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PEG SOLITAIRE ON LINE GRAPHS

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Abstract. In 2011, Beeler and Hoilman generalized the game of peg solitaire to arbitrary connected graphs. Since then peg solitaire and related games have been considered on many graph classes. One of the main goals is the characterization of solvable graphs. To this end, different graph operations, such as joins and Cartesian products, have been considered in the past. In this article, we continue this venue of research by investigating line graphs. Instead of playing peg solitaire on the line graph $L(G)$ of a graph G, we introduce a related game called stick solitaire and play it on G. This game is examined on several well-known graph classes, for example complete graphs and windmills. In particular, we prove that most of them are stick-solvable. We also present a family of graphs which contains unsolvable graphs in stick solitaire. Naturally, the fool's stick solitaire number is an object of interest, which we compute for the previously considered graph classes.

1. Introduction

In [4], Beeler and Hoilman introduced the game of peg solitaire on graphs as a generalization of the classical peg solitaire game:

Given a connected, undirected graph G with vertex set $V(G)$ and edge set $E(G)$, we can put pegs in the vertices of G. Given three vertices u, v, w with pegs in u and v and a hole in w such that $uv, vw \in E(G)$, we can jump with the peg from u over v into w, removing the peg in v (cf. Figure 1).

Keywords: peg solitaire, line graphs, stick solitaire, windmill, double star, caterpillar.

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This jump will be denoted as $u \cdot \vec{v} \cdot w$. In figures, vertices with pegs will be drawn filled while vertices without pegs are drawn unfilled.

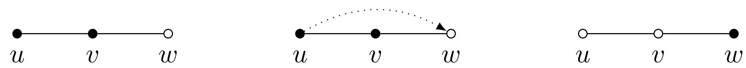

FIGURE 1. A jump in peg solitaire.

Peg solitaire has been considered for quite a few classes of graphs, including paths, complete graphs, stars, double stars and caterpillars (for more results and variants see [1, 3–8, 10–13]). The most interesting parameter of a graph G is its solitaire number.

Definition 1.1. *The* solitaire number *of a graph G, denoted with* $\mathrm{Ps}(G)$*, is the minimum number of pegs obtainable by a series of jumps such that no more jump is possible. A graph G is* solvable *if and only if* $\mathrm{Ps}(G) = 1$.

To get closer to the major goal, the characterization of solvable graphs, several graph operations, such as joins and Cartesian products, have been examined, see [2, 4, 11]. In this paper, we consider another graph operation.

Definition 1.2. *The* line graph $L(G)$ *of a graph G has vertex set $E(G)$ and $e_1, e_2 \in E(G)$ are adjacent in $L(G)$ if and only if they are incident in G (cf. Figure 2).*

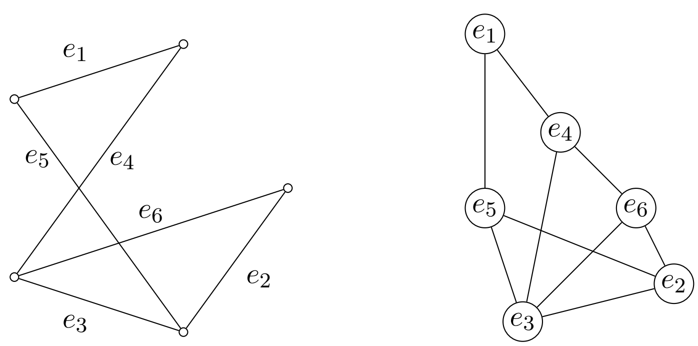

FIGURE 2. A graph G (left) and its line graph $L(G)$ (right).

Note that $L(G)$ contains a clique of size k for every vertex of G having degree k. Hence, $L(G)$ is a tree if and only if G is a path graph. Since cliques give a sizable amount of jump possibilities, we believe that a rather large proportion of line graphs is solvable in peg solitaire.

Instead of playing peg solitaire on $L(G)$, we define a different (but very closely related) game on G: given a graph $G = (V, E)$, we put *sticks* in the edges of G. Given three edges $e, f, g \in E$ with sticks in e and f and no stick (that is, a *hole*) in g and such that e is incident to f and f is incident to g, we can jump with the stick from e over f into g, removing the stick in f. This jump will be denoted

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as $e \cdot \overrightarrow{f} \cdot g$ (cf. Figure 3). In figures, edges with sticks will be drawn solid while edges without sticks are drawn dashed. The corresponding game will be called *stick solitaire*.

Note that, contrary to peg solitaire, there are different possibilities for e, f, g such that e and f as well as f and g are incident (cf. Figure 3):

- The edges e, f and g form (in this order) a path or a cycle of length 3.
- e, f and g are incident to the same vertex.

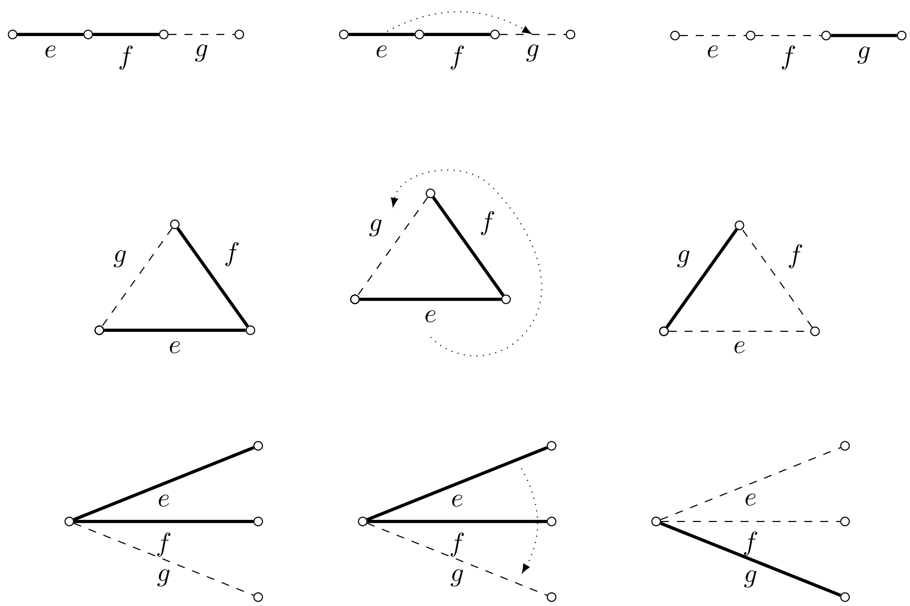

FIGURE 3. Possible jumps in stick solitaire.

A variant of stick solitaire, where the jump $e \cdot \overrightarrow{f} \cdot g$ is not allowed if e, f and g are incident to the same vertex, is considered in [9].

The following notations are similar to the respective notations for peg solitaire. We begin with a *starting state* $S \subset E$ of edges that are empty (that is, with a hole). A *terminal state* $T \subset E$ is a set of edges that contain sticks at the end of the game such that no more jumps are possible. A terminal state T is *associated* to a starting state S if T can be obtained from S by a series of jumps. We will always assume that the starting state S consists of a single edge.

Analogously to peg solitaire, we want to reach a terminal state consisting of a single edge. This is not possible for all graphs. Therefore, we use the following notations.

Definition 1.3. *A graph G is called*

- stick-solvable*, if there is some $e \in E$ such that the starting state $S = \{e\}$ has an associated terminal state consisting of a single edge.*
- freely stick-solvable*, if for all $e \in E$ the starting state $S = \{e\}$ has an associated terminal state consisting of a single edge.*

- k-stick-solvable, *if there is some $e \in V$ such that the starting state $S = \{e\}$ has an associated terminal state consisting of k edges.*
- strictly k-stick-solvable, *if G is k-stick-solvable but not ℓ-stick-solvable for any $\ell < k$.*

Definition 1.4. *If G is strictly k-stick-solvable, then we say that G has* stick solitaire number[1] $\mathrm{t}'(G) = k$.

Solving a graph in stick solitaire is described by the notion *stick-solve*. Using the respective definitions, we immediately see that a stick solitaire jump in G is the same as a peg solitaire jump in $L(G)$. Hence playing stick solitaire on G is the same as playing peg solitaire on $L(G)$. In particular, this gives the following result.

Proposition 1.5. *A graph G is (freely) stick-solvable if and only if $L(G)$ is (freely) peg-solvable. In particular, $\mathrm{t}'(G) = \mathrm{Ps}(L(G))$, where $\mathrm{Ps}(L(G))$ denotes the smallest number of pegs possible in a terminal state when peg solitaire is played on $L(G)$.*

Although stick solitaire does not yield a "new" game[2], since it is equivalent to peg solitaire on line graphs, stick solitaire is interesting for (at least) two reasons. First of all, stick solitaire can bring a new perspective to old questions (at least when considering line graphs). Secondly, it gives rise to questions about peg-solvability of graphs that where previously not considered, as we will see at the end of the following section.

Note that solvability for a graph G now has two meanings here: peg-solvability of G when playing peg solitaire or stick-solvability of G when playing stick solitaire. To avoid confusion, we will use the full terms and not drop the supplementary peg or stick in this paper.

The following notations and notions will be used throughout this paper. For a graph G and $W \subset V(G)$, the subgraph of G *induced* by W is the graph with vertex set W and edge set $\{uv \in E(G)\colon u, v \in W\}$; denote by $G - v$ the special case of the graph induced by $V(G) \setminus \{v\}$. Similarly, for $F \subset E(G)$, the subgraph of G *induced* by F is the graph with edge set F and vertex set $\{u \in V(G)\colon u \text{ is incident to some edge in } F\}$. The *path graph* resp. *cycle graph* on n vertices is denoted by P_n resp. C_n. Furthermore, $K_{m,n}$ is the *complete bipartite graph* on $m + n$ vertices, where the special case $K_{1,n}$ is called a *star graph*. We further consider the following graph classes.

Definition 1.6. *A* double star $\mathrm{DS}(L, R)$ *consists of two adjacent vertices c_L, c_R, where additionally c_L is adjacent to L pendant vertices and c_R is adjacent to R pendant vertices.*

[1]We use the notation $\mathrm{t}'(G)$ instead of the (maybe) more natural notation $\mathrm{t}(G)$ since $\mathrm{t}(G)$ is used for the variant described in [9].

[2]The variant of stick solitaire that is considered in [9] is not as closely related to peg solitaire and hence is a "new" game.

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Definition 1.7. *A* windmill variant $\mathrm{W}(P, B)$ *is a graph with a universal vertex u (that is adjacent to every other vertex), P pendant vertices, that are only adjacent to u, and B blades consisting of two vertices each, such that these two vertices are adjacent (cf. Figure 4).*

Definition 1.8. *A* generalized windmill $\mathrm{W}^*(P, B)$ *with B blade vertices and P pendant vertices is a graph G with a vertex u that is adjacent to exactly P pendant vertices and B vertices that lie in generalized blades, i.e., the induced subgraph defined by these vertices is a disjoint union of paths and cycles.*[3] *Figure 4 shows a generalized windmill $W^*(2, 8)$.*

FIGURE 4. The windmill variant $W(3, 2)$ (left) and a generalized windmill $W^*(2, 8)$ (right).

Definition 1.9. *A* caterpillar *is a tree obtained from the path graph on $n + 2$ vertices by appending pendant vertices to the existing vertices of degree 2 of the path graph P_{n+2}. The set of vertices of the original path graph is called* spine *of the caterpillar.*

Denote by $P_n(a_1, a_2, \dots, a_n)$ the caterpillar with spine of length $n+1$, such that $x_0, x_1, x_2, \dots, x_n, x_{n+1}$ are the vertices on the spine and, for $i \in [n]$[4]*, a_i is the number of vertices adjacent to x_i and not belonging to the spine (note that x_0 resp. x_{n+1} are not counted in a_1 resp. a_n). The pendant vertices incident to x_i are labeled $x_{i,1}, x_{i,2}, \dots, x_{i,a_i}$, the corresponding edges are $e_{i,1}, e_{i,2}, \dots, e_{i,a_i}$. Furthermore, the edges of the spine are denoted by $e_0, e_1, \dots, e_n$ (note $e_0 = x_0x_1, e_1 = x_1x_2$ and so on). The caterpillar $P_4(0, 1, 0, 2)$ is displayed in Figure 5.*

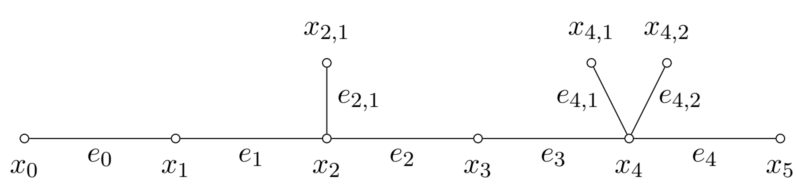

FIGURE 5. The caterpillar $P_4(0, 1, 0, 2)$.

[3]Note that all vertices in blades have degree 2 or 3. Moreover, every graph on n vertices, where some vertex has degree $n - 1$ and all other vertices have degree $1, 2$ or 3, is a generalized windmill.

[4]Using the standard notation $[m, n] = \{m, m + 1, \dots, n\}$ for positive integers $m < n$ and $[n] = [1, n]$.

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In this paper, we start by examining stick-solvability of several graph classes in Section 2. In Section 3, we consider trees, in particular caterpillars, and partly characterize their stick-solvability. Some general results are obtained in Section 4. We end with considering fool's stick solitaire in Section 5, i.e., we examine the cardinality of the largest possible terminal state for several types of graphs.

2. Graph classes

In this section, we determine the stick solitaire number of elements from several graph classes. The following proposition follows immediately from known results about $\mathrm{Ps}(P_n)$ and $\mathrm{Ps}(C_n)$ [4, Theorems 2.3 and 2.4], $L(P_n) = P_{n-1}, L(C_n) = C_n$, and the connection to peg solitaire given in Proposition 1.5.

Proposition 2.1. *Let $n \geq 3$ be an integer.*

- *We have $\mathrm{t}'(P_n) = 1$ if $n = 4$ or $n \equiv 1 \pmod 2$. In all other cases we have $\mathrm{t}'(P_n) = 2$. Moreover, P_n is never freely stick-solvable. More precisely, denote the edges of P_n by $e_1, \ldots, e_{n-1}$ and start with a hole in e_2. If P_n is stick-solvable, we can end with a stick in e_{n-2}, if P_n is not stick solvable, we can end with sticks in e_{n-3} and e_{n-1}.*
- *We have $\mathrm{t}'(C_n) = 1$ if $n = 3$ or $n \equiv 0 \pmod 2$ and $\mathrm{t}'(C_n) = 2$ otherwise. Moreover, C_n is freely stick-solvable if it is stick-solvable.*

Proposition 2.2. *Let n be an integer greater than 2. Then $\mathrm{t}'(K_n) = 1$. More precisely, K_n is freely stick-solvable.*

Proof. We prove this by induction on n. The statement is clearly true for $n = 3$ by Proposition 2.1. If $n > 3$, start with a hole in an arbitrary edge and let v be one of the vertices incident to this edge. Using only the edges incident to v, we can remove the sticks in all but one of these edges. Let vw contain the remaining stick, u be a neighbor of v other than w and x be a vertex different from u, v, w (such a vertex exists since $n \geq 4$). Then the jumps $uw \cdot \overrightarrow{vw} \cdot vu$ and $ux \cdot \overrightarrow{uv} \cdot uw$ yield a configuration where the subgraph of G induced by $V \backslash \{v\}$ has exactly one hole in one of its edges (cf. Figure 6). Since this subgraph is isomorphic to K_{n-1}, the result follows by induction. □

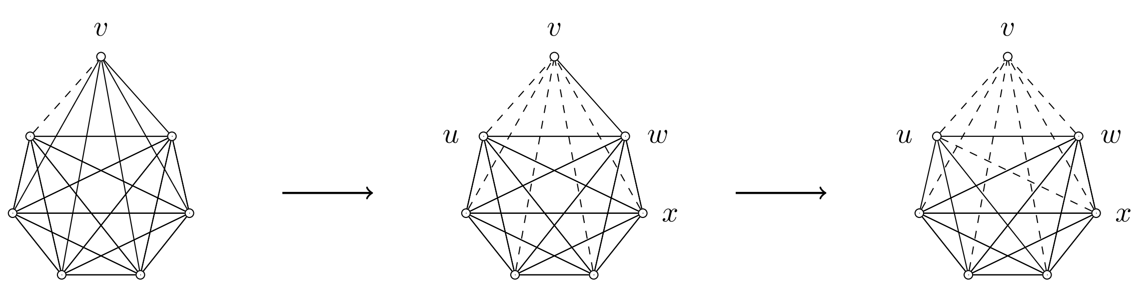

FIGURE 6. Proof idea of Proposition 2.2 illustrated.

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Proposition 2.3. *Let m and n be positive integers with $m+n \geq 3$. Then we have $\mathrm{t}'(K_{m,n}) = 1$. More precisely, $K_{m,n}$ is freely stick-solvable.*

Proof. Let w.l.o.g. $m \leq n$. If $m = 1$, then all but one of the sticks in the edges can be removed using the jump displayed at the bottom of Figure 3. Hence we consider $m \geq 2$ from now on. Let vw be the edge with the starting hole such that v is a vertex with n neighbors in $K_{m,n}$. Use the edges incident to v, as done in the Case $m = 1$, to remove all but one of the sticks in these edges (this is possible since $n \geq 2$). Let vu be the edge with the remaining stick, note that $u = w$ is possible. Choose $x \in N(u) \setminus \{v\}$ and $y \in N(v) \setminus \{u\}$, which is possible because of $n \geq m \geq 2$. We execute the jumps $xu \cdot \overrightarrow{uv} \cdot vy$ and $vy \cdot \overrightarrow{yx} \cdot xu$ and consider the subgraph induced by $V \setminus \{v\}$. This graph has exactly one edge with a hole and is isomorphic to $K_{m-1,n}$, hence the result follows by induction on m. □

In particular, the preceding result shows that the star $K_{1,n}$ is stick-solvable if $n \geq 2$ holds. An even stronger result, which we will use in Section 3, can be proved by choosing the jumps in the previous proof slightly more carefully.

Lemma 2.4. *Let n be an integer. If $n \geq 5$, then it is possible to stick-solve $K_{1,n}$ such that any one of its edges might contain the final stick. For $n = 4$, every edge except for the one with the starting hole might contain the final stick.*

Proof. Let u be the vertex of degree n and let the starting hole be in the edge uv.

Furthermore, let w and x be two arbitrary vertices of $K_{1,n}$, which do not lie in $\{u, v\}$. Jump $wu \cdot \overrightarrow{xu} \cdot uv$. The subgraph of $K_{1,n}$ induced by $V(G) \setminus \{v, w\}$ is solvable by Proposition 2.3. With the final stick in this subgraph we can jump over uv into uw.

It remains to show that uv can contain the final stick if $n \geq 5$. To this end, pick three vertices $w, x, y \in V(G) \setminus \{u, v\}$. Jump $wu \cdot \overrightarrow{xu} \cdot uv$ and $uv \cdot \overrightarrow{yu} \cdot uw$. Now proceed as before to solve subgraph of $K_{1,n}$ induced by $V(G) \setminus \{v, w\}$ followed by jump over uw into uv. □

Proposition 2.5. *Let P and B be non-negative integers satisfying $P+B \geq 2$. Then $\mathrm{t}'(W^*(P, B)) = 1$. More precisely, $W^*(P, B)$ is freely stick-solvable.*

Proof. Denote by A the set of the edges incident to the central vertex u and by C the set containing the other edges (lying in a blade).

First, we consider the case when the starting hole is in some element of A (cf. Figure 7). As long as some $b \in C$ containing a stick exists, we can jump $b \cdot \overrightarrow{a_1} \cdot a_2$ for certain $a_1, a_2 \in A$. This is possible since every edge from C is adjacent to two edges in A and these jumps will still result in a graph where exactly one edge from A has a hole. When all sticks from the edges in C have been removed, the subgraph of $W^*(P, B)$ induced by A is, being isomorphic to a star graph, solvable by Proposition 2.3.

If the starting hole is in some $b \in C$, which lies in a blade containing more than two vertices, we jump $b_1 \cdot \overrightarrow{a} \cdot b$ for certain $b_1 \in C$ and $a \in A$ and proceed as before.

Finally, suppose the starting hole is in some $b \in C$, which lies in a blade containing exactly two vertices. In case of $P = 0$ and $B = 2$, we have $W^*(0,2) \cong K_3$, which is stick-solvable by Proposition 2.2. Otherwise, we find some a_1 not being incident to b. Jump $a_1 \cdot \overrightarrow{a_2} \cdot b$ and $b \cdot \overrightarrow{a_3} \cdot a_1$, where a_2 and a_3 are the two edges incident to b. Using the same idea as in the first case for the subgraph of $W^*(P,B)$ induced by $E(W^*(P,B)) \setminus \{a_2, b\}$ stick-solves the graph. □

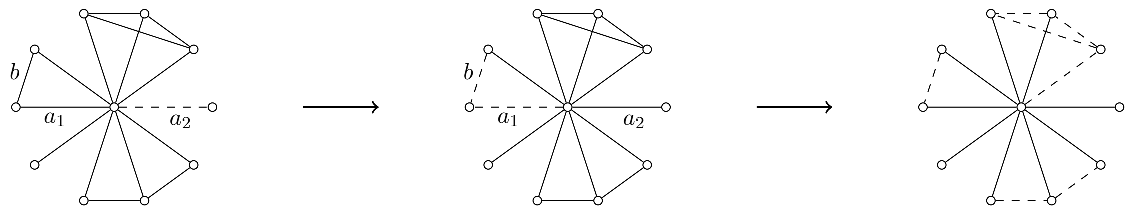

FIGURE 7. Proof idea of Proposition 2.5, with starting hole in some $a \in A$, illustrated.

An immediate consequence of the previous proposition is the following result.

Corollary 2.6. *Let P and B be non-negative integers satisfying $P + 2B \geq 2$. Then $\mathrm{t}'(W(P,B)) = 1$. More precisely, $W(P,B)$ is freely stick-solvable.*

Another graph class usually considered are double stars.

Proposition 2.7. *For $R, L \geq 1$ let $\mathrm{DS}(L,R)$ denote the double star with $L + R$ pendant vertices. Then $\mathrm{t}'(\mathrm{DS}(L,R)) = 1$. More precisely, $\mathrm{DS}(L,R)$ is freely stick-solvable if and only if $(L,R) \notin \{(1,1),(1,2),(2,1),(2,2)\}$.*

Proof. It is easy to see that the graphs $\mathrm{DS}(1,1), \mathrm{DS}(1,2)$ and $\mathrm{DS}(2,2)$ are not freely stick-solvable (for $\mathrm{DS}(1,1) \cong P_3$ this is clear, for $\mathrm{DS}(1,2)$ start with a hole in the edge connecting the left center with its pendant vertex, and for $\mathrm{DS}(2,2)$ start with a hole in the edge connecting the centers). Note also that all three of these graphs are not stick-solvable when starting with a hole in any other edge.

Let $e_1, e_2, \ldots, e_L$ denote the edges connecting pendant vertices with the left center, $f_1, f_2, \ldots, f_R$ the edges connecting pendant vertices with the right center, and let c be the remaining edge.

Consider the graphs $\mathrm{DS}(1,R)$ with $R \geq 3$ (by symmetry also $\mathrm{DS}(L,1)$ with $L \geq 3$). If the hole is in some f_i, we jump $e_1 \cdot \overrightarrow{c} \cdot f_i$. If the hole is in e_1, we jump $f_i \cdot \overrightarrow{c} \cdot e_1, f_j \cdot \overrightarrow{f_k} \cdot c, e_1 \cdot \overrightarrow{c} \cdot f_i$ for some $i \neq j \neq k \neq i$. If the hole is in c, we jump $f_i \cdot \overrightarrow{f_j} \cdot c, e_1 \cdot \overrightarrow{c} \cdot f_i$ for some $i \neq j$. In all three cases we stick-solve the remaining star (ignoring the left pendant vertex) to obtain the desired result.

Finally, let $L, R \geq 2$ with $(L,R) \neq (2,2)$. W.l.o.g. we may start with a hole in e_1 or c. In the first case, we stick-solve the left star ending with a stick in some e_i (and a hole in c). In the second case, w.l.o.g. let $L \geq R$, we also stick-solve the left star with a stick in some e_i (and a hole in c). Then we jump $f_1 \cdot \overrightarrow{f_2} \cdot c, e_i \cdot \overrightarrow{c} \cdot f_1$ and stick-solve the right star. □

For one result in Section 5, we need the following additional property of double stars:

Proposition 2.8. *Let $L \geq 2$ or $R \geq 2$ and $(L, R) \neq (2, 2)$. If we start with a hole in the edge between the centers of* $\mathrm{DS}(L, R)$*, the sequence of jumps can be chosen such that the terminal state consists of any edge other than the one connecting the two centers.*

Proof. We use the notation from the previous proof. Let w.l.o.g. $L \geq R$. Note that this implies $L \geq 3$. Start with a hole in the edge between the centers and stick-solve the left star, ending with an edge in some e_i and a hole in c. Note that we can choose the sequence of jumps such that we can choose in which of the edges e_i the stick is. Next, stick-solve the right star, ending with a stick in c (this is possible unless $R = 3$). From this configuration the last stick can be put in any f_j or e_k with $k \neq i$. Since i was arbitrary, the statement follows. It remains to consider the case $R = 3$. If $L \geq 4$, we just reverse the order and start stick-solving the right star before the left star, so we are left with the case $L = R = 3$. Let w.l.o.g. e_1 be the desired terminal state. In this case, jump $e_1 \cdot \overrightarrow{e_2} \cdot c, f_1 \cdot \overrightarrow{c} \cdot e_1, f_2 \cdot \overrightarrow{f_3} \cdot c, e_1 \cdot \overrightarrow{c} \cdot e_2, e_3 \cdot \overrightarrow{e_2} \cdot e_1$. □

We will now have an explicit look at the line graphs of $\mathrm{DS}(L, R)$ and $W(P, B)$. Recall that we can use the corresponding peg jumps, as given in the previous results, on these graphs to solve them in peg solitaire.

The left center of $\mathrm{DS}(L, R)$ is adjacent to all left pendants and the right center, hence this gives a K_{L+1} in the line graph. The analogue holds for the right center. Since the two centers are connected by an edge that is incident to all other edges, the line graph of $\mathrm{DS}(L, R)$ consists of the complete graphs K_{L+1} and K_{R+1} that are connected at one vertex. We will denote this graph by $K_{L+1 \cap R+1}$ (cf. Figure 8).

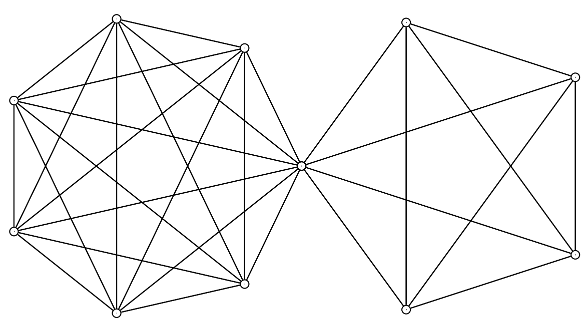

FIGURE 8. The graph $K_{7\cap 5}$.

Now we determine the line graph of $W(P, B)$. Since the central vertex is adjacent to all $P + 2B$ vertices, this gives a complete graph on $P + 2B$ vertices in $L(G)$. Additionally, for every blade we have another vertex in $L(G)$ that is adjacent to exactly two vertices of this complete graph. Note also that the neighbors are distinct for any two of these added vertices. We denote this graph by $K_n|\rangle B$ where $n = P + 2B$ (cf. Figure 9). This graph is only defined if $2B \leq n$.

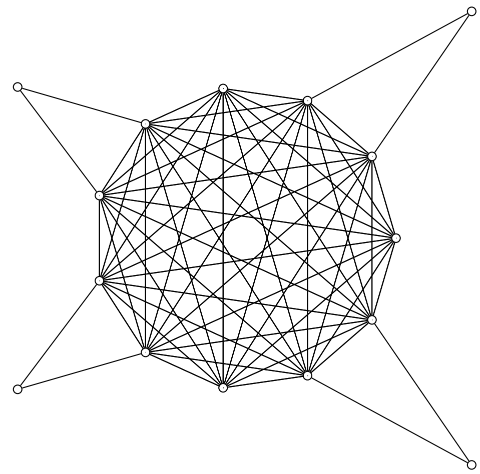

FIGURE 9. The graph $K_{11}|\rangle 4$.

Combining Propositions 1.5 and 2.7 as well as Corollary 2.6 yields the following result.

Proposition 2.9. *Let m, n, B be non-negative integers with $n \geq 2B$.*

- *The graph $K_{m \cap n}$ is peg-solvable. If at least one of m, n is greater than 2, $K_{m \cap n}$ is freely peg-solvable.*
- *The graph $K_n|\rangle B$ is freely peg-solvable.*

It is well known that $L(G)$ is Hamiltonian if G is Hamiltonian or G has an Eulerian cycle, implying the following result (for a proof use [4, Corollary 2.5]).

Proposition 2.10. *If G has an even positive number of edges and contains an Eulerian cycle or a Hamiltonian cycle, we have $\mathrm{t}'(G) = 1$. Moreover, G is freely stick-solvable in this case.*

3. Stick-solvability of trees

Characterizing all solvable trees is one of the main open problems in the research area of peg solitaire on graphs. We will start considering the same problem for stick solitaire, and therefore for peg solitaire on the corresponding line graphs, in this section. In particular, we show that almost all caterpillars are stick-solvable and provide a class of trees which are far away from being stick-solvable.

Lemma 3.1. *The caterpillar $P_n(0, 0, \ldots, 0, 1)$ is stick-solvable with starting hole in $e_{n,1}$ and final stick in e_1 and vice versa.*

Proof. Start with a hole in $e_{n,1}$. For odd n we jump $e_{n-1} \cdot \overrightarrow{e_n} \cdot e_{n,1}$. Then we stick-solve the path $x_0, x_1, \ldots, x_n, x_{n,1}$ using Proposition 2.1. If n is even, we jump $e_{n-2} \cdot \overrightarrow{e_{n-1}} \cdot e_{n,1}, e_n \cdot \overrightarrow{e_{n,1}} \cdot e_{n-1}$ and stick-solve the path $x_0, x_1, \ldots, x_n$. The other solution follows from the Duality Principle. □

Proposition 3.2. *The caterpillar $P_n(a_1, a_2, \ldots, a_n)$ is stick-solvable with the following two possible exceptions:*

- $n \geq 4$, $n \equiv 0 \pmod 2$, *and* $a_i = 0$ *for all* $i \in [n]$,
- $a_i \leq 1$ *for all* $i \in [n]$, $I = \{i \in [n] \colon a_i = 1\}$, *and one of the following holds:*
  - $|I| = 1, n \equiv 1 \pmod 2$, *and the element in I is an even integer,*
  - *consecutive (when I is naturally ordered) indices $i, j \in I$ fulfill $(i-j) \equiv 1 \pmod 2$ and we have* $5 < \min I < \max I < n-4$ *as well as* $\min I \equiv 0 \pmod 2$ *and* $(n - \max I) \equiv 1 \pmod 2$.

Proof. If all a_i equal 0, we can use Proposition 2.1. Hence let at least one a_i be larger than 0 from now on.

If $a_i \neq 1$ for all $i \in [n]$, we can stick-solve the graph with a hole in e_{n-1}:

(i) For odd n, we start stick-solving the spine with Proposition 2.1. During this process for every vertex x_i with $a_i \geq 2$ there is a configuration where exactly one of e_{i-1}, e_i contains a stick. In that situation, by Lemma 2.4, the star induced by x_i and its neighbors can be stick-solved by removing sticks from the edges incident to the leaves $x_{i,1}, x_{i,2}, \ldots, x_{i,a_i}$ and keeping the situation on the spine as it is, i.e., exactly one of e_{i-1}, e_i contains a stick and this is the same edge as before. We can proceed stick-solving the spine and emptying edges $e_{j,k}$ in the same manner.

(ii) For even n, we need to be slightly more careful. Let $i \in [n]$ be the largest index with $a_i \geq 2$. We stick-solve the spine until we reach a configuration, where exactly one of e_{i-1}, e_i contains a stick. To be more precise, we choose the configuration where the previous jump was either $e_{i+2} \cdot \overrightarrow{e_{i+1}} \cdot e_i$ or $e_{i+3} \cdot \overrightarrow{e_{i+2}} \cdot e_{i+1}$, or no jump occurred (in case of $i = n-1$). The case $i = 1$ is easy to stick-solve with Lemma 2.4 and Proposition 2.1, hence assume $i \geq 2$. If the stick lies in e_i, we jump $e_i \cdot \overrightarrow{e_{i,1}} \cdot e_{i-1}, e_{i-2} \cdot \overrightarrow{e_{i-1}} \cdot e_{i,1}$, stick-solve the star induced by x_i and its neighbors with final stick in e_{i-1}, and proceed as in the case of odd n (cf. Figure 10). If the stick lies in e_{i-1}, we also have a stick in e_{i+1}. We execute the jumps $e_{i-1} \cdot \overrightarrow{e_{i,1}} \cdot e_i, e_{i+1} \cdot \overrightarrow{e_i} \cdot e_{i,1}$, stick-solve the star induced by x_i and its neighbors with final stick in e_i, and, again, proceed as in the case of odd n.

We now consider the case where at least one a_i equals 1. Let $p \leq q$ be integers such that $a_p = a_q = 1$ and $a_i \neq 1 \neq a_j$ for all $i, j \in [n]$ with $i < p$ and $q < j$. We distinguish several cases.

- Case 1: p is odd or $n - q$ is even. W.l.o.g. let p be odd. If $p = 1$, we start with a hole in $e_{1,1}$. Otherwise, start with a hole in e_1 and, using Proposition 2.1, stick-solve the path $x_0, x_1, \ldots, x_p, x_{p,1}$, ending with a stick in e_{p-1}. If we, during this process (applies also to the rest of this case), reach some x_j with $a_j \geq 2$, we stick-solve the star induced by x_j and its neighbors "on the fly" whilst not changing the situation on the spine. We now identify the smallest $i > p$ with $a_i = 1$, or choose $i = n-1$ if $p = q$, and use Lemma 3.1 to stick-solve the caterpillar induced by $\{x_{p,1}, x_{i,1}, x_{p-1}, x_p, \ldots, x_i\}$. Continue in this manner to stick-solve the graph.

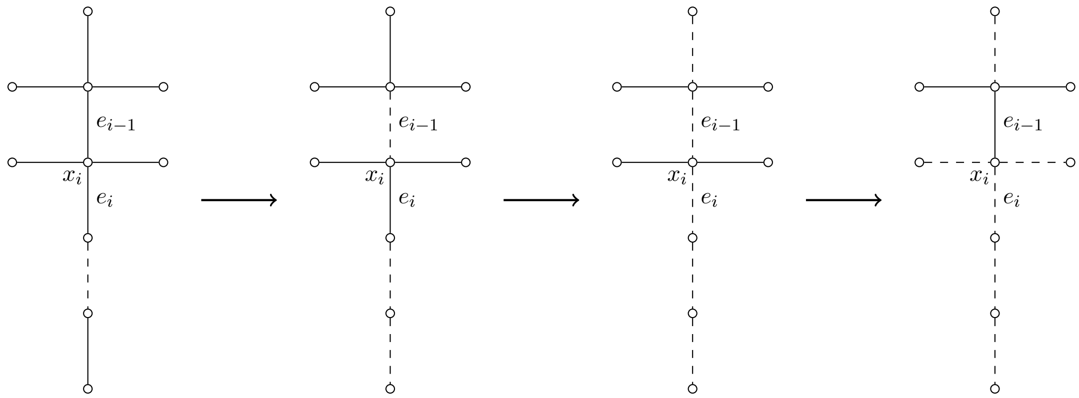

FIGURE 10. Proof idea of Proposition 3.2 Case (ii) illustrated.

- Case 2: p is even and $n-q$ is odd. W.l.o.g. we assume $p \geq n-q$ (otherwise we reverse the orientation of the caterpillar) and start with a hole in e_1.
  (a) Suppose we find some x_i with $a_i \geq 2$. Let i be minimal with this property. If $i < p$, we begin to stick-solve the spine until we reach a configuration, where exactly one of e_{i-1}, e_i contains a stick. Now, following the idea from (ii), we can modify the configuration on the spine to be able to proceed as in Case 1 (cf. Figure 11). If $i > p$, we stick-solve the path $x_0, x_1, \ldots, x_p$, necessarily ending with a stick in e_{p-2}. We execute the jumps $e_p \cdot \overrightarrow{e_{p,1}} \cdot e_{p-1}, e_{p-2} \cdot \overrightarrow{e_{p-1}} \cdot e_{p,1}$ and, by induction, reach a situation as before.

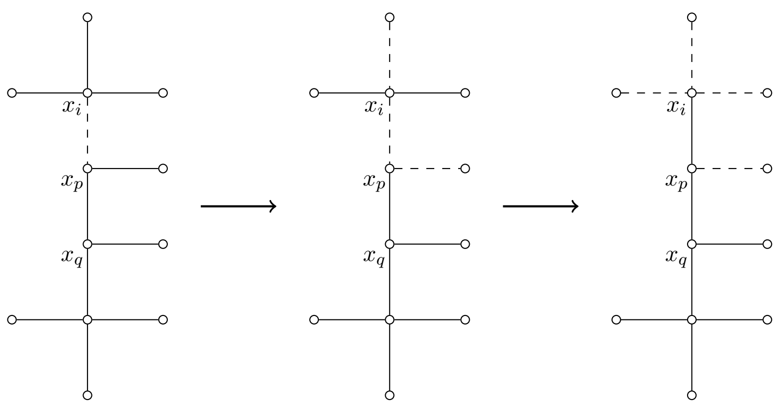

FIGURE 11. Proof idea of Proposition 3.2 Case 2(a) illustrated. Here we have $p = 2, q = 3, n = 4, n-q = 1, i = 1$.

  (b) Now consider the case where no $i \in [n]$ with $a_i \geq 2$ exists. In this case, only one i with $a_i \neq 0$ cannot exist by our assumption. Hence, either we find a pair $i < j$ of indices such that $j-i$ is even, $a_i = 1 = a_j$, and $a_k = 0$ for all $i < k < j$ or we don't, but then we have $n-q \leq 4$ since the other options are excluded in the statement of the proposition.
  In the first case (cf. Figure 12), let i be minimal with the given property. If $i \neq p$, stick-solve the caterpillar induced by $x_0, x_1, \ldots, x_p, x_{p+1}, x_{p,1}$ using Lemma 3.1, note that the

final stick is in $e_{p,1}$. Next consider the caterpillar $P_{n-p+1}(0, a_{p+1}, a_{p+2}, \dots, a_n)$ as a subgraph of the original caterpillar (by ignoring $x_0, x_1, \dots, x_{p-1}$ and all pendant vertices adjacent to them). By induction we reach a subgraph isomorphic to $P_\ell(0, 0, \dots, 0, a_i, a_{i+1}, \dots, a_n)$ with sticks in every edge except for the second edge on the spine (every other edge of the original caterpillar is empty). Using Proposition 2.1, note that the considered path has the correct parity, until we reach x_i, we can get to a configuration, where exactly the edges incident to $x_{i-1}, x_i, \dots, x_n$ except for e_{i-1} contain a stick (note that if $a_{i-1} = 1$, $e_{i-1,1}$ will contain a stick instead of e_{i-2}). After executing the jumps $e_i \cdot \overrightarrow{e_{i,1}} \cdot e_{i-1}, e_{i-2} \cdot \overrightarrow{e_{i-1}} \cdot e_{i,1}$, we can stick-solve the resulting configuration as in Case 1.

In the remaining case we proceed as before (using the same induction idea) except that we now reach a configuration with sticks exactly in the edges $e_q, e_{q+1}, \dots, e_n$ and $e_{q-2}, e_{q,1}$, which can easily be reduced to a configuration with only one stick. This works since $n - q$ is sufficiently small.

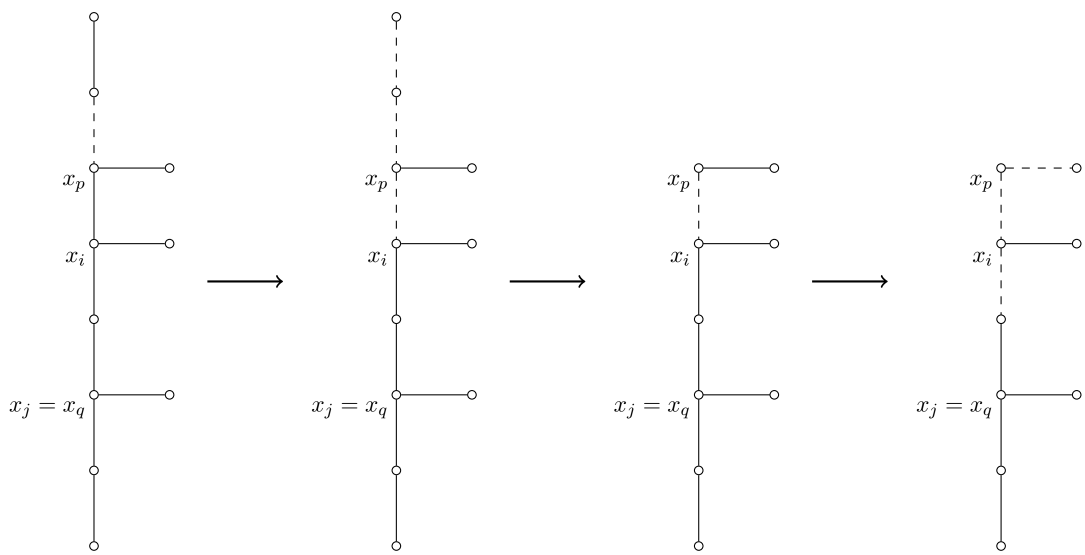

FIGURE 12. Proof idea of Proposition 3.2 Case 2(b) illustrated. Here we have $p = 2, q = 5, n = 6, n - q = 1, i = 3, j = 5$.

□

Remark 3.3. *The excluded paths in Proposition 3.2 are not stick-solvable. For the other special class of caterpillars, it is possible to find a terminal state with two pegs, but, apart from small examples (that are known to be unsolvable using exhaustive computer search), it is unknown, whether they are stick-solvable. We conjecture that they are indeed not stick-solvable. Figure 13 shows a caterpillar that is unsolvable in stick solitaire.*

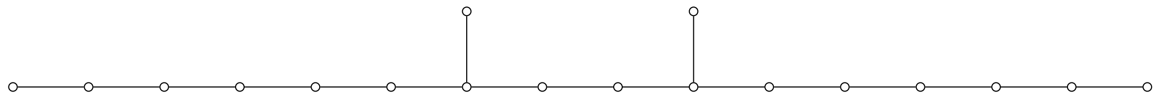

FIGURE 13. A caterpillar which is not stick-solvable.

Line graphs usually appear to be peg-solvable. The reason seems to be the existence of many cliques of size at least 3 in $L(G)$ (for every vertex having degree larger than 2 in G). Nevertheless, for arbitrary k, a line graph $L(G)$ with $\mathrm{Ps}(L(G)) \geq k$ exists. We now present a family of such graphs.

Definition 3.4. *The* generalized star graph $K_{1,3}(n)$ *is obtained from* $K_{1,3}$ *by replacing every vertex except for the center by a path of length* n *(cf. Figure 14 for the graph* $K_{1,3}(9)$*).*

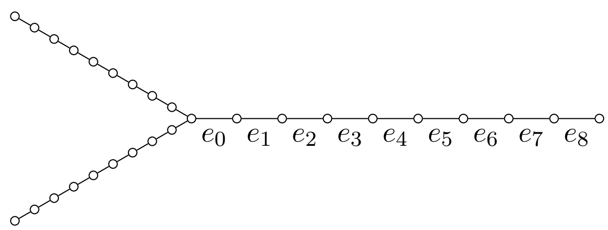

FIGURE 14. The graph $K_{1,3}(9)$.

Proposition 3.5. *For every integer* n *with* $n \geq 9$*, we have* $\mathrm{t}'(K_{1,3}(n)) \geq \frac{n-8}{2}$.

Proof. Start with a hole in one of the paths (the exact position of the hole does not matter) and start stick-solving. To stick-solve any of the other paths, since there are no holes in outer direction (i.e., away from the center), for any edge e in this new path the first jump involving e must be in inner direction (i.e., towards the center). Consider the edges e_3, e_4, e_5 with edge distance $3, 4$ and 5 from the center (cf. Figure 14). Suppose we jump with e_5 over e_4 (if this is not the case we jump with e_4 over e_3 and the arguments are the same but possibly one more stick remains). This results in a gap of size 2 (this is known as an empty bridge in peg solitaire). To empty the edges in outer direction from these two edges, it does not suffice to jump only with sticks from these outer edges, because this will result in a terminal configuration *stick, empty, stick, empty,...*, containing at least $\frac{n-6}{2}$ sticks.

So we have to also jump with sticks from the edges in inner direction to close the gap. Doing so results in two new gaps of size 2 with two sticks between them. Using the same arguments as above, we have to use the two sticks to close the outer gap. This results in a gap of size 4 in the edges e_2, e_3, e_4, e_5 or farther out. Such a gap cannot be closed: With the sticks from outer edges, only one hole can be filled. Concerning the inner edges and those in the other paths: even if the other two paths had all possible sticks to jump with (which is not true for at least one of the paths), we could only first jump with a stick from an edge of another path incident to the center over the edge of the current path incident to the center, filling the first hole. Jumping with sticks from two incident edges of the other path into the now empty edge incident to the center and then with this stick over the just filled hole, we fill the second hole. From now on, to make progress, we have to jump with the sticks of the other paths in direction of the center. As shown above, this cannot close a gap of size 2. Hence, we can only reach the edge incident to the center of the respective path with one of the paths, we cannot reach it with the path with which we have done the jump to fill the second hole. Consequently we cannot fill the edge incident to the center of the path we are currently in and thus we cannot close the gap of size 4. Hence at least $\frac{n-8}{2}$ sticks remain in a terminal state. □

It appears that long paths in graphs, as given in the previous proposition, in general form a problem for peg-solvability as well as stick-solvability. The reason for this becomes visible in the proof above. If the path is sufficiently long, at least one peg (resp. stick) remains at the end of the path which is far away from the rest of the graph. To remove this peg (resp. stick), several pegs (resp. sticks) have to be pushed in its direction. Therefore, at least some large degree vertices (for peg solitaire) seem to exist in such cases to have enough jump possibilities. As several examples show, this phenomenon is still not as easy to handle as it may appear here. Nevertheless, it seems to be reasonable to also focus on this structural property when characterizing solvable graphs.

4. General results

In this section, we prove two general results about stick solitaire. The first one is an analogue of a result in peg solitaire [2, Theorem 4.2].

Proposition 4.1. *Let G be a graph on at least three vertices with a universal vertex, i.e., a vertex that is adjacent to every other vertex. Then G is stick-solvable.*

Proof. Since K_3 and P_3 are stick-solvable, let G have at least four vertices from now on. Let u be a universal vertex of G. If all vertices except u have degree at most 3, we have a generalized windmill, which is stick-solvable by Proposition 2.5. Hence, considering the other option, we choose a vertex $v \in V(G) \setminus \{u\}$ of degree d, where $d = \deg(v) \geq 4$, and start with a hole in uv. We will successively remove the sticks in edges incident to those (large degree) vertices and delete them until we obtain a generalized windmill. To this end, we iterate the following steps (defining $G' = G$, see also Figure15):

(1) Let $u, v_1, v_2, \ldots, v_{d-1}$ be the neighbors of v in G', where $\deg(v_i) \geq \deg(v_j)$ for $i \leq j$.
(2) Perform the jumps $uv_i \cdot \overrightarrow{v_iv} \cdot vu$ and $v_{i+1}v \cdot \overrightarrow{vu} \cdot uv_i$ for $i = 1, 3, \ldots, 2\lfloor \frac{d}{2} \rfloor - 3$, removing exactly the sticks from the edges vv_j for all $j \in [2\lfloor \frac{d}{2} \rfloor - 2]$.
(3) Let w be a vertex in $V(G') \setminus \{u, v, v_{2\lfloor \frac{d}{2} \rfloor - 1}, v_{d-1}\}$ which is incident to at least 4 edges containing sticks. If no such vertex exists, end the iteration process and proceed with the rest of the proof.
(4) Execute the jumps $wu \cdot \overrightarrow{uv_{d-1}} \cdot uv$ and either (if d is even) $v_{d-1}v \cdot \overrightarrow{uv} \cdot v_{d-1}u$ or (if d is odd) $v_{d-2}u \cdot \overrightarrow{uv} \cdot v_{d-1}u, v_{d-1}v \cdot \overrightarrow{vv_{d-2}} \cdot v_{d-2}u$. Note that we have holes in exactly the edges incident to v and in the edge wu.

(5) Redefine $G' \leftarrow G' - v$ and $v \leftarrow w$ while keeping the stick situation as is.

Eventually we will leave the procedure at Step 3 because $V(G)$ is finite. Denote by G'' the graph obtained from G' by deleting the edges vv_j for all $j \in [2\lfloor \frac{d}{2} \rfloor - 2]$ and keep the placement of sticks as in G'. By construction, G'' contains at most three vertices of degree larger than 3, namely u, v_{d-2}, v_{d-1} (not all of them need to have such a large degree). Denote by W the set of vertices of degree 4 in G''. We distinguish the cases $W = \{u\}$, $W = \{u, a\}$, and $W = \{u, a, b\}$, where $a, b \in \{v_{d-2}, v_{d-1}\}$. Note that W cannot be empty since u has at least four neighbors, namely v and its neighbors apart from u, which were at least three in G'.

In the first case, the graph G'' (in the current state) is a generalized windmill which can be stick-solved with Proposition 2.5.

The second case can be stick-solved in the following way. Jump $va \cdot \overrightarrow{au} \cdot uv$ and successively remove all sticks from edges incident to a (as done with a $K_{1,n}$) ending with the last stick in av. The graph obtained from G'' by deleting all edges incident to a except for av and au is a generalized windmill which is stick-solvable by Proposition 2.5.

The third case is similar to the second one. We start by jumping $va \cdot \overrightarrow{au} \cdot uv$ and perform the same jumps as in the previous case ending with the last stick in au (instead of in av). Next, we jump $au \cdot \overrightarrow{uv} \cdot va, vb \cdot \overrightarrow{bu} \cdot uv, av \cdot \overrightarrow{vu} \cdot ua$. Now we eliminate all sticks from edges incident to b ending with a stick in bu. Again, we obtain a generalized windmill which is stick-solvable by Proposition 2.5. □

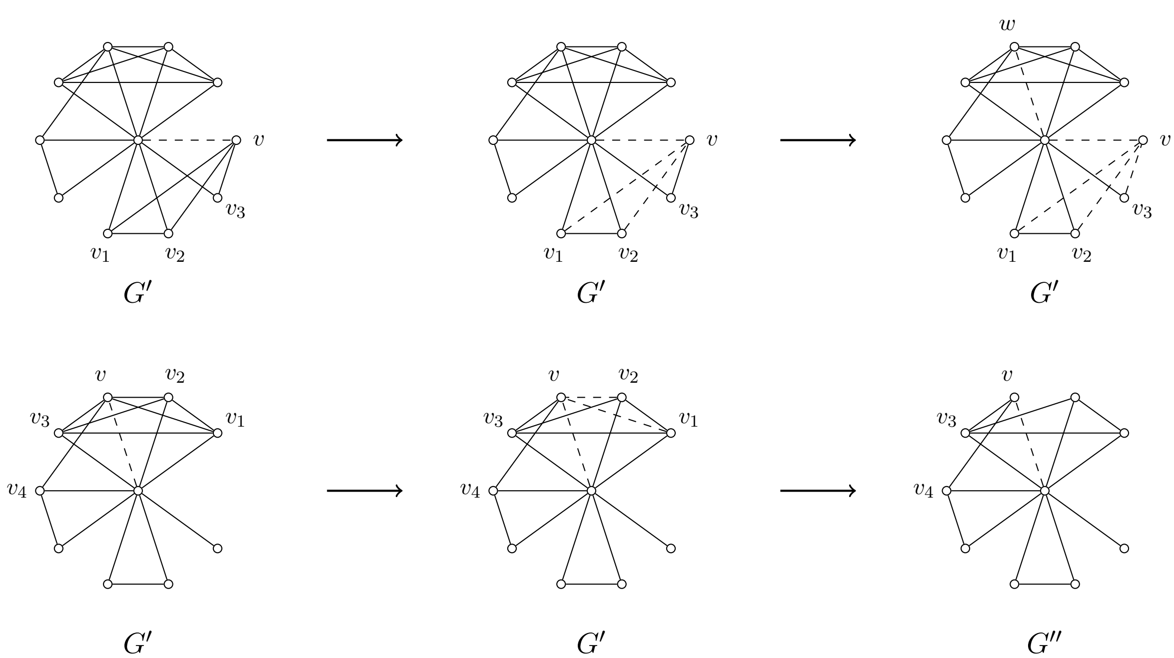

FIGURE 15. Iteration steps of Proposition 4.1 illustrated.

When considering peg solitaire, there is a helpful result about the stick-solvability of subgraphs, since additional edges increase the number of possible moves. An analogue result can be shown for stick solitaire when increasing the number of possible moves. This can be done via the contraction of a graph.

Definition 4.2. *Let G be a graph and $v, w \in V(G)$. The* contraction *of G with respect to v and w, denoted by $G[v = w]$, is the graph where the vertices v and w are identified (cf. Figure 16), that is, $G[v = w] = (V', E')$ with $V' = (V(G)\backslash\{v, w\}) \cup \{x\}$ (where $x \notin V(G)$) and*

$$E' = (E(G)\backslash(\{vu\colon\ u \in V(G)\} \cup \{wu\colon\ u \in V(G)\})) \cup \{xu\colon\ vu \in E(G) \text{ or } wu \in E(G)\}.$$

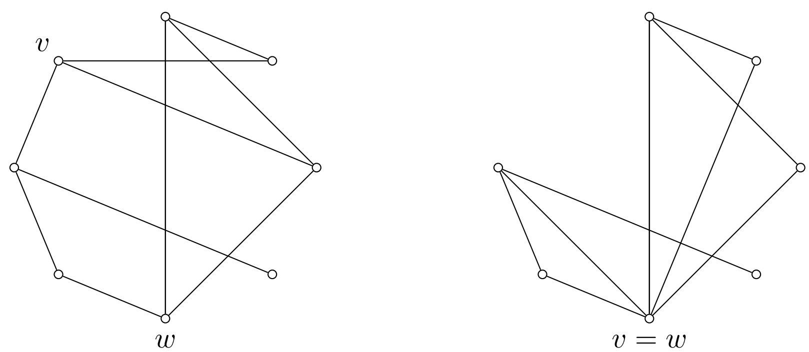

Figure 16. A graph G (left) and the contraction $G[v = w]$ (right).

We get the following straightforward result.

Proposition 4.3. *Let G be a graph and let $v, w \in V(G)$ such that $vw \notin E$ and v and w have no common neighbors. Then $\mathrm{t}'(G[v = w]) \le \mathrm{t}'(G)$.*

Proof. Note that incidences of all edges not incident to v or w remain the same, while all edges that where incident to v or w are now incident. Since all moves which are possible in G can also be executed in $G[v = w]$ (in general even more), the statement follows. □

5. Fool's stick solitaire

In this section, we examine the game of fool's stick solitaire. As in the original peg solitaire, Fool's stick solitaire is the game of playing stick solitaire in the worst possible way, i.e., we are interested in the cardinality of a largest possible terminal state. This cardinality will be denoted by $\mathrm{Ft}'(G)$ (analogously $\mathrm{Fs}(G)$ is the fool's peg solitaire number of G). Since stick solitaire on G is the same as peg solitaire on $L(G)$, we clearly have $\mathrm{Ft}'(G) = \mathrm{Fs}(L(G))$. This already gives the number $\mathrm{Ft}'(G)$ for path graphs and cycle graphs when using the following known result about $\mathrm{Fs}(P_n)$ and $\mathrm{Fs}(C_n)$.

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Proposition 5.1. *[6, Theorem 3.4, Theorem 3.5]*

- *For the path on n vertices, if $n > 3$, then* $\mathrm{Fs}(P_n) = \lfloor \frac{n}{2} \rfloor$.
- *For the cycle on n vertices,* $\mathrm{Fs}(C_n) = \lfloor \frac{n-1}{2} \rfloor$.

Proposition 5.2. *Let n be an integer greater than 2.*

- *If $n > 4$, then we have* $\mathrm{Ft}'(P_n) = \lfloor \frac{n-1}{2} \rfloor$.
- *We have* $\mathrm{Ft}'(C_n) = \lfloor \frac{n-1}{2} \rfloor$.

Proof. Use $L(P_n) = P_{n-1}, L(C_n) = C_n$ and Proposition 5.1. □

A trivial upper bound for $\mathrm{Fs}(G)$ is the independence number of G. Note that any maximum independent set in $L(G)$ corresponds to a maximum matching in G. Let $\nu(G)$ be the matching number of G, that is, the number of edges in a maximum matching of G. Together with well-known bounds [6, Theorem 2.1, Theorem 2.3] for $\mathrm{Fs}(G)$, we get the following result.

Proposition 5.3. *We have $\mathrm{Ft}'(G) \leq \nu(G)$. If for any maximum matching M of G the complement $E \backslash M$ is a matching with at least two edges, then $\mathrm{Ft}'(G) \leq \nu(G) - 1$.*

We will use this bound to determine $\mathrm{Ft}'(G)$ for the graph classes considered before.

Proposition 5.4. *We have $\mathrm{Ft}'(K_n) = \lfloor \frac{n}{2} \rfloor$ for every positive integer n.*

Proof. If $n = 3$, this is obvious, so suppose $n \geq 4$. Denote the vertices of K_n by $v_1, \ldots, v_n$ such that the hole is in v_1v_2. Consider the subgraph defined by the edges v_1v_2, v_1v_i and v_2v_i for $i \geq 3$, which can be stick-solved analogously to double stars with terminal stick in v_1v_3. Next, jump $v_3v_4 \cdot \overrightarrow{v_1v_3} \cdot v_1v_2$. Now the subgraph induced by the vertices $v_3, \ldots, v_n$ is isomorphic to K_{n-2} and has exactly one hole. Iterating the process yields $\mathrm{Ft}'(K_n) \geq \lfloor \frac{n}{2} \rfloor$. Since $\nu(K_n) = \lfloor \frac{n}{2} \rfloor$, the result follows from Proposition 5.3. □

Proposition 5.5. *Given positive integers m and n with $m + n \geq 3$, we have*

$$\mathrm{Ft}'(K_{m,n}) = \begin{cases} 1 & \text{if } m = n = 2, \\ \min\{m, n\} & \text{otherwise.} \end{cases}$$

Proof. First note that Proposition 5.3 yields $\mathrm{Ft}'(K_{m,n}) \leq \min\{m, n\}$.

Let w.l.o.g. $m \leq n$. If $m = n = 2$, then $K_{m,n} = C_4$, hence the result follows from Proposition 5.2. Let $m = 2, n = 3$ and suppose V partitions into sets $\{v, w\}$ and $\{a, b, c\}$. Start with a hole in va and jump $bw \cdot \overrightarrow{wa} \cdot av, vc \cdot \overrightarrow{cw} \cdot wb, vb \cdot \overrightarrow{bw} \cdot wc$. This results in a state where sticks are in the non-incident edges va and wc. If $m = 2, n \geq 4$, start playing on a subgraph isomorphic to $K_{m,n-2}$, ending in a terminal state with exactly one stick. The remaining sticks together with an empty edge form a subgraph isomorphic to $K_{2,3}$, which can be stick-solved to a terminal state with exactly two sticks. This gives the result if $m = 2$.

If $m = n = 3$, denote the vertices by $v_1, v_2, v_3, w_1, w_2, w_3$, start with a hole in v_1w_1 and execute the jumps

$$w_2v_3 \cdot \overrightarrow{v_3w_1} \cdot w_1v_1, w_1v_2 \cdot \overrightarrow{v_2w_2} \cdot w_2v_3, v_1w_3 \cdot \overrightarrow{w_3v_2} \cdot v_2w_1, v_1w_1 \cdot \overrightarrow{w_1v_2} \cdot v_2w_2, v_3w_2 \cdot \overrightarrow{w_2v_1} \cdot v_1w_1.$$

Let now $m, n \geq 3$ with $(m, n) \neq (3, 3)$. Denote the vertices by $v_1, \ldots, v_m$ resp. $w_1, \ldots, w_n$ such that the hole is in the edge v_1w_1. Consider the subgraph consisting of the edges v_1w_i for $i \in [n]$ and v_jw_1 for $j \in [m]$. This is a double star $\mathrm{DS}(m-1, n-1)$ with centers v_1 and w_1. According to Proposition 2.8 this can be stick-solved with the terminal stick in the edge v_2w_1. Next jump $w_2v_2 \cdot \overrightarrow{v_2w_1} \cdot w_1v_1$ to get a stick in v_1w_1. Now consider the subgraph of $K_{m,n}$ induced by all edges that are neither incident to v_1 nor incident to w_1. This is isomorphic to $K_{m-1,n-1}$ and there is a stick in all edges but one. Iterating the above process and using the previous results gives the desired statement. □

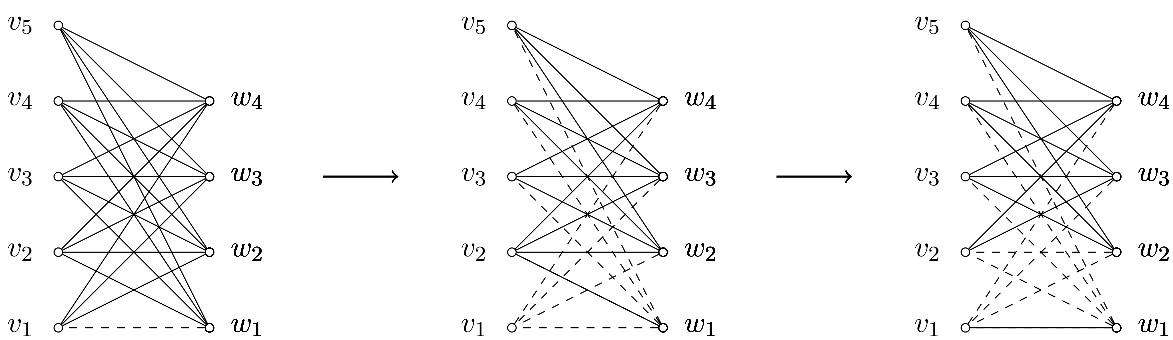

FIGURE 17. Proof idea of Proposition 5.5 for the case $m, n \geq 3$ illustrated.

Proposition 5.6. *For non-negative integers P, B with $P + 2B \geq 2$ we have*

$$\mathrm{Ft}'(W(P, B)) = \begin{cases} B & \text{if } P = 0, B \neq 0, \\ B + 1 & \text{else.} \end{cases}$$

Proof. Denote the edges incident to the universal vertex with e_i and the other edges with f_i.

For $B = 0$ the statement follows from Proposition 5.5, so assume $B \neq 0$.

If $P = 0$, we start with a hole in some e_i. Let e_j be the edge that is incident to the same f_k as e_i. Stick-solve the inner star ending in $e \in \{e_i, e_j\}$. After $e \cdot \overrightarrow{f_k} \cdot e'$, where $e' \in \{e_i, e_j\} \setminus \{e\}$, no more jumps are possible (cf. Figure 18). Therefore, we have $\mathrm{Ft}'(W(0, B)) \geq B$, which proves the statement since $\nu(W(0, B)) = B$.

In the last case $(P, B \geq 1)$ we have $\nu(W(P, B)) = B + 1$. If $P = B = 1$, we start with a hole in the edge, say e_1, which is not incident to f_1 and jump $e_2 \cdot \overrightarrow{e_3} \cdot e_1$ to obtain a terminal state with two sticks. Otherwise we start with a hole in an edge that is incident to some f_i. We stick-solve the inner star ending with a stick in an edge that is not incident to some f_j (which is possible since the inner star has either at least five edges or four edges of which two do not lie in a blade). This procedure results in a terminal state with $B+1$ sticks. □

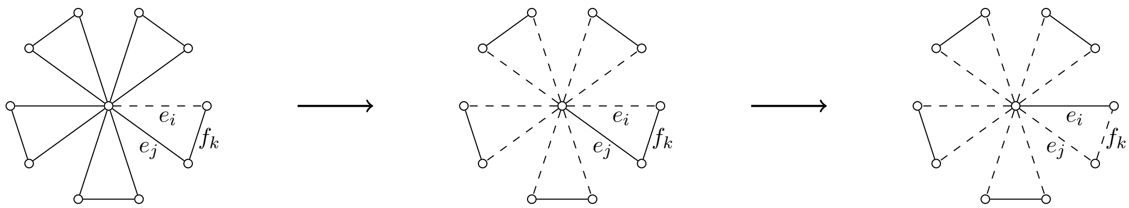

FIGURE 18. Proof idea of Proposition 5.6 for $P = 0$ illustrated.

Proposition 5.7. *We have* $\mathrm{Ft}'(\mathrm{DS}(L,R)) = 2$.

Proof. First note $\nu(\mathrm{DS}(L,R)) = 2$, hence $\mathrm{Ft}'(\mathrm{DS}(L,R)) \leq 2$. We have already seen in Proposition 2.7 that $\mathrm{Ft}'(\mathrm{DS}(L,R)) = 2$ if $(L,R) \in \{(1,1),(2,1),(1,2),(2,2)\}$. So assume $L, R \geq 2$ with $(L,R) \neq (2,2)$. W.l.o.g. assume $L \geq 3$ and adopt the notation from the proof of Proposition 2.7. Start with a hole in f_1. Stick-solve the right star ending with a stick in some f_i (and a hole in c). Then stick-solve the left star ending with a stick in some e_j (which is possible because of $L \geq 3$). We arrive at a terminal state with two sticks. □

6. Open problems

Certainly many more questions or problems could be considered, such as the determination of $\mathrm{t}'(G)$ and $\mathrm{Ft}'(G)$ for other classes of graphs. As for the original game of peg solitaire, it would be nice to have a characterization of stick-solvable trees (or at least for certain classes of trees). Another, more general, question is the characterization of peg-solvable line graphs (i.e. stick-solvable graphs).

It would also be interesting to find all graphs which are peg-solvable as well as stick-solvable. This would give a characterization of peg-solvable graphs, whose line graphs are also peg-solvable.

Is it possible that adding an edge between existing vertices decreases the stick-solvability number? Under which circumstances will an edge increase or decrease the stick-solvability number? What is the minimal number of edges one has to add to obtain a stick-solvable graph? How much can the stick-solvability number change?

Another approach would be the concept of contracting edges that we briefly discussed in Section 4. Thus, one should rather ask if contracting appropriate edges always yields (after some steps) a stick-solvable graph and if so, how many edges we have to contract to obtain a stick-solvable graph. It would also be interesting to characterize those graphs which are obtainable by the contraction defined in Section 4 (since our type of contraction poses additional restraints on the graph, known results on contractions do not apply). Another intriguing question is whether Proposition 4.3 also holds if non-adjacent vertices that do have a common neighbor are contracted.

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