambda^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 a\big] t^2 - \big[d\lambda + (\frac{d+1}{2})^2 (t-1)\big] t\Big\} \\ &\quad \times \Big\{\lambda^2 - \big[\Delta(d) + (\frac{d+3}{2})^2 a\big] t^2 + \big[d\lambda + (\frac{d+1}{2})^2 (t-1)\big] t\Big\}. \end{aligned}$$

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Let ρ be the largest root of $f_a(\lambda) = 0$. By calculation, one has

$$\rho = \frac{1}{2}\Big(dt + \sqrt{d^2t^2 + \big[4\Delta(d)t + (d+3)^2at + (d+1)^2(t-1)\big]t}\Big).$$

□

Theorem 3.7. *Among $\Phi_t(n', d)$ with even $d \geq 6$, the minimum ε-spectral radius is achieved by the t-clique tree $T_t(n', d, \lfloor \frac{n-d-1}{2} \rfloor, 0, \lceil \frac{n-d-1}{2} \rceil)$.*

Proof. Let t-clique tree corresponding to tree $T \in \Phi(n, d)$ have the minimum ε-spectral radius when $d \geq 6$ is even, it's easy to get $T \cong T(n, d, a, b, c)$ and $c \geq a \geq 1$, $b \geq 0$.

Let $P_{d+1} = v_0, v_1, \ldots, v_d$ be a diametrical path of tree $T(n, d, a, b, c)$. All edges of tree $T(n, d, a, b, c)$ are replaced by clique $K_{t+1} (t \geq 2)$, the t-clique tree corresponding to $T(n, d, a, b, c)$ is denoted by $T_t(n', d, a, b, c)$.

Let

$$\begin{aligned}
T_i &= N_{T_t(n',d,a,b,c)}(v_i) \cap N_{T_t(n',d,a,b,c)}(v_{i+1}),\\
U &= N_{T_t(n',d,a,b,c)}(v_{\frac{d-2}{2}}) \backslash \big(T_{\frac{d-4}{2}} \cup T_{\frac{d-2}{2}} \cup \{v_{\frac{d-4}{2}}, v_{\frac{d}{2}}\}\big),\\
Y &= N_{T_t(n',d,a,b,c)}(v_{\frac{d}{2}}) \backslash \big(T_{\frac{d-2}{2}} \cup T_{\frac{d}{2}} \cup \{v_{\frac{d-2}{2}}, v_{\frac{d+2}{2}}\}\big),\\
W &= N_{T_t(n',d,a,b,c)}(v_{\frac{d+2}{2}}) \backslash \big(T_{\frac{d}{2}} \cup T_{\frac{d+2}{2}} \cup \{v_{\frac{d}{2}}, v_{\frac{d+2}{2}}\}\big).
\end{aligned}$$

Clearly, $|U| = at$, $|Y| = bt$, $|W| = ct$, $|T_i| = t - 1 (\ 0 \leq i \leq d-1)$.

By the definition of eccentricity matrix, $\varepsilon\big(T_t(n', d, a, b, c)\big)$ is written as

$$\begin{array}{c|cccccccccccccccccc}
 & v_0 & v_1 & \cdots & v_{\frac{d}{2}} & v_{\frac{d+2}{2}} & \cdots & v_{d-1} & v_d & U & Y & W & T_0 & T_1 & \cdots & T_{\frac{d-2}{2}} & T_{\frac{d}{2}} & \cdots & T_{d-1}\\
\hline
v_0 & 0 & 0 & \cdots & \frac{d}{2} & \frac{d+2}{2} & \cdots & d-1 & d & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \frac{d+4}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \cdots & d\mathbf{J}\\
v_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & d-1 & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & (d-1)\mathbf{J}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
v_{\frac{d}{2}} & \frac{d}{2} & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{d}{2} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \frac{d}{2}\mathbf{J}\\
v_{\frac{d+2}{2}} & \frac{d+2}{2} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
v_{d-1} & d-1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \mathbf{O} & \mathbf{O} & \mathbf{O} & (d-1)\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O}\\
v_d & d & d-1 & \cdots & \frac{d}{2} & 0 & \cdots & 0 & 0 & \frac{d+4}{2}\mathbf{J} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & d\mathbf{J} & (d-1)\mathbf{J} & \cdots & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O}\\
U & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \frac{d+4}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \frac{d+4}{2}\mathbf{J}\\
Y & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \frac{d+2}{2}\mathbf{J}\\
W & \frac{d+4}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d+4}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O}\\
T_0 & \mathbf{O} & \mathbf{O} & \cdots & \frac{d}{2}\mathbf{J} & \frac{d+2}{2}\mathbf{J} & \cdots & (d-1)\mathbf{J} & d\mathbf{J} & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \frac{d+4}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \cdots & d\mathbf{J}\\
T_1 & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & (d-1)\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & (d-1)\mathbf{J}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
T_{\frac{d-2}{2}} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \frac{d+2}{2}\mathbf{J}\\
T_{\frac{d}{2}} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
T_{d-1} & d\mathbf{J} & (d-1)\mathbf{J} & \cdots & \frac{d}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \frac{d+4}{2}\mathbf{J} & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & d\mathbf{J} & (d-1)\mathbf{J} & \cdots & \frac{d+2}{2}\mathbf{J} & \mathbf{O} & \cdots & \mathbf{O}
\end{array}$$

Let $\mathbf{x}$ be a Perron eigenvector cornensponding to $\rho := \varepsilon_1\big(T_t(n', d, a, b, c)\big)$, whose coordinate with respect to vertex v is x_v. For $u, u' \in U$, $y, y' \in Y$, $w, w' \in W$, $t_j, t'_j \in T_j (0 \leq j \leq d-1)$, we have $x_u = x_{u'}$, $x_y = x_{y'}$,

$x_w = x_{w'}$, $x_{t_j} = x_{t'_j}$. Then we can obtain

$$\begin{cases} \rho x_{v_0} & = \rho x_{t_0} = \frac{d}{2}x_{v_{\frac{d}{2}}} + \cdots + dx_{v_d} + \frac{d+2}{2}\sum_{y\in Y} x_y + \frac{d+4}{2}\sum_{w\in W} x_w + \sum_{\frac{d}{2}\le j\le d-1}(j+1)\sum_{t_j\in T_j} x_{t_j}; \\ \rho x_{v_1} & = \rho x_{t_1} = (d-1)x_{v_d} + (d-1)\sum_{t\in T_{d-1}} x_t; \\ & \vdots \\ \rho x_{v_{\frac{d-2}{2}}} & = \rho x_{t_{\frac{d-2}{2}}} = \frac{d+2}{2}x_{v_d} + \frac{d+2}{2}\sum_{t\in T_{d-1}} x_t; \\ \rho x_{v_{\frac{d}{2}}} & = \frac{d}{2}(x_{v_0} + x_{v_d}) + \frac{d}{2}\sum_{t\in T_0\cup T_{d-1}} x_t; \\ \rho x_{v_{\frac{d+2}{2}}} & = \rho x_{t_{\frac{d}{2}}} = \frac{d+2}{2}x_{v_0} + \frac{d+2}{2}\sum_{t\in T_0} x_t; \\ & \vdots \\ \rho x_{v_{d-1}} & = \rho x_{t_{d-2}} = (d-1)x_{v_0} + (d-1)\sum_{t\in T_0} x_t; \\ \rho x_{v_d} & = \rho x_{t_{d-1}} = dx_{v_0} + \cdots + \frac{d}{2}x_{v_{\frac{d}{2}}} + \frac{d+4}{2}\sum_{u\in U} x_u + \frac{d+2}{2}\sum_{y\in Y} x_y + \sum_{0\le j\le \frac{d-2}{2}}(d-j)\sum_{t_j\in T_j} x_{t_j}; \\ \rho x_u & = \frac{d+4}{2}x_{v_d} + \frac{d+4}{2}\sum_{t\in T_{d-1}} x_t; \\ \rho x_y & = \frac{d+2}{2}(x_{v_0} + x_{v_d}) + \frac{d+2}{2}\sum_{t\in T_0\cup T_{d-1}} x_t; \\ \rho x_w & = \frac{d+4}{2}x_{v_0} + \frac{d+4}{2}\sum_{t\in T_0} x_t. \end{cases}$$

Simplify, we have

$$\begin{cases} \rho x_{v_0} & = \rho x_{t_0} = \frac{d}{2}x_{v_{\frac{d}{2}}} + \cdots + dx_{v_d} + \frac{d+2}{2}btx_y + \frac{d+4}{2}ctx_w + \sum_{\frac{d}{2}\le j\le d-1}(j+1)\sum_{t_j\in T_j} x_{t_j}; \\ \rho x_{v_1} & = \rho x_{t_1} = (d-1)tx_{v_d}; \\ & \vdots \\ \rho x_{v_{\frac{d-2}{2}}} & = \rho x_{t_{\frac{d-2}{2}}} = \frac{d+2}{2}tx_{v_d}; \\ \rho x_{v_{\frac{d}{2}}} & = \frac{d}{2}t(x_{v_0} + x_{v_d}); \\ \rho x_{v_{\frac{d+2}{2}}} & = \rho x_{t_{\frac{d}{2}}} = \frac{d+2}{2}tx_{v_0}; \\ & \vdots \\ \rho x_{v_{d-1}} & = \rho x_{t_{d-2}} = (d-1)tx_{v_0}; \\ \rho x_{v_d} & = \rho x_{t_{d-1}} = dx_{v_0} + \cdots + \frac{d}{2}x_{v_{\frac{d}{2}}} + \frac{d+4}{2}atx_u + \frac{d+2}{2}btx_y + \sum_{0\le j\le \frac{d-2}{2}}(d-j)\sum_{t_j\in T_j} x_{t_j}; \\ \rho x_u & = \frac{d+4}{2}tx_{v_d}; \\ \rho x_y & = \frac{d+2}{2}t(x_{v_0} + x_{v_d}); \\ \rho x_w & = \frac{d+4}{2}tx_{v_0}. \end{cases}$$

Let $\Delta(d) := (\frac{d+2}{2})^2 + (\frac{d+4}{2})^2 + \cdots + (d-1)^2 = \frac{d(d-2)(7d-1)}{24}$. Hence,

$$\begin{aligned}
\rho^2 x_{v_0} &= \rho\Big[\frac{d}{2}x_{v_{\frac{d}{2}}} + \cdots + (d-1)x_{v_{d-1}} + dx_{v_d} + \frac{d+2}{2}btx_y + \frac{d+4}{2}ctx_w + \sum_{\frac{d}{2}\le j\le d-1}(j+1)\sum_{t_j\in T_j}x_{t_j}\Big] \\
&= \Big\{(\frac{d}{2})^2 t(x_{v_0}+x_{v_d}) + \Big[(\frac{d+2}{2})^2 + \cdots + (d-1)^2\Big]tx_{v_0} + d\rho x_{v_d}\Big\} + (\frac{d+2}{2})^2bt^2(x_{v_0}+x_{v_d}) \\
&\quad + (\frac{d+4}{2})^2ct^2x_{v_0} + (t-1)\Big\{\Big[(\frac{d+2}{2})^2 + \cdots + (d-1)^2\Big]tx_{v_0} + d\rho x_{v_d}\Big\} \\
&= \big[(\frac{d}{2})^2t + \Delta(d)t^2 + (\frac{d+2}{2})^2bt^2 + (\frac{d+4}{2})^2ct^2\big]x_{v_0} + \big[\rho dt + (\frac{d}{2})^2t + (\frac{d+2}{2})^2bt^2\big]x_d.
\end{aligned}$$

and

$$\begin{aligned}
\rho^2 x_{v_d} &= \rho\Big[dx_{v_0} + (d-1)x_{v_1} + \cdots + \frac{d}{2}x_{v_{\frac{d}{2}}} + \frac{d+4}{2}atx_u + \frac{d+2}{2}btx_y + \sum_{0\le j\le \frac{d-2}{2}}(d-j)\sum_{t_j\in T_j}x_{t_j}\Big] \\
&= \Big\{d\rho x_{v_0} + \Big[(d-1)^2 + \cdots + (\frac{d+2}{2})^2\Big]tx_{v_d} + (\frac{d}{2})^2t(x_{v_0}+x_{v_d})\Big\} + (\frac{d+4}{2})^2at^2x_{v_d} \\
&\quad + (\frac{d+2}{2})^2bt^2(x_{v_0}+x_{v_d}) + (t-1)\Big\{d\rho x_{v_0} + \Big[(d-1)^2 + \cdots + (\frac{d+2}{2})^2\Big]tx_{v_d}\Big\} \\
&= \big[(\frac{d}{2})^2t + \Delta(d)t^2 + (\frac{d+2}{2})^2bt^2 + (\frac{d+4}{2})^2at^2\big]x_{v_d} + \big[\rho dt + (\frac{d}{2})^2t + (\frac{d+2}{2})^2bt^2\big]x_{v_0}.
\end{aligned}$$

That is,

$$(3.3)\qquad \begin{cases} \rho^2x_{v_0} - \big[(\frac{d}{2})^2t + \Delta(d)t^2 + (\frac{d+2}{2})^2bt^2 + (\frac{d+4}{2})^2ct^2\big]x_{v_0} - \big[\rho dt + (\frac{d}{2})^2t + (\frac{d+2}{2})^2bt^2\big]x_d; \\ \\ \rho^2x_{v_d} - \big[(\frac{d}{2})^2t + \Delta(d)t^2 + (\frac{d+2}{2})^2bt^2 + (\frac{d+4}{2})^2at^2\big]x_{v_d} - \big[\rho dt + (\frac{d}{2})^2t + (\frac{d+2}{2})^2bt^2\big]x_{v_0}. \end{cases}$$

Since $x_{v_0} \neq 0$ and $x_{v_d} \neq 0$, ρ is the largest root of $f_{a,b,c}(\lambda) = 0$. Let

$$(3.4)\qquad f_{a,b,c}(\lambda) = \begin{vmatrix} \lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2bt + (\frac{d+4}{2})^2ct\big]t & -\big[d\lambda + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t \\ \\ -\big[d\lambda + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t & \lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2bt + (\frac{d+4}{2})^2at\big]t \end{vmatrix}.$$

Claim 3.1. *$c - a \le 1$.*

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Suppose that $c-a\geq 2$. Let ρ' be the largest root of $f_{a+1,b,c-1}(\lambda)=0$. By a direct calculation, we obtain

$$\begin{aligned}
&f_{a,b,c}(\rho')\\
&=f_{a,b,c}(\rho')-f_{a+1,b,c-1}(\rho')\\
&=\Big[{\rho'}^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2ct\big]t\Big]\Big[{\rho'}^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2at\big]t\Big]\\
&\quad-\Big[{\rho'}^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2(c-1)t\big]t\Big]\Big[{\rho'}^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt\\
&\quad+(\frac{d+4}{2})^2(a+1)t\big]t\Big]\\
&=(a-c+1)(\frac{d+4}{2})^2t^4\\
&<0.
\end{aligned}$$

Hence, which implies $\rho'<\rho$, a contradiction.

Claim 3.2. $b=0$.

Next, we discuss in two cases.

Case 1. $n-d-1-b$ is even.

One has $a=c$ by Claim 3.1. Let ρ be the largest root of $f_{a,b,a}(\lambda)=0$. Note that

$$\begin{aligned}
&f_{a,b,a}(\lambda)\\
&=\Big[\lambda^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2at\big]t\Big]^2-\big[d\lambda+(\frac{d}{2})^2+(\frac{d+2}{2})^2bt\big]^2t^2\\
&=\Big\{\lambda^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2at\big]t-\big[d\lambda+(\frac{d}{2})^2+(\frac{d+2}{2})^2bt\big]t\Big\}\\
&\quad\times\Big\{\lambda^2-\big[(\frac{d}{2})^2+\Delta(d)t+(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2at\big]t+\big[d\lambda+(\frac{d}{2})^2+(\frac{d+2}{2})^2bt\big]t\Big\}\\
&=\Big\{\lambda^2-dt\lambda-\big[2(\frac{d}{2})^2+\Delta(d)t+2(\frac{d+2}{2})^2bt+(\frac{d+4}{2})^2at\big]t\Big\}\\
&\quad\times\Big\{\lambda^2+dt\lambda-\big[\Delta(d)+(\frac{d+4}{2})^2a\big]t^2\Big\}.
\end{aligned}$$

It is easy to obtain

$$\begin{aligned}
\rho&=\frac{1}{2}\Big(dt+\sqrt{d^2t^2+\big[2d^2+2(d+2)^2bt+(d+4)^2at+4\Delta(d)t\big]t}\Big)\\
&=\frac{1}{2}\Big(dt+\sqrt{d^2t^2+\big[2d^2+2(d+2)^2(n-d-1-2a)t+(d+4)^2at+4\Delta(d)t\big]t}\Big)\\
&=\frac{1}{2}\Big(dt+\sqrt{d^2t^2+\big[2d^2+2(d+2)^2(n-d-1)t+\big[(d+4)^2-4(d+2)^2\big]at^2+4\Delta(d)t\big]t}\Big)\\
&=\frac{1}{2}\Big(dt+\sqrt{d^2t^2+\big[2d^2+2(d+2)^2(n-d-1)t+4\Delta(d)t\big]t-(3d+8)dt^2a}\Big).
\end{aligned}$$

Since $b \geq 0$, we get $a = c = \frac{n-d-1-b}{2} \leq \frac{n-d-1}{2}$. Thus, the ε-spectral radius of t-clique tree $T_t(n', d, a, b, a)$ achieves the minimal value when $b = 0$ and $a = c = \frac{n-d-1}{2}$.

Case 2. $n - d - 1 - b$ is odd.

We have $c = a + 1$ by Claim 3.1. Suppose that $b \geq 1$ and ρ' is the the ε-spectral radius of graph $T_t(n', d, a + 1, b - 1, a + 1)$. Note that

$$\begin{aligned}
& f_{a+1,b-1,a+1}(\lambda) \\
=& \begin{vmatrix} \lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2(b-1)t + (\frac{d+4}{2})^2(a+1)t\big]t & -\big[d\lambda + (\frac{d}{2})^2 + (\frac{d+2}{2})^2(b-1)t\big]t \\ -\big[d\lambda + (\frac{d}{2})^2 + (\frac{d+2}{2})^2(b-1)t\big]t & \lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2(b-1)t + (\frac{d+4}{2})^2(a+1)t\big]t \end{vmatrix} \\
=& \Big(\lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2(b-1)t + (\frac{d+4}{2})^2(a+1)t\big]t\Big)^2 - \Big(\big[d\lambda + (\frac{d}{2})^2 + (\frac{d+2}{2})^2(b-1)t\big]t\Big)^2 \\
=& \Big\{\lambda^2 - dt\lambda - \big[2(\frac{d}{2})^2 + \Delta(d)t + 2(\frac{d+2}{2})^2(b-1)t + (\frac{d+4}{2})^2(a+1)t\big]t\Big\}\Big\{\lambda^2 + dt\lambda - \big[\Delta(d) + (\frac{d+4}{2})^2(a+1)\big]t^2\Big\}.
\end{aligned}$$

It is obvious that ρ' is the largest root of $\lambda^2 - dt\lambda - \big[2(\frac{d}{2})^2 + \Delta(d)t + 2(\frac{d+2}{2})^2(b-1)t + (\frac{d+4}{2})^2(a+1)t\big]t = 0$. Thus, one has

$$\begin{aligned}
& {\rho'}^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2bt + (\frac{d+4}{2})^2(a+1)t\big]t \\
=& dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2.
\end{aligned}$$

So we can compute

$$\begin{aligned}
& f_{a,b,a+1}(\rho') \\
=& \begin{vmatrix} {\rho'}^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2bt + (\frac{d+4}{2})^2(a+1)t\big]t & -\big[d\rho' + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t \\ -\big[d\rho' + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t & {\rho'}^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+2}{2})^2bt + (\frac{d+4}{2})^2at\big]t \end{vmatrix} \\
=& \begin{vmatrix} dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2 & -\big[d\rho' + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t \\ -\big[d\rho' + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t & dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2 + (\frac{d+4}{2})^2t^2 \end{vmatrix} \\
=& \begin{vmatrix} dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2 & -\big[d\rho' + (\frac{d}{2})^2 + (\frac{d+2}{2})^2bt\big]t \\ -2(\frac{d+2}{2})^2t^2 & -2(\frac{d+2}{2})^2t^2 + (\frac{d+4}{2})^2t^2 \end{vmatrix} \\
=& \begin{vmatrix} dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2 & -2(\frac{d+2}{2})^2t^2 \\ -2(\frac{d+2}{2})^2t^2 & -4(\frac{d+2}{2})^2t^2 + (\frac{d+4}{2})^2t^2 \end{vmatrix} \\
=& \Big(dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2\Big)\Big(-4(\frac{d+2}{2})^2t^2 + (\frac{d+4}{2})^2t^2\Big) - 4(\frac{d+2}{2})^4t^4.
\end{aligned}$$

We obtain $\rho' \geq d$ by Lemma 2.3. Note that $b \geq 1$ and $t \geq 1$. Then we have

$$-4(\frac{d+2}{2})^2t^2 + (\frac{d+4}{2})^2t^2 \leq -4(\frac{d+2}{2})^2 + (\frac{d+4}{2})^2 = 4 - \frac{d^2}{2} < 0,$$

and

$$dt\rho' + (\frac{d}{2})^2t + (\frac{d+2}{2})^2(b-2)t^2 \geq d\rho' + (\frac{d}{2})^2 + (\frac{d+2}{2})^2(b-2) \geq d^2 + (\frac{d}{2})^2 - (\frac{d+2}{2})^2 = d(d-1) - 1 > 0.$$

Hence, $f_{a,b,a+1}(\rho') < 0$ which implies $\rho' < \varepsilon\big(T_t(n', d, a, b, a+1)\big)$, a contradiction.

Combining Claims 3.1 and 3.2, we know that the minimum ε-spectral radius is achieved by the t-clique tree $T_t(n', d, \lfloor\frac{n-d-1}{2}\rfloor, 0, \lceil\frac{n-d-1}{2}\rceil)$ when $d \geq 6$ is even. □

The next result immediately follows from Theorem 3.7.

Corollary 3.8. *Let n be odd and T_t be an n'-vectex t-clique tree with even $d \geq 6$. Let*

$$\Delta(d) := (\frac{d+2}{2})^2 + (\frac{d+4}{2})^2 + \cdots + (d-1)^2 = \frac{d(d-2)(7d-1)}{24} \text{ and } n' = (n-1)t + 1.$$

Then

$$\varepsilon_1(T_t) \geq \frac{1}{2}\Big(dt + \sqrt{d^2t^2 + \big[d^2 + 4\Delta(d)t + (d+4)^2at\big]t}\Big).$$

with equality if and only if $T_t \cong T_t(n', d, a, 0, a)$ and $a = \frac{n-d-1}{2}$.

Proof. If $n-d$ is odd and $a = \frac{n-d-1}{2}$. By equation (3.4), we have

$$\begin{aligned} & f_{a,0,a}(\lambda) \\ = & \begin{vmatrix} \lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+4}{2})^2at\big]t & -\big[d\lambda + (\frac{d}{2})^2\big]t \\ -\big[d\lambda + (\frac{d}{2})^2\big]t & \lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+4}{2})^2at\big]t \end{vmatrix} \\ = & \Big(\lambda^2 - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+4}{2})^2at\big]t\Big)^2 - \Big(\big[d\lambda + (\frac{d}{2})^2\big]t\Big)^2 \\ = & \Big(\lambda^2 + dt\lambda - \big[\Delta(d) + (\frac{d+4}{2})^2a\big]t^2\Big)\Big(\lambda^2 - dt\lambda - \big[(\frac{d}{2})^2 + \Delta(d)t + (\frac{d+4}{2})^2at\big]t\Big). \end{aligned}$$

Let ρ be the largest root of $f_{a,0,a}(\lambda) = 0$. Then

$$\rho = \frac{1}{2}\Big(dt + \sqrt{d^2t^2 + \big[d^2 + 4\Delta(d)t + (d+4)^2at\big]t}\Big).$$

□

Remark 3.9. *The t-clique tree is a tree with order n when $t = 1$, Wei et al. [22] have idetified the tree with given order and diameter having the minimum eccentricity spectral radius (see [22]). Furthermore, our results Theorems 3.5 and 3.7 are consistent with Wei when $t = 1$.*

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References

[1] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Springer, New York, 2012.

[2] A. T. Balaban, D. Ciubotariu and M. Medeleanu, Topological indices and real number vertex invariants based on graph eigenvalues or eigenvectors, J. Chem. Inf. Comput. Sci., **31** (1991) 517–523.

[3] B. Liu, *Combinatorial matrix theory*, Second Edition, Science Press, Beijing, 2005.

[4] D. Cvetković, P. Rowlinson and S. Simić, *An introduction to the theory of graph spectra*, Cambridge University Press, Cambridge, 2010.

[5] H. Lin, R. Liu and X. Lu, The intertia and energy of the distance matrix of a connected graph, *Linear Algebra Appl.*, **467** (2015) 29–39.

[6] H. Minc, *Nonnegative matrices*, J. Wiley, New York, 1988.

[7] I. Mahato, R. Gurusamy, M. R. Kannan and S. Arockiaraj, Spectra of eccentricity matrices of graphs, *Discrete Appl. Math.*, **285** (2020) 252–260.

[8] I. Gutman and M. Medeleanu, On the structure-dependence of the largest eigenvalue of the distance matrix of an alkane, *Indian J. Chem.*, **37** (1998) 569–573.

[9] J. A. Bondy and U. S. R. Murty, *Graph theory*, Springer, New York, 2008.

[10] J. R. S. Blair and B. Peyton, An introduction to chordal graphs and clique trees, Graph theory and sparse matrix computation, *Math. Appl.*, **56** (1993) 1–29.

[11] J. Wang, M. Lu, F. Belardo and M. Randić, The anti-adjacency matrix of a graph: Eccentricity matrix, *Discrete Appl. Math.*, **251** (2018) 299–309.

[12] J. Wang, L. Lu, M. Randić and G. Li, Graph energy based on the eccentricity matrix, *Discrete Math.*, **342** (2019) 2636–2646.

[13] J. Wang, M. Lu, L. Lu and F. Belardo, Specrtal properties of the eccentricity matrix of graphs, *Discrete Appl. Math.*, **279** (2020) 168–177.

[14] J. Wang, X. Lei, S. Li, W. Wei and X. Luo, On the eccentricity matrix of graphs and its applications to the boiling point of hydrocarbons, *Chemom. Intell. Lab. Syst.*, **207** (2020).

[15] J. Wang, M. Lu, M. Brunetti, L. Lu and X. Huang, Spectral determinations and eccentricity matrix of graphs, *Adv. Appl. Math.*, **139** (2022) 25 pp.

[16] J. Wang, X. Lei, M. Lu, S. Sorgun and H. Kücük, On graphs with exactly one anti-adjacency eigenvalue and beyond, *Discrete Math.*, **346** (2023) 15 pp.

[17] M. Randić, D_{MAX}-matrix of dominant distances in a graph, *MATCH Commun. Math. Comput. Chem.*, **70** (2013) 221–238.

[18] M. Randić, R. Orel and A. T. Balaban, D_{MAX} matrix invariants as graph descriptors, Graphs having the same Balaban index, *MATCH Commun. Math. Comput. Chem.*, **70** (2013) 239–258.

[19] P. B. Henderson and Y. Zalcstein, A Graph-Theoretic Characterization of the PV class of Synchronizing Primitives, *SIAM J. Comput.*, **6** (1977) 88–108.

[20] R. B. Bapat, *Graphs and Matrices*, Springer, London, 2014.

[21] S. Sorgun and H. Kücük, On two problems related to anti-adjacency (eccentricity) matrix, *Discrete Appl. Math.*, **328** (2023) 1–9.

[22] W. Wei, X. He and S. Li, Solutions for two conjectures on the eigenvalues of the eccentricity matrix, and beyond, *Discrete. Math.*, **343** (2020) 21 pp.

[23] W. Wei and S. Li, On the eccentricity spectra of complete multipartite graphs, *Appl. Math. Comput.*, **424** (2022) 12 pp.

[24] X. Lei, J. Wang and G. Li, On the eigenvalues of eccentricity matrix of graphs, *Discrete Appl. Math.*, **295** (2021) 134–147.

[25] X. Zhang and J. Zhou, The Distance Laplacian Spectral Radius of Clique Trees, *Discrete Dyn. Nat. Soc.*, (2020) 8 pp. https://doi.org/10.1155/2020/8855987.

[26] X. Yang and L. Wang, The eccentricity matrix of a digraph, *Discrete Appl. Math.*, **322** (2022) 61–73.

[27] X. Lei and J. Wang, Spectral determination of graphs with one positive anti-adjacency eigenvalue, *Appl. Math. Comput.*, **422** (2022) 13 pp.

[28] Y. Jin, R. Gu and X. Zhang, The distance energy of clique trees, *Linear Algebra Appl.*, **615** (2021) 1–10.

[29] Z. Qiu and Z. Tang, On the eccentricity spectra of threshold graphs, *Discrete Appl. Math.*, **310** (2022) 75–85.

[30] Z. Qiu, Z. Tang and Q. Li, Eccentricity spectral radius of clique trees with given diameter, *Discrete Appl. Math.*, **337** (2023) 202–217 .

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