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ON THE sd_b-CRITICAL GRAPHS

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Abstract. A b-coloring of a graph G is a proper coloring of its vertices such that each color class contains a vertex that has a neighbor in every other color classes. The b-chromatic number of a graph G, denoted by $b(G)$, is the largest integer k such that G admits a b-coloring with k colors. Let G_e be the graph obtained from G by subdividing the edge e. A graph G is sd_b-critical if $b(G_e) < b(G)$ holds for any edge e of G. In this paper, we first present several basic properties of sd_b-critical graphs and then we give a characterization of sd_b-critical P_4-sparse graphs and sd_b-critical quasi-line graphs.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let G be a graph with vertex-set $V = V(G)$ and edge-set $E = E(G)$. For a vertex $v \in V$, we let $N_G(v)$ denote the neighborhood of v. The *degree* of a vertex v of G is $d_G(v) = |N_G(v)|$. A vertex with degree zero is called an *isolated vertex.* Whereas, a vertex with degree one is called a *pendent vertex,* and its neighbor is called its *support.* The *maximum* and *minimum* degree of G is denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a non-empty subset $A \subseteq V(G)$, the subgraph induced by A is denoted by $G[A]$. An *independent set* in a graph is a subset of its vertices that does not induce any edge. A set of pairwise adjacent vertices in a graph is called a *clique.* The *join of two graphs* G and H, denoted by $G \vee H$, is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H. In particular, $G \vee H$ is called a

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complete split graph if $V(G)$ is a clique and $V(H)$ is an independent set. As usual, C_n and K_n denote a chordless cycle and a complete graph on n vertices, respectively.

A *proper coloring* of a graph G is a function $c : V(G) \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ for every edge uv of G. The smallest integer k for which G admits a proper coloring with k colors is the *chromatic number* of G, denoted by $\chi(G)$.

A *b-coloring* of G is a proper coloring so that for each color class there exists a vertex having neighbors in all the other color classes. We call any such vertex a b-vertex. The *b-chromatic number* of G, denoted by $b(G)$, is the largest integer k for which G has a b-coloring with k colors. This parameter was initially defined and studied by Irving and Manlove [12] in 1999, where they proved that determining $b(G)$ is an NP-complete problem in the general case, even when restricted to bipartite graphs. However, they established that the determination of this parameter can be achieved in polynomial time in trees. Subsequently, several results were obtained on this parameter for different classes of graphs, see [7, 14, 15, 16]. In particular, it was observed that

$$b(G) \leq \Delta(G) + 1. \tag{1.1}$$

Further investigations into the effects of particular graph operations, such as edge or vertex deletion and edge contraction, on the b-chromatic number have been explored in various classes of graphs. The authors of [3, 5, 9] investigated graphs where the b-chromatic number decreases upon the removal of any edge. Blidia et al. [6] provided a characterization of trees in which the b-chromatic number increases upon the deletion of any vertex. Ikhlef Eschouf and Blidia [4] have considered graphs for which the b-chromatic number increases with the removal of any edge (or any vertex). In [2], the authors characterized the trees whose b-chromatic number decreases on the contraction of any edge of G. For further works, see [1, 17]

In the remainder of this section, we introduce few additional definitions and notation. Consider a graph G and an edge uv of G. A *subdivision* of uv consists of removing uv from G and introducing a new vertex that connects to both u and v. We denote by G_{uv} the graph obtained from G by subdividing the edge uv and we denote by $\overline{uv}$ the new vertex so obtained.

Definition 1. *A graph G is called sd_b-*critical *if $b(G_{uv}) < b(G)$ holds for every edge uv of G.*

The main aim of this paper is to characterize some classes of sd_b-critical graphs. We begin, in section 2, by introducing specific properties related to sd_b-critical graphs. In section 3, we provide a characterization of sd_b-critical P_4-sparse graphs and sd_b-critical quasi-line graphs.

Note that a graph G is *sd_b-critical* if and only if the graph obtained from G by deleting all isolated vertices is *sd_b-critical*. For this reason, we assume that all graphs considered in this paper are without isolated vertices.

2. Preliminary results

We begin this section with some properties of sd_b-critical graphs that play an important role in our investigation.

Theorem 2. *Let $G = (V, E)$ be a sd_b-critical graph, c be a b-coloring of G with $b(G)$ colors and B be the set of all b-vertices of c. Then following statements hold.*

(i) $b(G) \geq 3$.

(ii) *If any two vertices of G both are in B or both are adjacent to some vertex in B, then they have a different color.*

(iii) $|B| = b(G)$ *and* $d_G(u) = b(G) - 1$ *for every vertex* $u \in B$.

(iv) $V - B$ *is an independent set.*

(v) $\delta(G) \geq 2$ *and if* $v \in V - B$, *then* $d_G(v) \leq b(G) - 2$.

(vi) $b(G) = \Delta(G) + 1$.

Proof. (i) Suppose (i) is not true. Therefore, as G contains at least one edge, we see that $b(G) = 2$. Let G_e be the graph obtained from G by subdividing any edge e of G. The same argument shows that $b(G_e) \geq 2 = b(G)$, which is impossible.

(ii) Suppose on the contrary that G has two vertices, say x and y of the same color.

To prove the first part assume that x and y both are in B. Suppose that one neighbor of x, say u is $V - B$. Let π be a coloring of G_{xu} obtained from c as follows. Color $\overline{xu}$ by one missing color in its neighborhood (this is possible since $b(G) \geq 3$ and $d_{G_{bx}}(\overline{xu}) = 2$)) and all the other vertices of G_{ux} keep their colors initially given by c. It is easy to see that π is a b-coloring of G_{ux} using $b(G)$ colors, a contradiction. Assume now that all neighbors of x are in B and let v be any neighbor of y. By coloring vertices of G_{yv} as before (by replacing x by y and u by v), we get a b-coloring of G_{yv} using $b(G)$ colors, a contradiction.

To prove the second part assume that x and y have a common neighbor, say v in B. By the first part, we can assume that one of x and y, say y is in $V - B$. The previous argument (applied to G_{yv}) leads to the same contradiction as before.

(iii) Is a direct consequence of item (ii).

(iv) Suppose to the contrary that $V - B$ contains two adjacent vertices, say x and y. Let G_{xy} be the graph obtained from G by subdividing the edge xy. Color $\overline{xy}$ by any color of c different from $c(x)$ and $c(y)$, while the other vertices keep their colors initial given by c. This gives a b-coloring of G_{xy} with $b(G)$ colors, a contradiction.

(v) We start by proving the first part. Let $v \in V(G)$ and suppose that $d_G(v) \leq 1$. Since G is without isolated vertices and $b(G) \geq 3$, respectively, we have $d_G(v) = 1$ and $v \in V - B$. Let u be the support vertex adjacent to v in G. By (iv), $u \in B$. Let G_{uv} be the graph obtained from G by subdividing the edge uv. Define a coloring of G_{uv} from c as follows. Color $\overline{uv}$ by $c(v)$ and color v by $c(u)$. All the other vertices keep their colors initially given by c. We obtain a b-coloring of G_{uv} using $b(G)$ colors, a contradiction. Thus $d_G(v) \geq 2$.

To prove the second part, let $v \in V - B$. From (iv), all neighbors of v are in B. This implies that $d_G(v) \leq |B| - 1$. If $d_G(v) = |B| - 1$, then v would be a b-vertex of c, a contradiction. Thus $d_G(v) \leq |B| - 2$. This together with the first part of item (iii) imply that $d_G(v) \leq b(G) - 2$.

(vi) Follows as a direct consequence of (iii) and (v). □

Corollary 3. *Let $G = (V, E)$ be a sd_b-critical graph. Then $b(G) = 3$ if and only if $G = K_3$.*

Proof. Let c be a b-coloring of G with 3 colors and B be the set of all b-vertices of c. By the first part of Theorem 2-(iii), $|B| = 3$. We shall show that $V - B$ is empty. To the contrary, suppose that $V - B$ contains at least one vertex say a. By the second part of Theorem 2-(v) shows that a is pendant vertex, which contradicts the first part of the same item. Thus $V - B$ is empty and so $G = K_3$. The converse is obvious. □

3. sd_b-critical P_4-sparse graphs

In [10], Hoàng introduced the class of P_4-sparse graphs as those in which every set of five vertices induces at most one P_4. This class of graphs is strictly contained in the class of P_5-free graphs and strictly contains P_4-free graphs.

A graph G is called a spider if its vertex-set admits a partition into three sets S, K and R such that:

1- S is a stable set and K is a clique with $|S| = |K| \geq 2$.

2- Each vertex belonging to R is adjacent to all vertices of K and to no vertex of S.

3- There exists a bijection $f : S \to K$ such that exactly one of the following statements holds

- $i)$ for each vertex $v \in S$, $N_G(v) \cap K = \{f(v)\}$.
- $ii)$ for each vertex $v \in S$, $N_G(v) \cap K = K \backslash \{f(v)\}$.

If case 3-$i)$ holds, then G is called a thin spider. While if case 3-$ii)$ holds, then G is called a thick spider.

The P_4-sparse graphs have been characterized independently by Hoàng [10] and Jamison and Olariu [13].

Theorem 4. [10, 13] *For a graph G, the following conditions are equivalent.*

(i) *G is P_4-sparse graph,*

(ii) *for every induced subgraph H of G with at least two vertices, exactly one of the following statements is true:*

(ii.1) *H is disconnected;*

(ii.2) *$\overline{H}$ is disconnected;*

(ii.3) *H is isomorphic to a spider.*

In order to characterize sd_b-critical P_4-sparse graphs, we need to define the class of graphs $\mathcal{F}_{p,k}$ as follows.

Definition 5 (Class $\mathcal{F}_{p,k}$)**.** *Let $p \geq 2$ and k be two integers. A graph G is in a class $\mathcal{F}_{p,k}$ if G has p connected components $G_1, G_2, \ldots, G_p$, such each G_i, $(i \in \{1, 2, \ldots, p\})$ satisfies the following conditions*

(i) *G_i is a complete split graph with vertex-set $A_i \cup B_i$.where A_i and B_i are , respectively, independent and clique, each of order at least two.*

(ii) $|A_i| + |B_i| = |B_1| + |B_2| + \cdots + |B_p| = k \geq 2p$.

Figure 1 shows two graphs belonging to $\mathcal{F}_{p,k}$

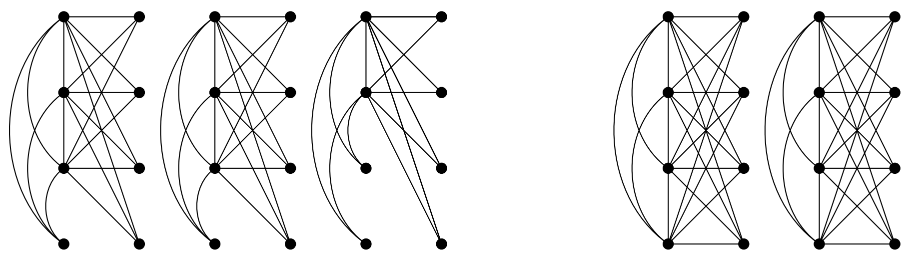

FIGURE 1. Two graphs belonging to $\mathcal{F}_{p,k}$

The following observation follows from the definition of $\mathcal{F}_{p,k}$.

Observation 6. *Let G be a member of $\mathcal{F}_{p,k}$ and G_e be the graph obtained from G by subdividing any edge e of G. Then, we have*

(i) $\triangle(G_e) = \triangle(G) = k - 1 \geq 3$.

(ii) *Each vertex in $\cup_{i=1}^{p} B_i$ has degree exactly $k-1$ in G_e and each vertex in $\cup_{i=1}^{p} A_i$ has degree at most $k-2$ in G_e.*

Lemma 7. *If $G \in \mathcal{F}_{p,k}$, then $b(G) = \triangle(G) + 1$.*

Proof. Form Observation 6 and (1.1) it is enough to exhibit a b-coloring of G with $\triangle(G)+1$ colors as follows. Assign distinct colors to each vertex in $\cup_{j=1}^{p} B_j$ and for each i in $\{1, 2, \ldots, p\}$, color all vertices in A_i differently using colors that appear in $(\cup_{j=1}^{p} B_j) - B_i$. □

From Observation 6, we remark that Definition 5-(ii) remains valid for G_{uv}.

Lemma 8. *If $G \in \mathcal{F}_{p,k}$, then G is sd_b-critical graph.*

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Proof. Suppose to the contrary that G is not sd_b-critical graph. Then for some edge uv in some component G_t, $t \in \{1, 2, \ldots, p\}$, we have $b(G_{uv}) \geq b(G)$. By Lemma 7, $b(G_{uv}) \geq \triangle(G)+1$. Combining this together with (1.1) and Observation 6-(i), we get

$$b(G_{uv}) = \triangle(G_{uv}) + 1 = k \geq 4. \tag{3.1}$$

Let c be a b-coloring of G_{uv} using $\triangle(G_{uv}) + 1$ colors and let B the set of all b-vertices of c. Next, we assert that any two b-vertices of c have different colors. That is

$$|B| = k. \tag{3.2}$$

Indeed, since $|B| \geq b(G_{uv})$, (3.1) yields $|B| \geq k$. Let us now prove $|B| \leq k$. To this end, observe first that $\overline{uv}$ is not a b-vertex since its degree is two, while $b(G_{uv}) \geq 4$. Also, in view of (3.1) and Observation 6-(ii), we see that the only vertices in G_{uv} that are candidates to be b-vertices are those in $\cup_{i=1}^{p} B_i$. This gives $|B| \leq |B_1| + |B_2| + \cdots + |B_p|$. This and the remark before Lemma 8 imply the upper bound. Thus, the assertion holds.

Next, we consider two cases.

Case 1. $u, v \in B_t$. Notice that in this case B_t is not a clique in G_{uv}. Since u is adjacent to all vertices of A_t, color $c(u)$ cannot appear in A_t. By the previous assertion, $c(u)$ cannot appear in $B_t - \{u\}$. In particular, $c(u) \neq c(v)$. But this is impossible since v is a b-vertex that needs the color $c(u)$ in its neighborhood.

Case 2: $u \in B_t$ and $v \in A_t$. From (3.1), we see that the neighbors of any b-vertex of c have distinct colors. As v is adjacent to all vertices of $B_t - \{u\}$, the color $c(v)$ cannot appear in $B_t - \{u\}$. Therefore, since u is a b-vertex, color $c(v)$ must appear in $A_t - \{v\}$. But in this case, color $c(v)$ appears twice in the neighborhood of any vertex in $B_t - \{u\}$, which is impossible.

In each case, we have a contradiction, implying that G is sd_b-critical graph. □

Now we give a characterization of the sd_b-critical P_4-sparse graphs.

Theorem 9. *Let G be a P_4-sparse graph. Then G is sd_b-critical if only if G is either a complete graph or $G \in \mathcal{F}_{p,k}$.*

Proof. Necessity is easy when G is a complete graph. Also, by Lemma 8, any member of $\mathcal{F}_{p,k}$ is sd_b-critical. So, let us prove the sufficiency. Let G be a sd_b-critical P_4-sparse graph and consider a b-coloring c of G with $b(G)$ colors. Let B be the set of all b-vertices of c. Next, we distinguish between two cases:

Case 1. $\overline{G}$ is not connected. Then G is the join of two graphs G_1 and G_2. We claim that $V - B$ is an empty set. Suppose for the sake of contradiction that $V - B$ contains at least one vertex say x. Assume that $x \in V(G_1)$. Since x is adjacent to all vertices of G_2, the color of x cannot appear in G_2

and further, by Theorem 2-(iv)), all vertices of G_2 are in B. Let $y \in B$ having the same color as x. Then clearly $y \in G_1$ and therefore the color of x appears twice in the neighborhood of each vertex of G_2, which contradicts Theorem 2-(ii). Thus all vertices of G are b-vertices. Using Theorem 2-(iii) and the fact that every b-vertex requires all colors except its own within its neighborhood, we conclude that G is a complete graph.

Case 2: $\overline{G}$ is connected.

Let H be any component of G (possibly $H = G$). So, we claim the following.

Claim 1. *H is not a spider.*

Proof of Claim 1. Suppose that H is a spider and let $V(H) = S \cup K \cup R$. Since $\delta(H) \geq 2$ (by Theorem 2-(v)), H must be a thick spider with $|S| = |K| \geq 3$. In such a case, every vertex of $R \cup S$ has degree at most $|R| + |K| - 1$ and every vertex of K has degree exactly $|K| + |S| + |R| - 2$. Therefore, as $|S| \geq 3$, it follows that

$$\triangle(H) = |K| + |S| + |R| - 2. \tag{3.3}$$

Observe that $\triangle(H) = \triangle(G)$, for otherwise all vertices of H belong to $V - B$ and are isolated (by Theorem 2-(vi)), which is impossible. This together with (3.3) and Theorem 2-(vi) imply that

$$b(G) = |K| + |S| + |R| - 1 = |V(H)| - 1. \tag{3.4}$$

It follows from (3.4), that H contains exactly two vertices having the same color. These two vertices both cannot be in K (since K is a clique). But then K contains some vertex having a repeated color in its neighborhood, which contradicts the Theorem-(ii)). This finishes the proof of Claim 1. □

Claim 1 together with Theorem 4 imply that G has $p \geq 2$ components such that each component G_i of G is the join of two graphs G_i^1 and G_i^2. Let $k = \Delta(G) + 1$ and $B = B_1 \cup B_2 \cup \cdots B_p$,where for $i \in \{1, 2, \ldots, p\}$, $B_i = B \cap V(G_i)$ and $A_i = V(G_i) - B_i$. Observe that $A_i \subseteq V(G) - B$ is independent according to Theorem 2-(iv)).

Claim 2. *Conditions (i) and (ii) of Definition 5 are fulfilled.*

Proof of Claim 2. We first show that Definition 5-(i) is fulfilled. To do this, we start by proving that B_i is a clique of order at least two. To the contrary, suppose that B_i contains at least two non-adjacent vertices, say b_1 and b_2. Since G_i is the join of G_i^1 and G_i^2, both b_1 and b_2 must belong to either G_i^1 or G_i^2, say G_i^1 (up to symmetry). Given that b_1 is a b-vertex, it must be adjacent to some vertex having the same color as b_2, say a. Since all colors in G_i^1 cannot appear in G_i^2, it follows that $a \in V(G_i^1)$. Furthermore, by Theorem 2-(ii), a is not a b-vertex and so $a \in A_i$. Since a is adjacent to all vertices of G_i^2 and A_i is independent, all vertices of G_i^2 are b-vertices and so $V(G_i^2) \subseteq B_i$. This means that the color of b_2 appears twice in the neighborhood of every b-vertex in $V(G_i^2)$, which contradicts Theorem

2-(ii). Thus B_i is a clique. Since $\delta(G) \geq 2$ and A_i is independent, it follows that $|B_i| \geq 2$.

Next, we shall show that A_i contains at least two vertices. Suppose first that A_i is empty; so $B_i = V(G_i)$. Since each b-vertex needs all colors (except its own color) in its neighborhood, all colors of c appear in G_i. This fact together with Theorem 2-(ii) imply that $B_i = B$ and so $V(G - G_i) = V - B$. But then, according to Theorem 2-(iv), all vertices of $G - G_i$ are isolated, which is impossible since $\delta(G) \geq 2$. Thus A_i is nonempty. Suppose now that A_i contains exactly one vertex, say a. Since B_i is a clique and all vertices of B_i have the same degree (by Theorem 2-(iii)), a must be adjacent to all vertices of B_i. But this implies that a is a b-vertex of c, contradiction. Thus $|A_i| \geq 2$.

Finally, we prove that G_i is a complete split graph with vertex-set $A_i \cup B_i$. Since A_i is independent and G_i is the join of G_i^1 and G_i^2, we can assume, up to symmetry, that $A_i \subseteq V(G_i^1)$ and therefore $V(G_i^2) \subseteq B_i$. Let $C_i = V(G_i^1) - A_i$ (possibly empty). Observe that $V(G_i^2) \cup C_i = B_i$. Since B_i is a clique and every vertex in B_i has degree $\Delta(G)$ (by Theorem 2-(iii) and (vi)), it follows that each vertex in C_i is adjacent to all vertices of A_i and vice versa. Thus G_i is a complete split graph with vertex-set $A_i \cup B_i$.

We prove now that Definition 5-(ii) is fulfilled. Let u be any vertex in B_i. Using items (iii) and (vi) of Theorem 2, we get

$$d_{G_i}(u) = \Delta(G) \text{ and } |B| = b(G) = \Delta(G) + 1. \tag{3.5}$$

Furthermore, since G_i is a complete split graph with vertex-set $A_i \cup B_i$,

$$d_{G_i}(u) = |A_i| + |B_i| - 1. \tag{3.6}$$

By combining (3.5) with (3.6) and taking into consideration that $\Delta(G) = k - 1$ and $|B| = |B_1| + |B_2| + \cdots + |B_p|$, we get

$$|A_i| + |B_i| = k.$$

and

$$|B_1| + |B_2| + \cdots + |B_p| = k. \tag{3.7}$$

Since $|B_i| \geq 2$ for all i, (3.7) gives

$$k \geq 2p.$$

, so Claim 2 is proved. □

From our previous discussion we conclude that G is either a complete graph or it is in $\mathcal{F}_{k,p}$. □

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4. sd_b-critical quasi-line graphs

A graph G is a quasi-line graph if for every vertex $v \in V(G)$, the set of neighbors of v in G can be expressed as the union of two cliques. By definition, quasi-line graphs are claw-free.

Let $\mathcal{F}$ be the set of two graphs depicted in Figure2.

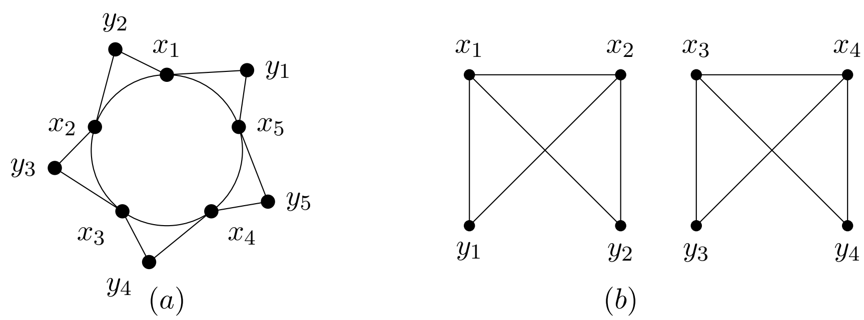

FIGURE 2. Family $\mathcal{F}$

Lemma 10. *Let G be a sd_b-critical quasi-line graph with vertex-set V. Let c be a b-coloring of G with $b(G)$ colors and B be the set of all b-vertices of c. Then all colors appearing in $V - B$ are mutually distinct.*

Proof. To the contrary, suppose there exist two vertices $u, v \in V - B$ such that $c(u) = c(v)$. Since $\delta(G) \geq 2$ (by Theorem 2-(v)), we can let u_1 and u_2 (respectively, v_1 and v_2) be the neighbors of u (respectively, v). By Theorem 2-(iv), u_1, u_2, v_1, v_2 are in B. In addition, $u_i \neq v_j$ for all $i, j \in \{1, 2\}$, otherwise if $u_t = v_s$ for some $t, s \in \{1, 2\}$, then c remain a b-coloring of G_{uv_s}, a contradiction. Therefore, since $|B| = b(G)$ (by Theorem 2-(iii)), u_1, u_2, v_1, v_2 have distinct colors. Let $x \in B$ a vertex of color $c(u)$. By Theorem 2-(ii), x has no neighbor in $\{u_1, u_2, v_1, v_2\}$. As x is a b-vertex it must be adjacent to at least four vertices, say x_1, x_2, x_3, x_4 of color $c(u_1)$, $c(u_2)$, $c(v_1)$, $c(v_2)$, respectively. By Theorem 2-(ii), x_1, x_2, x_3, x_4 belong to $V - B$ and are independent. But in this case, x, x_1, x_2, x_3 induce a star centered at x, which is impossible. □

Lemma 11. *If $G \in \mathcal{F}$, then G is sd_b-critical graph.*

Proof. We first prove that

$$b(G) = \Delta(G) + 1. \tag{4.1}$$

From (1.1), it suffices to show that $b(G) \geq \Delta(G) + 1$. To do this, we exhibit a b-coloring of G with $\Delta(G) + 1$ colors as follows. If $G = G_1$, then we assign color i to x_i and color $i + 2$ (modulo 5) to y_i

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for each $i \in \{1, \ldots, 5\}$. While if $G = G_2$, then we assign color i to x_i for each $i \in \{1, \ldots, 4\}$ and assign colors 3, 4, 1, 2 to y_1, y_2, y_3, y_4, respectively. Thus (4.1) holds.

To validate the criticality of G, we shall prove that for any edge e, $b(G_e) \leq b(G) - 1$. Using (4.1) and the fact $\Delta(G) = \Delta(G_e)$, we get

$$b(G_e) \leq \Delta(G_e). \tag{4.2}$$

To the contrary, suppose that (4.2) is not true. Then by (1.1), we have $b(G_e) = \Delta(G_e) + 1$. Let c a b-coloring of G_e using $\Delta(G_e)+1$ colors and let B the set of all b-vertices of c. Since $b(G_e) = \Delta(G_e)+1$ and every b-vertex has degree equal to $\Delta(G_e)$, $B = \{x_1, x_2, \ldots, x_{\Delta(G)+1}\}$ and all the color appearing in B (or in the neighborhood of x_i for each i) are mutually distinct. From this, we can assume, without loss of generality, that for $i \in \{1, \ldots, \Delta(G)\}$, $c(x_i) = i$. By symmetry, we can take $e \in \{x_1x_2, x_1y_1\}$. Next, we consider two cases.

Case 1. $G = G_1$.

Assume first that $e = x_1x_2$. Since x_1 is a b-vertex and color 2 cannot appear in $\{y_2, \overline{x_1x_2}\}$, it follows that $c(y_1) = 2$. On other hand, as all colors are not repeated in the neighborhood of x_2, $\overline{x_1x_2}$ and y_2 cannot be colored by color 3. This means that color 3 cannot appear in the neighborhood of x_1, which is impossible.

Assume now that $e = x_1y_1$. In this case, $c(y_2) \neq 3$, for otherwise x_2 would have a repeated color in its neighborhood. Therefore $c(\overline{x_1y_1}) = 3$ since x_1 is a b-vertex that needs this color in its neighborhood. Since x_5 is a b-vertex, it needs colors 2 and 3 in its neighborhood. As y_1 cannot be colored by 3, it follows that $c(y_1) = 2$ and $c(y_5) = 3$. But in this case, the color 3 is repeated twice in the neighborhood of x_4, a contradiction.

Case 2. $G = G_2$.

If $e = x_1x_2$, then color 2 cannot appear in the neighborhood of x_1 since x_1 and x_2 have the same neighbors in G_e and $c(x_2) = 2$. But this is impossible since x_1 is b-vertex, a contradiction.

If $e = x_1y_1$, then since x_2 a b-vertex, we may assume, without loss of generality, that y_1 and y_2 are colored with colors 3 and 4, respectively. But in this case the color 3 cannot appear in the neighborhood of x_1, which is impossible.

In each case, we have a contradiction. Thus (4.2) holds, implying that G is sd_b-critical graph. □

Now we characterize sd_b-critical quasi-line graphs.

Theorem 12. *Let G be a quasi-line graph. Then G is sd_b-critical if and only if $G = K_n$ or $G \in \mathcal{F}$.*

Proof. If $G = K_n$, then it is easy to check that G is sd_b-critical. If $G \in \mathcal{F}$, then by Lemma 11, G is sd_b-critical. To show the converse, let G be a sd_b-critical quasi-line graph. Consider a b-coloring c of

G using $b(G)$ colors and let $B = \{b_1, b_2, \ldots, b_k\}$ be the set of all b-vertices of c. By items (iii) and (iv) of Theorem 2, we have $k = \triangle(G) + 1 = b(G)$. Again, in view of Theorem 2-(iii), we can assume that each b_i has color i for each $i \in \{1, 2, \ldots, k\}$.

Consider first the case when B is a clique. Since $d_G(b_i) = \triangle(G)$ (by items (iii) and (v) of Theorem 2), b_i has no neighbor in $V - B$. This together with the fact G is without isolated vertices imply that $V - B$ is an empty set, implying that $G = K_n$.

From now on, we can assume that B is not a clique. Therefore $V - B$ is a nonempty set since every vertex in B has degree exactly $\Delta(G)$. Let $V - B = \{a_1, a_2, \ldots, a_t\}$ with $t \geq 1$. From Lemma 10 and without loss of generality, we may assume that a_i has color i for each $i \in \{1, 2, \ldots, t\}$. By combining Theorem 2-(i) with Corollary 3, we get $k = |B| = b(G) \geq 4$.

Claim 3. *For each $i \in \{1, 2, \ldots, k\}$, $|N_G(b_i) \cap (V - B)| = 2$.*

Proof of Claim 3. To the contrary, suppose that for some i in $\{1, 2, \ldots, k\}$, we have $|N(b_i) \cap (V - B)| \neq 2$. If b_i has three neighbors in $V - B$, then since $V - B$ is independent (by Theorem 2-(iv)), these three vertices together with b_i induce a star $K_{1,3}$ centered at b_i, a contradiction. So, b_i has most one neighbor in $V - B$. Next, there are two cases to consider.

Case 1. b_i has no neighbor in $V - B$.
Since b_i is a b-vertex, it must be adjacent to all vertices of B. As $V - B \neq \emptyset$, we can let $a_s \in V - B$. By items (iv) and (v) of Theorem 2, a_s has at least two neighbors in B, say b_j and b_k. The Theorem-(ii) shows that $s \neq i$. Let $b_s \in B$ (such a vertex exist since $b(G) \geq 4$). The above argument shows that b_s is not adjacent neither to b_j nor to b_k. Therefore, since b_s is a b-vertex, it must be adjacent to a_j and a_k (two vertices in $V - B$ of color j and k, respectively). Since b_i has no neighbor in $V - B$ and $V - B$ is independent (by Theorem 2-(iii)), b_i, b_s, a_j, a_k induce a star $K_{1,3}$ centered at b_s, a contradiction.

Case 2. b_i has exactly one neighbor in $V - B$, say a_s.
Let $b_s \in B$ be a vertex of color s. By the Theorem 2-(ii), b_s is not adjacent to b_i. Since b_i is a b-vertex and has degree $\Delta(G)$, it must be adjacent to all vertices of $B - \{b_s\}$. Since b_s needs the color i in its neighborhood, it must be adjacent to a_i (vertex of color i in $V - B$). By items (iv) and (v) of Theorem 2, a_i has another neighbor in B, distinct from b_s, say b_j. But in this case, the color i appears twice in the neighborhood of b_j, which contradicts the Theorem2-(ii). This completes the proof of Claim 3. □

Claim 4. *For each $i \in \{1, 2, \ldots, k\}$, $3 \leq d_G(b_i) \leq 4$.*

Proof of Claim 4. The lower bound holds by combining Theorem 2-(v) and Corollary 3. So, let us prove the upper bound. To this end, suppose that $d_G(b_i) \geq 5$ for some $i \in \{1, 2, \ldots, k\}$. Since G is quasi-line graph, we can let $N_G(b_i) = A_1 \cup A_2$, where A_1 and A_2 are two disjoint cliques. Since $d_G(b_i) \geq 5$, we can assume, without loss of generality, that $|A_1| \geq 3$. As b_i has exactly two neighbors in $V - B$ (by Claim 3) and $V - B$ is independent (by Theorem 2-(iv)), A_1 contains exactly one vertex, say a_s in $V - B$ and at least two vertices, say b_j, b_k in B. The Theorem 2-(ii) shows that the b-vertex of color

s (i.e., b_s) has no neighbor in $\{b_i, b_j, b_k\}$. This means that b_s is not in $A_1 \cup A_2$. Since b_s is a b-vertex, it must be adjacent to three vertices of color i, j, k. These three vertices are in $V - B$ (according to Theorem 2-(ii)) and so they must be a_i, a_j, a_k. But in this case b_s, a_i, a_j, a_k induce a star centered at b_s, contradiction. This complete the proof of Claim 4. □

We assert that

$$\text{all colors of } c \text{ appear in } V - B. \tag{4.3}$$

Indeed, if there is a color, say i that does not appear in $V - B$, then b_i must be the unique vertex of color i and therefore every b-vertex b_j $(j \neq i)$ must be adjacent to b_i. But in this case, since $d_G(b_i) = b(G) - 1$, b_i has no neighbor in $V - B$, which is impossible according to Claim 3. Thus, (4.3) holds.

Combining (4.3) with Lemma 10, we get

$$|V - B| = |B| = b(G). \tag{4.4}$$

By Claim 4, we distinguish between two cases.

Case 1. $d_G(b_i) = 3$. Then $b(G) = 4$ and so by Theorem 2-(iii) and (4.4), respectively, we have $B = \{b_1, b_2, b_3, b_4\}$ and $V - B = \{a_1, a_2, a_3, a_4\}$. By Claim 3, each b-vertex has exactly two neighbors in $V - B$ and so it has exactly one neighbor in B. Therefore B induces two copies of K_2, say $b_1 b_2$ and $b_3 b_4$. Clearly $a_i b_i \notin E(G)$ since a_i and b_i have the same color. Also, by the Theorem 2-(ii), we conclude that $a_1 b_2$, $a_2 b_1, a_3 b_4, a_4 b_3$ are not edges of G. This together with the fact that $V - B$ is independent (by Theorem 2-(iv)) and every vertex of $V - B$ has degree at least two (Theorem 2-(v)) imply that $N_G(a_1) = N_G(a_2) = \{b_3, b_4\}$ and $N_G(a_3) = N_G(a_4) = \{b_1, b_2\}$. Thus G is isomorphic to the graph illustrated in Figure 2-(b)

Case 2. $d_G(b_i) = 4$; so $b(G) = 5$. Using a similar argument as in Case 1, we have $B = \{b_1, b_2, b_3, b_4, b_5\}$, $V - B = \{a_1, a_2, a_3, a_4, a_5\}$ and each b-vertex has exactly two neighbors in B. Therefore B induces a cycle C_5, say b_1-b_2-b_3-b_4-b_5 (in this order). Since c is a proper coloring $a_i b_i$ is not an edge of G. Also, by the Theorem 2-(ii), a_1 and a_i $(i \neq 1)$, respectively, has no neighbor in $\{b_2, b_5\}$ and $\{b_{i-1}, b_{i+1}\}$ (modulo 5). Combining this together with items (iv) and (v) of Theorem 2, we get $N_G(a_i) = \{b_{i+2}, b_{i+3}\}$ (modulo 5). Thus G is isomorphic the graph illustrated in Figure 2-(a) □

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