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AN EXISTENCE THEOREM OF PERFECT MATCHING ON k-PARTITE k-UNIFORM HYPERGRAPHS VIA DISTANCE SPECTRAL RADIUS

LEI ZHANG AND HAIZHEN REN*

Abstract. Let $n_1, n_2, \dots, n_k$ be integers and $V_1, V_2, \dots, V_k$ be disjoint vertex sets with $|V_i| = n_i$ for each $i = 1, 2, \dots, k$. A k-partite k-uniform hypergraph on vertex classes $V_1, V_2, \dots, V_k$ is defined to be the k-uniform hypergraph whose edge set consists of the k-element subsets S of $V_1 \cup V_2 \cup \cdots \cup V_k$ such that $|S \cap V_i| = 1$ for all $i = 1, 2, \dots, k$. We say that it is balanced if $n_1 = n_2 = \cdots = n_k$. In this paper, we give a distance spectral radius condition to guarantee the existence of perfect matching in k-partite k-uniform hypergraphs, this result generalize the result of Zhang and Lin [Perfect matching and distance spectral radius in graphs and bipartite graphs, *Discrete Appl. Math.*, **304** (2021) 315-322].

1. Introduction

A hypergraph H is a pair (V, E), where $E \in P(V)$ and $P(V)$ stands for the power set of V. The elements of $V = V(H)$, labeled as $[n] = \{1, \dots, n\}$, are referred to as vertices and the elements of $E = E(H)$ are called edges. A hypergraph H is said to be k-uniform for an integer $k \geq 2$ if, for all $e \in E(H)$, $|e| = k$. A subhypergraph H' of H is an k-uniform hypergraph such that $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$. An induced subhypergraph H' of H is an k-uniform hypergraph such that $V(H') \subseteq V(H)$ and $E(H') = \{e \in E(H) : e \subseteq V(H')\}$. We write $H[A]$ for the induced subhypergraph

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*Corresponding author.

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of H on the vertex set A. We denote by $H_1 \cup H_2$ the disjoint union of two graphs H_1 and H_2, which is the graph with $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$.

A hypergraph H is called a linear hypergraph, if each pair of the edges of H has at most one common vertex. The hypergraph H considered in this paper always is simple, i.e. $e_i \neq e_j$ if $i \neq j$ for any e_i, $e_j \in E(H)$. In a hypergraph, two vertices are said to be adjacent if there is an edge that contains both of these vertices. Two edges are said to be adjacent if their intersection is not empty. A vertex v is said to be incident to an edge e if $v \in e$. A path of length q in a hypergraph H is defined to be an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, v_q, e_q, v_{q+1}$ such that

(1) $v_1, \ldots, v_{q+1}$ are all distinct vertices of H,
(2) $e_1, \ldots, e_q$ are all distinct edges of H,
(3) $v_r, v_{r+1} \in e_r$ for $r = 1, \ldots, q$.

If $q \geq 1$ and $v_1 = v_{q+1}$, then this path is called a cycle of length q. A hypergraph H is connected if there exists a path starting at v and terminating at u for all $v, u \in V$, and is called acyclic if it contains no cycle. The other undefined definitions here can refer to [2] and [3].

Let H be a connected k-uniform hypergraph with $V(H) = \{v_1, \ldots, v_n\}$. For $u, v \in V(H)$, the distance between u and v is the length of a shortest path from u to v in H, denoted by $d_H(u, v)$. In particular, $d_H(u, u) = 0$. The diameter of H is the maximum distance between all vertex pairs of H. The distance matrix of H is the $n \times n$ matrix $D(H) = (d_H(u, v))_{u,v \in V(H)}$. The eigenvalues of $D(H)$ are called the distance eigenvalues of H. Since $D(H)$ is real and symmetric, the distance eigenvalues of H are real. The distance spectral radius of H, denoted by $\mu(H)$, is the largest distance eigenvalue of H. Note that $D(H)$ is an irreducible nonnegative matrix. The Perron-Frobenius theorem implies that $\mu(H)$ is simple, and there is a unique positive unit eigenvector corresponding to $\rho(H)$, which is called the distance Perron vector of H, denoted by $x(H)$.

The study of distance eigenvalues can be traced back to 1971 by Graham and Pollack [7] and they described a relationship between the number of negative distance eigenvalues and the addressing problem in data communication system. Since then, the study of distance eigenvalues of a graph has become a research subject of enormous interest, some latest results see [12, 16]. Especially, Zhang and Lin [22] presented a distance spectral radius condition to guarantee the existence of a perfect matching in graphs. For more results on the distance matrix and its spectral properties, we refer the reader to the excellent surveys ([1, 9, 11]).

The distance spectral properties of hypergraphs have received attention recently. Sivasubramanian [19] gave a formula for the inverse of a few q-analogs of the distance matrix of a 3-uniform hypertree. Lin and Zhou [13] studied the distance spectral radius of k-uniform hypergraphs and determined the k-uniform hypertrees with maximum and minimum distance spectral radius. Lin and Zhou [14] determined the unique k-uniform unicyclic hypergraph of size $m \geq 2$ with minimum and second

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minimum distance spectral radius, and discussed the possible structure of the k-uniform unicyclic hypergraphs of fixed size with maximum distance spectral radius, respectively. Wang and Zhou [21] determined the unique hypertrees with minimum and maximum distance spectral radius among hypertrees on n vertices with m edges.

A matching in H is a collection of vertex-disjoint edges of H. A perfect matching in H is a matching that covers $V(H)$. Clearly a perfect matching in H exists only if k divides the number of vertices of H. While a maximum matching in a graph can be found in polynomial time [5], it is NP-hard to find it even for 3-uniform hypergraphs [10]. Much effort has been devoted to finding good sufficient conditions for the existence of a large matching in uniform hypergraphs, including Dirac type conditions. A celebrated result in this area is due to Rödl, Ruciński, and Szemerédi [18]. They determined the minimum co-degree threshold function that ensures a perfect matching in n-vertex k-uniform hypergraphs. Some other results see [8, 20]. These results are all about the existence condition of perfect matching through the structure of the hypergraphs.

Let $n_1, n_2, \ldots, n_k$ be integers and $V_1, V_2, \ldots, V_k$ be disjoint vertex sets with $|V_i| = n_i$ for each $i = 1, 2, \ldots, k$. A k-partite k-uniform hypergraph on vertex classes $V_1, V_2, \ldots, V_k$ is defined to be the k-uniform hypergraph whose edge set consists of the k-element subsets S of $V_1 \cup V_2 \cup \cdots \cup V_k$ such that $|S \cap V_i| = 1$ for all $i = 1, 2, \ldots, k$. We say that it is balanced if $n_1 = n_2 = \cdots = n_k$. Note that if a connected k-uniform k-partite hypergraph is not balanced, then it has no perfect matching. Let $B_{n-1,n-1,\ldots,n-1,n-2}$ be the hypergraph obtained from $K_{n,n,\ldots,n,n-2}$ by attaching two pendent vertices to a vertex in the n-vertex part.

In this paper, we obtained sufficient conditions for the existence of perfect matching in uniform hypergraphs in term of the perspective of the spectrum of hypergraphs. In the following, we obtained a distance spectral radius condition to guarantee the existence of a perfect matching in k-partite k-uniform hypergraphs.

Theorem 1.1. *Let H be a connected k-uniform k-partite hypergraph with order kn where n is an integer and $n \geq 3$. If $\mu(D(H)) \leq \mu(D(B_{n-1,n-1,\ldots,n-1,n-2}))$, then H has a perfect matching unless $G \cong B_{n-1,n-1,\ldots,n-1,n-2}$ (see Figure 1).*

2. Main results

We first present a fundamental relation to compare the distance spectral radius of a hypergraph and its spanning subhypergraph.

Lemma 2.1. [15] *Let H be a connected hypergraph with $u, v \in V(H)$. If u and v are not adjacent in H, then $\mu(H + e) < \mu(H)$, where e is a subset of $V(H)$ containing u and v.*

An automorphism of a hypergraph H is a bijection on $V(H)$ which induces a bijection on $E(H)$.

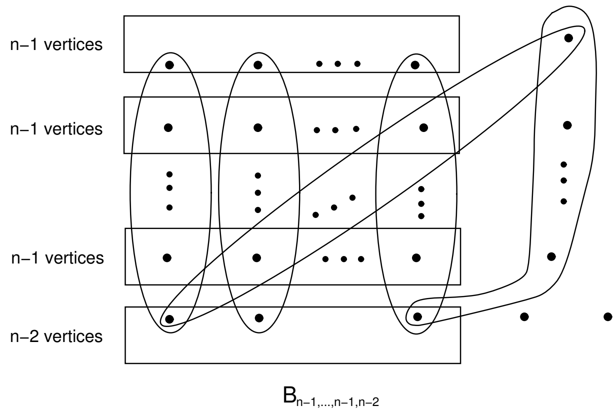

FIGURE 1. The extremal hypergraph $B_{n-1,n-1,\ldots,n-1,n-2}$ of Theorem 1.1.

Lemma 2.2. [13] *Let H be a connected k-uniform hypergraph with σ being an automorphism of H. Let $x = x(H)$. Then $\sigma(u) = v$ implies that $x_u = x_v$.*

In [17], Rodriguez introduced the concept of the Wiener index of a hypergraph H, $W(H)$, which is the sum of distances between all pairs of vertices of a hypergraph:

$$W(H) = \sum_{\{u,v\}\subseteq E(H)} d(u,v).$$

Note that $\mu(D(H)) = \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\mathbf{x}^t D(H)\mathbf{x}}{\mathbf{x}^t\mathbf{x}}$. Then we have

$$\mu(D(H)) = \max_{\mathbf{x}\in\mathbb{R}^n} \frac{\mathbf{x}^t D(H)\mathbf{x}}{\mathbf{x}^t\mathbf{x}} \geq \frac{\mathbf{1}^t D(H)\mathbf{1}}{\mathbf{1}^t\mathbf{1}} \geq \frac{2W(H)}{n},$$

where $\mathbf{1} = (1, 1, \ldots, 1)^t$.

Let H be a k-uniform k-partite hypergraph with some ordering on parts, as $V_1, V_2, \ldots, V_k$, such that the subhypergraph generated on $\cup_{i=1}^{k-1} V_i$ has a unique perfect matching M. In this case, the authors of [6] give a necessary and sufficient condition for having a matching of size $t = |V_1|$ in H.

Lemma 2.3. [6] *Let H be a k-uniform k-partite hypergraph with some ordering on parts, as $V_1, V_2, \ldots, V_k$, such that the subhypergraph generated on $\cup_{i=1}^{k-1} V_i$ has a unique perfect matching M. Then H has a matching of size $t = |V_1|$, if and only if for every subset S of M, $|N(S)| \geq |S|$.*

The following lemma give a version of Hall's theorem for hypergraphs.

Lemma 2.4. [6] *Let H be a k-uniform k-partite hypergraph with some ordering on parts as $V_1, V_2, \ldots, V_k$ where $|V_1| = |V_2| = \cdots = |V_k|$ such that the subhypergraph generated on $\cup_{i=1}^{k-1} V_i$ has a unique perfect matching M. Then H has a perfect matching if and only if for every subset S of M, $|N(S)| \geq |S|$.*

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Let U be a real $n\times n$ matrix, and let $P=\{1,2,\ldots,n\}$. Given a partition $\prod : P = P_1\cup P_2\cup\cdots\cup P_k$, the matrix U can be correspondingly partitioned as

$$U=\begin{pmatrix} U_{1,1} & U_{1,2} & \cdots & U_{1,k} \\ U_{2,1} & U_{2,2} & \cdots & U_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ U_{k,1} & U_{k,2} & \cdots & U_{k,k} \end{pmatrix}$$

The quotient matrix of U with respect to $\prod$ is defined as the $k\times k$ matrix $Q_{\prod}=(b_{i,j})_{i,j=1}^{k}$ where $b_{i,j}$ is the average value of all row sums of $U_{i,j}$. The partition $\prod$ is called equitable if each block $U_{i,j}$ of U has constant row sum $b_{i,j}$. Also, we say that the quotient matrix $Q_{\prod}$ is equitable if $\prod$ is an equitable partition of U.

Lemma 2.5. [4] *Let U be a real symmetric matrix, and let $\mu(U)$ be the largest eigenvalue of U. If $Q_{\prod}$ is an equitable quotient matrix of U, then the eigenvalues of $Q_{\prod}$ are also eigenvalues of U. Furthermore, if U is nonnegative and irreducible, then $\mu(U)=\mu(Q_{\prod})$.*

Let $n_1,n_2,\ldots,n_k$ be integers and $V_1,V_2,\ldots,V_k$ be disjoint vertex sets with $|V_i|=n_i$ for each $i=1,2,\ldots,k$. A k-partite k-uniform complete hypergraph on vertex classes $V_1,V_2,\ldots,V_k$, denoted by $K_{n_1,n_2,\ldots,n_k}$, is defined to be the k-uniform hypergraph whose edge set consists of all the k-element subsets S of $V_1\cup V_2\cup\cdots\cup V_k$ such that $|S\cap V_i|=1$ for all $i=1,2,\ldots,k$. Let $B_{n-1,n-1,\ldots,n-1,n-2}$ be the hypergraph obtained from $K_{n,n,\ldots,n,n-2}$ by attaching two pendent vertices to a vertex in the n-vertex part (see Figure 1). Then we have the following result.

Theorem 2.6. *Let H be a connected k-uniform k-partite hypergraph with order kn where n is an integer and $n\geq 3$. If $\mu(D(H))\leq\mu(D(B_{n-1,n-1,\ldots,n-1,n-2}))$, then H has a perfect matching unless $G\cong B_{n-1,n-1,\ldots,n-1,n-2}$ (see Fig. 1).*

Proof. Assume to the contrary that H has no perfect matching with the minimum distance spectral radius. Let $H=H[X,Y]$ be a connected k-uniform k-partite hypergraph, where $|V_1|=|V_2|=\cdots=|V_k|=n$. By Lemma 2.4, since H has no perfect matching, there exist $S\subseteq M$ and $|N(S)|<|S|$(Attention, S is an edge set). Notice that there exist no edges between S and $V_k-N(S)$, otherwise, we can find a vertex $v\in S$ and $V_k-N(S)$ contains its neighbors, a contradiction. Let $B_{s,s,\ldots,s,p}$ be the connected balanced k-uniform k-partite hypergraph obtained from H by joining S and $N(S)$, $M-S$ and $V_k-N(S)$ and by adding all possible edges between $M-S$ and $N(S)$ where $|S|=s$, $|N(S)|=p$ and $1\leq p<s\leq n-1$, so that $B_{s,s,\ldots,s,p}\cong K_{s,s,\ldots,s,p}\cup K_{n-s,n-s,\ldots,n-s,n-p}+e(N(S),M-S)$, i.e. $B_{s,s,\ldots,s,p}\cong K_{n,n,\ldots,n,n}-e(S,V_k-N(S))$ (see Figure 2). Note that $H\cong B_{s,s,\ldots,s,p}$, otherwise, we have $\mu(B_{s,s,\ldots,s,p})\leq\mu(D(H))$ by Lemma 2.1, a contradiction.

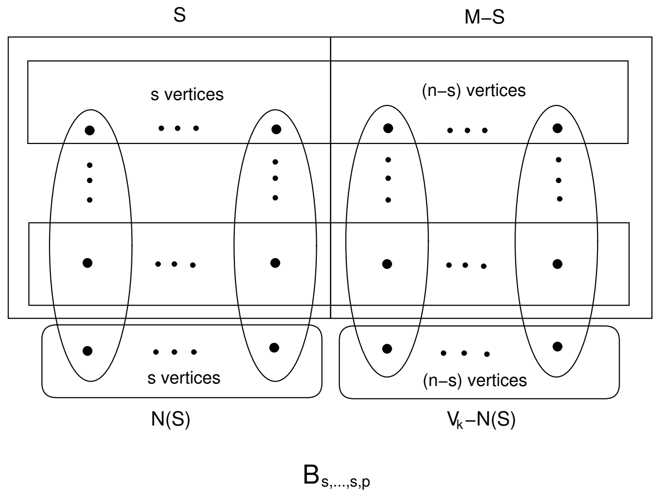

FIGURE 2. The hypergraph $B_{s,s,\dots,s,p}$

Claim 1. $\mu(D(B_{s,s,\dots,s,p})) \geq \mu(D(B_{s,s,\dots,s,s-1}))$ with equality if and only if $B_{s,s,\dots,s,p} \cong B_{s,s,\dots,s,s-1}$.

If $p = s-1$, then $B_{s,s,\dots,s,p} \cong B_{s,s,\dots,s,s-1}$. Now we consider $1 \leq p \leq s-2$. Denote the vertex set of $B_{s,s,\dots,s,s-1}$ by $V(B_{s,s,\dots,s,s-1}) = S \cup (M-S) \cup N(S) \cup (V_k - N(S))$ where $|S| = s$ and $|N(S)| = s-1$. Referring to the Perron-Frobenius theorem, suppose that a positive vector X is the Perron vector of $D(B_{s,s,\dots,s,s-1})$, and let $X(v)$ denote the entry of X corresponding to the vertex $v \in V(B_{s,s,\dots,s,s-1})$. By symmetry, it is easy to see that all vertices of S (resp. $M-S$, $N(S)$ and $V_k - N(S)$) have the same entries in X. Thus we can suppose $X(u) = z_1$ for any $u \in S$, $X(v) = z_2$ for any $v \in M-S$, $X(w) = z_3$ for any $w \in N(S)$ and $X(z) = z_4$ for any $z \in V_k - N(S)$. Therefore, we have

$$\begin{aligned}
& \mu(D(B_{s,s,\dots,s,p})) - \mu(D(B_{s,s,\dots,s,s-1})) \\
\geq\ & X^t(D(B_{s,s,\dots,s,p}) - D(B_{s,s,\dots,s,s-1}))X \\
=\ & 4s(s-1-p)z_1z_3 \\
>\ & 0.
\end{aligned}$$

So Claim 1 holds.

Therefore, $H \cong B_{s,s,\dots,s,s-1}$, or there will be a contradiction.

Claim 2. $\mu(D(B_{s,s,\dots,s,s-1})) \geq \mu(D(B_{n-1,n-1,\dots,n-1,n-2}))$ with equality if and only if $B_{s,s,\dots,s,s-1} \cong B_{n-1,n-1,\dots,n-1,n-2}$.

If $s = 2$ or $s = n-1$, then $B_{s,s,\dots,s,s-1} \cong B_{n-1,n-1,\dots,n-1,n-2}$. Now consider $3 \leq s \leq n-2$. The quotient matrix of the partition $S, M-S, N(S), V_k - N(S)$ of $B_{s,s,\dots,s,s-1}$ where $|S| = s \leq n-2$ and

$|N(S)| = s - 1$ is

$$U = \begin{pmatrix} sk-2 & s-1 & 2(k-1)(n-s) & 3(n-s+1) \\ (k-1)s & 2(s-2) & (k-1)(n-s) & 2(n-s+1) \\ 2s(k-1) & s-1 & k(n-s)-2 & n-s+1 \\ 3(k-1)s & 2(s-1) & (k-1)(n-s) & 2(n-s) \end{pmatrix}$$

the characteristic polynomial of the matrix is

$$\begin{aligned} f(\lambda) &= \lambda^4 + (8 - kn - 2n)\lambda^3 \\ &\quad +(-3k^2ns + 3k^2s^2 + kn^2 - 6kn - 8ks + n^2 + 4ns - 12n - 4s^2 + 8s + 24)\lambda^2 \\ &\quad +(4k^2n^2s - 4k^2ns^2 - 12k^2ns + 12k^2s^2 - 2kn^2s + 4kn^2 + 2kn^2s \\ &\quad +4kn^2 + 2kns^2 - 12kn + 8ks^2 - 40ks - 4n^2s + 4n^2 + 4n^2 + 4ns^2 \\ &\quad +16ns - 24n - 24s^2 + 40s + 32)\lambda \\ &\quad -4k^2n^2s^2 + 12k^2n^2s + 8k^2ns^3 - 20k^2ns^2 - 8k^2ns - 4k^2s^4 + 8k^2s^3 \\ &\quad +8k^2s^2 - 4kn^2s + 4kn^2 + 4kns^2 - 8kn + 16ks^2 - 48ks + 4n^2s^2 \\ &\quad -12n^2s + 4n^2 - 8ns^3 + 20ns^2 + 12ns - 16n + 4s^4 - 8s^3 - 28s^2 + 48s + 16 \end{aligned}$$

We know that $\mu(D(B_{s,s,\dots,s,s-1}))$ is the largest root of $f(\lambda) = 0$ by the above process. Since $\mu(D(B_{n-1,n-1,\dots,n-1,n-2}))$, written by ρ, is the largest root of the equation

$$\begin{aligned} q(\lambda) &= \lambda^4 + (8 - kn - 2n)\lambda^3 + (-3k^2n + 3k^2 + kn^2 - 14kn + 8k + n^2 + 12)\lambda^2 \\ &\quad +(4k^2n^2 - 16k^2n + 12k^2 + 10kn^2 - 66kn + 48k - 8n^2 + 52n - 32)\lambda \\ &\quad 4k^2n - 4k^2 + 16kn^2 - 84kn + 64k - 12n^2 + 64n - 48 \end{aligned}$$

we have

$$\begin{aligned} h(\rho) &= f(\rho) - q(\rho) \\ &= (n - s - 1)[(8k + 4s - 3k^2s + 3k^2 - 12)\rho^2 \\ &\quad +2(24k + 6n + 12s - 3kn - 4ks - 2ns - 2k^2n - 6k^2s \\ &\quad +6k^2 + 2k^2ns - kns - 32)\rho - 4(3kn - 4n - 4s - 16k + 4ks + 3ns \\ &\quad -k^2s - ns^2 + k^2 - 3s^2 + s^3 + 3k^2s^2 - k^2s^3 - 3k^2ns + k^2ns^2 + kns + 16)] \end{aligned}$$

Now we need to state $h(\rho) < 0$. Firstly, we give a lower bound on ρ,

$$\frac{2W(B_{n-1,n-1,\dots,n-1,n-2})}{kn} = (k+1)n + (3k+1) - \frac{3k^2 + k - 4}{kn} - \frac{1}{k} \le \rho,$$

It is easy to see $(n-s-1)>0$ and $-\frac{2(24k+6n+12s-3kn-4ks-2ns-2k^2n-6k^2s+6k^2+2k^2ns-kns-32)}{2(8k+4s-3k^2s+3k^2-12)}$ $<(k+1)n<\rho$. Thus, $h(\rho)$ is monotonically decreasing for $\rho>(k+1)n$ and $h(\rho)<h((k+1)n)$. Let $h((k+1)n)$ be written by $-(n-s-1)g(s)$ and we only need to prove $g(s)>0$ for $3\leq s\leq n-2$ in the following steps. Based on Matlab programming, we have

$$\begin{aligned} g(s) &= (3k^4s+3k^4+2k^3s-10k^3-3k^2s+3k^2-2ks+10k)n^2 \\ &\quad +(12k^3s-12k^3+4k^2s^2+8k^2s-60k^2-12ks+28k-12s+48)n \\ &\quad -4k^2s^3+12k^2s^3-4k^2s+4k^2+16ks-64k+4s^3-12s^2-16s+64 \end{aligned}$$

we can easily get $g(s)$ is monotonically increasing for $3\leq s\leq n-2$, so

$$g(s)>g(3)=(12k^4-4k^3-6k^2+4k)n^2+(24k^3-8k+12)n-8k^2-16k+16.$$

By using simple calculation and $n\geq k\geq 3$, we have $g(3)>0$.

Therefore, $h(\rho)<h((k+1)n)=-(n-s-1)g(s)<0$. So Claim 2 holds.

In conclusion, $H\cong B_{n-1,n-1,\dots,n-1,n-2}$ has the minimum distance spectral radius among all kn-vertex balanced k-uniform k-partite hypergraph without a perfect matching. □

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Lei Zhang

Department of Mathematics and Statistics, Qinghai Normal University, 810008, Xining, China

The State Key Laboratory of Tibetan Information Processing and Application, Xining, China

Academy of Plateau, Science and Sustainability, 810008, Xining, China

Email: shuxuezhanglei@163.com

Haizhen Ren

Department of Mathematics and Statistics, Qinghai Normal University, 810008, Xining, China

The State Key Laboratory of Tibetan Information Processing and Application, Xining, China

Academy of Plateau, Science and Sustainability, 810008, Xining, China

Email: haizhenr@126.com

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