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CAYLEY HYPERGRAPH OVER POLYGROUPS

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Abstract. Comer introduced a class of hypergroups, using the name of polygroups. He emphasized the importance of polygroups, by analyzing them in connections to graphs, relations, Boolean and cylindric algebras. Indeed, polygroups are multi valued systems that satisfy group like axioms. Given a polygroup with a finite generating set, we can form a Cayley hypergraph for that polygroup with respect to that generating set. This helps us to better understand and investigate polygroup structures. More precisely, in this paper, we introduce the construction of Cayley hypergraphs over polygroups, say $CH(\mathbf{P}, S)$ such that $\mathbf{P}$ is a polygroup and $\langle S\rangle = P$. We investigate some properties of them. It is well known to give a constructing for building a big polygroup from two small ones. This structure is called extension of polygroups. In particular, we describe the connection between Cayley hypergraphs over extension of two polygroups and Cartesian product of two Cayley hypergraphs.

1. **Introduction**

The concept of a hypergraph appeared around 1960 and one of the primary concerns was to extend some classical results of graph theory. A very good presentation of hypergraph theory appeared in [1, 2]. The book [2] presents the theory of hypergraphs in its most original aspects, while also introducing and assessing the latest concepts on hypergraphs. Indeed, a *hypergraph* is a pair $\Gamma = (H, E)$, where H is a finite set of vertices and $E = \{E_1, \ldots, E_n\}$ is a set of hyperedges which are nonempty subsets of H such that $\bigcup_{i=1}^{m} E_i = H$.

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The notion of a hypergroup is introduced by Marty [11] in 1934. In a hypergroup the product of two elements is a non-empty set. In [5], Corsini associated to every hypergraph a commutative quasi-hypergroup and found a necessary and sufficient condition so that the hyperoperation associative. Then many researchers worked on this subject, for instance see [7, 8, 9, 10, 13]. An important class of hypergroups is polygroups [6].

There are frequent occasions for which hypergraphs with a lot of symmetry are required. One such family of hypergraphs can be constructed using polygroups. So, in this work, we introduce the construction of Cayley hypergraphs over polygroups. Cayley hypergraphs gives a way of encoding information about polygroup in a hypergraph. In particular, we are going to develop Cayley hypergraphs to study various kinds of polygroups.

2. Preliminaries and basic definitions

Let $\Gamma = (H, E)$ be a hypergraph and $(x, y) \in H^2$. According to [1], a hyperedge sequence $(E_1, \ldots, E_k)$ is called a path of length k from x to y if the following conditions are satisfied:

(1) $x \in E_1$ and $y \in E_k$,
(2) $E_i \neq E_j$ for $i \neq j$,
(3) $E_i \cap E_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$.

In a hypergraph Γ, two vertices x and y are called *connected* if Γ contains a path from x to y. If two vertices are connected by a path of length 1, i.e., by a single hyperedge, the vertices are called *adjacent*. We use the notation $x \sim y$ to denote the adjacency of vertices x and y. A hypergraph is said to be connected if every pair of vertices in the hypergraph is connected. Let $\Gamma = (H, E)$ and $\Gamma' = (H', E')$ be two hypergraphs. We say that Γ' is a *subhypergraph* of Γ if $H' \subseteq H$ and $E' \subseteq E$. Hypergraph Γ' is an *induced subhypergraph*, if $H' \subseteq H$ and $E'_i = \{E_i \cap H' \mid |E_i| = 1 \vee |E_i \cap H'| \geqslant 2\}$.

Now, we recall the definition of a polygroup.

Definition 2.1. [4, 6] *A polygroup is a system* $\boldsymbol{P} =< P, \circ, e, ^{-1} >$, *where* $e \in P$, $^{-1}$ *is a unitary operation on* P, $\circ$ *maps* $P \times P$ *into the non-empty subsets of* P, *and the following axioms hold for all* $x, y, z \in P$:

(I) $(x \circ y) \circ z = x \circ (y \circ z)$,
(II) $e \circ x = x \circ e = x$,
(III) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ *and* $z \in y^{-1} \circ x$.

Clearly, every group is a polygroup, $e \in x \circ x^{-1} \cap x^{-1} \circ x$, $(x^{-1})^{-1} = x$, and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$, where $A^{-1} = \{a^{-1} | \ a \in A\}$.

Let $\mathbf{P} =< P, \circ, e, ^{-1} >$ be a polygroup and S be a non-empty subset of P. Then smallest sub-polygroup of $\mathbf{P}$ containing S is called the sub-polygroup generates by S and denoted by $\langle S \rangle$. In fact:

$$\langle S \rangle = \{x \in P \ : \ x \in s_1 \circ s_2 \circ \cdots \circ s_k, \ s.t. \ k \in \mathbb{N}, s_i \in S \cup S^{-1}\}.$$

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The definition of Cayley graph is introduced by Arthur Cayley to explain the concept of abstract groups which are described by a set of generators.

Definition 2.2. [12] *Let G be any finite group, with identity 1_G; and suppose $S \subset G$, where $1_G \notin S$ and $s \in S$ implies that $s^{-1} \in S$. We define a* **Cayley graph***, denoted $Cay(G;S)$, to have elements of G as vertices and for any two vertices $(x,y) \in G^2$, an edge $[x,y]$ if $x = s \cdot y$ for some $s \in S$.*

Definition 2.3. [3] *Let G be a group, Ω a subset of $G \setminus \{1\}$ and t an integer verifying $2 \leqslant t \leqslant max\{o(\omega) \mid \omega \in \Omega\}$. Then Buratti introduced a* ***t-Cayley hypergraph*** *$H = t - Cay(G, \Omega)$ as a hypergraph with vertex set G and edge set*

$$E(H) = \{\{g, g\omega, \dots, g\omega^{t-1}\} | g \in G, \omega \in \Omega\}.$$

One can easily see that any 2-Cayley hypergraph is a Cayley graph.

Definition 2.4. [9] *Let G be a group and let X be a set of subsets $\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_d}$ of $G \setminus \{1_G\}$ such that $\bigcup_{i=1}^{d} \mathbf{x_i}$ generate G, where 1_G is the identity element in G. A* **Cayley hypergraph** *$CH(G, X)$ has vertex set G and edge set $\{\{g, g\mathbf{x}\} \mid g \in G, \mathbf{x} \in X\}$, where an edge $\{g, g\mathbf{x}\}$ is the set $\{g\} \cup \{gx | x \in \mathbf{x}\}$. For all $\mathbf{x} \in X$, if $|\mathbf{x}| = 1$, then the Cayley hypergraph is a Cayley graph.*

For any $\mathbf{x_i} \in X$, if $\mathbf{x_i} = \{\omega, \dots, \omega^{t-1}\}$ for some $\omega \in G \setminus \{1_G\}$, then the Cayley hypergraph $CH(G, X)$ is a t-Cayley hypergraph $t - Cay(G, \Omega)$. Hence a Cayley hypergraph is a generalization of a t-Cayley hypergraph.

3. Cayley hypergraphs constructed from polygroups

Now we define a Cayley hypergraph over a polygroup that is a generalization of a Cayley hypergraph.

Definition 3.1. *Let $\boldsymbol{P} =< P, \circ, e, ^{-1} >$ be a polygroup, with identity e; and suppose that S is a family of subsets S_1, S_2,... S_k of $P \setminus \{e\}$ such that $\bigcup_{i=1}^{k} S_i$ generate P and for each $1 \leqslant i \leqslant k$, $S_i^{-1} = S_i$. We define a Cayley hypergraph over polygroup, denoted $CH(\boldsymbol{P}, S)$, to have elements of P as vertices and*

$$\forall x \in P \text{ and } 1 \leqslant i \leqslant k,\ E_i(x) = \{x\} \cup \{x \circ s \mid s \in S_i\}.$$

It is clear that for each $1 \leqslant i \leqslant k$, $E_i(e) = \{e\} \cup S_i$.

Example 3.2. *Suppose that* **P** *is a polygroup with the following multiplication table:*

$\circ$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	$\{1,3\}$	$\{2,4\}$
4	4	3	$\{2,4\}$	$\{1,3\}$

If $S = \{2, 4\}$ and $S_* = \{\underbrace{\{2\}}_{S_1}, \underbrace{\{4\}}_{S_2}\}$, then we have the Cayley hypergraphs in Figure 3.2.

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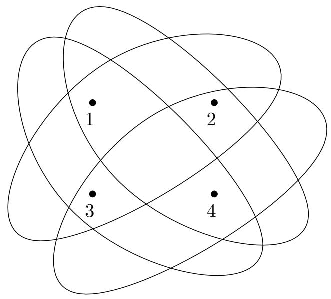
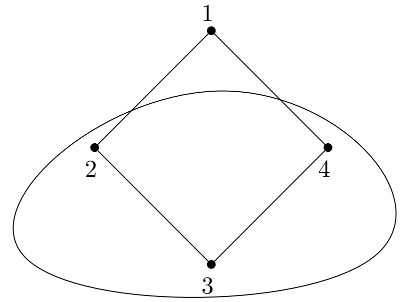


(a) $CH(\mathbf{P}, S)$



(b) $CH(\mathbf{P}, S_*)$

FIGURE 1. The Cayley hypergraphs defineded in Example 3.2

Lemma 3.3. *Let $CH(\boldsymbol{P}, S)$ be a Cayley hypergraph and $(x, y) \in P^2$. Then x and y are adjacent $(x \sim y)$ if and only if have at least one of the following conditions:*

(1) *There exists $i \in \{1, \ldots, k\}$ such that $x \in E_i(y)$;*
(2) *There exists $j \in \{1, \ldots, k\}$ such that $y \in E_j(x)$;*
(3) *There exist $z \in P$ and $l \in \{1, \ldots, k\}$ such that $\{x, y\} \subset E_l(z)$.*

Proof. It is straightforward, by Definition 3.1. □

Note that for every $(x, y) \in P^2$, we have:

$$x \in E_i(y) \Leftrightarrow \exists s \in S_i \text{ } s.t. \text{ } x \in y \circ s$$
$$\Leftrightarrow y \in x \circ s^{-1}$$
$$\Leftrightarrow y \in E_i(x).$$

Theorem 3.4. *Let $\boldsymbol{P} =< P, \circ, e,^{-1} >$ be a polygroup and $S = \{S_1, S_2, \ldots, S_k\}$ such that for each $1 \leqslant i \leqslant k$, $e \notin S_i$ and $S_i^{-1} = S_i$. Then Cayley hypergraph $CH(\boldsymbol{P}, S)$ is connected if and only if $\langle \bigcup_{i=1}^{k} S_i \rangle = P$.*

Proof. Let $x \in P$, $CH(\mathbf{P}, S)$ be a connected hypergraph and

$$x_0 = e \sim x_1 \sim x_2 \sim \cdots \sim x_n = x$$

be a path from e to x. Then by induction the result follows. So, we prove the claim for $n = 2$, i.e., if $x_0 = e \sim x_1 \sim x_2 = x$, then we show that $x \in \langle \bigcup_{i=1}^{k} S_i \rangle$.

Since $e \sim x_1$, then $x_1 \in E_i(e)$ or $e \in E_j(x_1)$ or there exists $z \in P$ such that $\{e, x_1\} \subset E_l(z)$, by Lemma 3.3. If $x_1 \in E_i(e) = S_i \cup \{e\}$, then $x_1 \in S_i$. If $e \in E_j(x_1)$, then there exists $s_{1j} \in S_j$ such that $e \in x_1 \circ s_{1j}$. So $x_1 \in e \circ s_{1j}^{-1} = s_{1j}^{-1} \in S_j$. If $\{e, x_1\} \subset E_l(z)$, then there exists $(s_{1l}, s_{2l}) \in S_l^2$ such that $e \in z \circ s_{1l}$ and $x_1 \in z \circ s_{2l}$. So $z \in e \circ s_{1l}^{-1} = s_{1l}^{-1}$. Hence $x_1 \in s_{1l}^{-1} \circ s_{2l} \subset \langle \bigcup_{i=1}^{k} S_i \rangle$. Therefore in all

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cases $x_1 \in \langle \bigcup_{i=1}^{k} S_i \rangle$. On the other hand, $x_1 \sim x$. So $x_1 \in E_p(x)$ or $x \in E_m(x_1)$ or there exists $w \in P$ such that $\{x_1, x\} \subset E_n(w)$, by Lemma 3.3.

If $x_1 \in E_p(x)$, then there exists $s_{1p} \in S_p$ such that $x_1 \in x \circ s_{1p}$. So $x \in x_1 \circ s_{1p}^{-1} \subset \langle \bigcup_{i=1}^{k} S_i \rangle$.

If $x \in E_m(x_1)$, then there exists $s_{1m} \in S_m$ such that $x \in x_1 \circ s_{1m} \subset \langle \bigcup_{i=1}^{k} S_i \rangle$.

If $\{x_1, x\} \subset E_n(w)$, then there exists $(s_{1n}, s_{2n}) \in S_n^2$ such that $x_1 \in w \circ s_{1n}$ and $x \in w \circ s_{2n}$. So $w \in x_1 \circ s_{1n}^{-1} \subset \langle \bigcup_{i=1}^{k} S_i \rangle$. Hence $x \in \langle \bigcup_{i=1}^{k} S_i \rangle$.

Conversely, let S generates P. So for each $x \in P$ there exists $(s_1, s_2, \ldots, s_n) \in S^n$ such that $x \in s_1 \circ s_2 \circ \cdots \circ s_n$. Thus, there exists $x_1 \in s_1 \circ s_2 \circ \cdots \circ s_{n-1}$ such that $x \in x_1 \circ s_n$. Hence $x \sim x_1$. Similarly, there exists $x_2 \in s_1 \circ s_2 \circ \cdots \circ s_{n-2}$ such that $x_1 \in x_2 \circ s_{n-1}$. Hence $x_1 \sim x_2$. By continuing this process the following a path is obtained:

$$x \sim x_1 \sim x_2 \sim \cdots \sim x_{n-3} \sim x_{n-2}.$$

So $x_{n-3} \in x_{n-2} \circ s_3$ and also $x_{n-2} \in s_1 \circ s_2$. Thus $s_1 \in x_{n-2} \circ s_2^{-1}$. Therefore $x_{n-2} \sim s_1$. On the other hand, it is clear that all members of S are adjacent to e. So

$$x \sim x_1 \sim x_2 \sim \cdots \sim x_{n-2} \sim s_1 \sim e.$$

Therefore, $CH(\mathbf{P}, S)$ is a connected hypergraph. □

Example 3.5. *Suppose that* $\mathbf{P}$ *is a polygroup with the following multiplication table*

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	2	$\{1,2\}$	$\{3,4,5\}$	$\{3,4,5\}$	$\{3,4,5\}$
3	3	$\{3,4,5\}$	$\{1,2\}$	2	2
4	4	$\{3,4,5\}$	2	$\{1,2\}$	2
5	5	$\{3,4,5\}$	2	2	$\{1,2\}$

If $S = \{2\}$*, then* $S\rangle \neq P$*, and we have the Cayley hypergraph in Figure 2.*

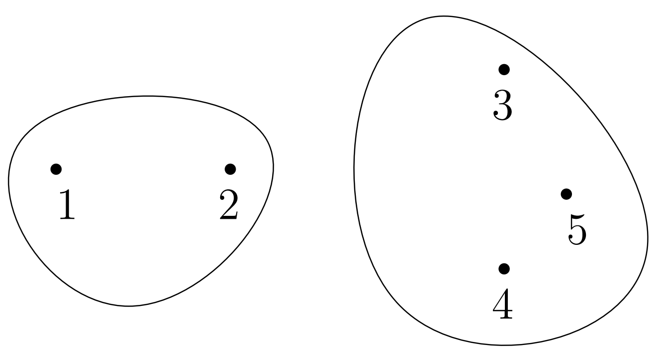

FIGURE 2. The Cayley hypergraph defineded in Example 3.5.

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4. Cayley hypergraphs constructed from extension of polygroups

Let $\mathbf{P_1} =< P_1, \circ, e_1, ^{-1} >$ and $\mathbf{P_2} =< P_2, \bullet, e_2, ^{-I} >$ be two polygroups. Then, on $P_1 \times P_2$ we can define a hyperproduct as follows:

$$(x_1, y_1) * (x_2, y_2) = \{(x, y) \mid x \in x_1 \circ x_2 \ ; \ y \in y_1 \bullet y_2\}, \quad \forall (x_1, y_1), (x_2, y_2) \in P_1 \times P_2.$$

We recall this the *direct hyperproduct* of $\mathbf{P_1}$ and $\mathbf{P_2}$ and denoted by $\mathbf{P_1} \times \mathbf{P_2}$. Clearly, $\mathbf{P_1} \times \mathbf{P_2} =< P_1 \times P_2, *, (e_1, e_2), ^{(-1,-I)} >$ is a polygroup.

Note that if $\mathbf{P_1} =< P_1, \circ, e_1, ^{-1} >$ and $\mathbf{P_2} =< P_2, \bullet, e_2, ^{-I} >$ be two polygroups and $S_1 \subset P_1$ and $S_2 \subset P_2$, where $e_1 \notin S_1$, $e_2 \notin S_2$, $S_1^{-1} = S_1$, $S_2^{-1} = S_2$, $\langle S_1 \rangle = P_1$ and $\langle S_2 \rangle = P_2$, then $\langle S_1 \times S_2 \rangle$ is not a set of generators for $P_1 \times P_2$ ($\langle S_1 \times S_2 \rangle \neq P_1 \times P_2$).

Proposition 4.1. *Let $\mathbf{P_1} =< P_1, \circ, e_1, ^{-1} >$ and $\mathbf{P_2} =< P_2, \bullet, e_2, ^{-I} >$ be two polygroups, $\langle S_1 \rangle = P_1$ and $\langle S_2 \rangle = P_2$. If $S = \{ \underbrace{\{\{e_1\} \times S_2\}}_{S_\alpha}, \underbrace{\{S_1 \times \{e_2\}\}}_{S_\beta} \}$, then $\langle S_\alpha \cup S_\beta \rangle = P_1 \times P_2$.*

Proof. Let $(x, y) \in P_1 \times P_2$ that $x \in \langle S_1 \rangle$ and $y \in \langle S_2 \rangle$. Then

$$\begin{cases} \exists (s_1, \ldots, s_n) \in S_1^n \ s.t. \ x \in s_1 \circ \cdots \circ s_n, \\ \\ \exists (s'_1, \ldots, s'_m) \in S_2^m \ s.t. \ y \in s'_1 \bullet \cdots \bullet s'_m. \end{cases}$$

We consider for n and m, the following three cases:

If $n = m$, then $(x, y) \in (s_1, s'_1) * \cdots * (s_n, s'_m) \subseteq \langle S_\alpha \cup S_\beta \rangle$.

If $n > m$, then $y \in s'_1 \bullet \cdots \bullet s'_m \bullet \underbrace{e_2 \bullet \cdots \bullet e_2}_{(n-m) \text{ times}}$. Therefore

$$(x, y) \in (s_1, s'_1) * \cdots * (s_m, s'_m) * (s_{m+1}, e_2) * \cdots * (s_n, e_2) \in \langle S_\alpha \cup S_\beta \rangle.$$

If $n < m$, then similar to the above. □

Definition 4.2. *Let $\Gamma_1 = (H_1, E_1)$ and $\Gamma_2 = (H_2, E_2)$ be hypergraphs. The Cartesian product of Γ_1 and Γ_2 is the hypergraph $\Gamma_1 \ \square \ \Gamma_2$ with set of vertices $H_1 \times H_2$ and set of hyperedges:*

$$E_1 \ \square \ E_2 = \big\{\{x\} \times F \mid x \in H_1, F \in E_2\big\} \cup \big\{G \times \{y\} \mid G \in E_1, y \in H_2\big\}.$$

Theorem 4.3. *Let $\Gamma_1 = CH(P_1, S_1)$ and $\Gamma_2 = CH(P_2, S_2)$ be Cayley hypergraphs that each of the sets S_1 and S_2 contains a set. Then $\Gamma_1 \ \square \ \Gamma_2 = CH(P, S)$ such that $P = P_1 \times P_2$ and $S = \{ \underbrace{\{\{e_1\} \times S_2\}}_{S_\alpha}, \underbrace{\{S_1 \times \{e_2\}\}}_{S_\beta} \}$.*

Proof. Let $K \in E(\Gamma_1 \ \square \ \Gamma_2)$. We consider the following two cases:

Case 1. If $K = \{x\} \times F$ such that $x \in P_1$ and $F \in E(\Gamma_2)$, then there exists $y \in P_2$ such that $F = \{y\} \cup \{y \bullet s \mid s \in S_2\}$. Let $z = (x, y) \in P_1 \times P_2$. Hence

$$\{z\} \cup \{z * s_\alpha \mid s_\alpha \in S_\alpha\} = \{(x, y)\} \cup \{(x, y) * (e_1, s) \mid s \in S_2\} \in E(CH(P, S)).$$

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On the other hand

$$\begin{aligned}\{(x,y)\}\cup\{(x,y)*(e_1,s)\mid s\in S_2\} &= \{(x,y)\}\cup\{(x,y\bullet s)\mid s\in S_2\}\\ &= \big\{\{x\}\times\{\{y\}\cup\{y\bullet s\mid s\in S_2\}\}\big\}\\ &= \{x\}\times F = K.\end{aligned}$$

Therefore $K\in E(CH(P,S))$.

Case 2. If $L=G\times\{w\}$ such that $w\in P_2$ and $G\in E(\Gamma_1)$, then there exists $h\in P_1$ such that $G=\{h\}\cup\{h\circ s\mid s\in S_1\}$. Let $q=(h,w)\in P_1\times P_2$. Hence

$$\{q\}\cup\{q*s_\beta\mid s_\beta\in S_\beta\}=\{(h,w)\}\cup\{(h,w)*(s,e_2)\mid s\in S_1\}\in E(CH(P,S)).$$

On the other hand

$$\begin{aligned}\{(h,w)\}\cup\{(h,w)*(s,e_2)\mid s\in S_1\} &= \{(h,w)\}\cup\{(h\circ s,w)\mid s\in S_1\}\\ &= \big\{\{\{h\}\cup\{h\circ s\mid s\in S_1\}\}\times\{w\}\big\}\\ &= G\times\{w\}=L.\end{aligned}$$

Therefore $L\in E(CH(P,S))$.

Conversely, let $E\in E(CH(P,S))$. Then there exists $(p_1,p_2)\in P$ such that $E=\{(p_1,p_2)\}\cup\{(p_1,p_2)*s_{\alpha\beta}\mid s_{\alpha\beta}\in S\}$. So there are two cases:

Case 1. If $s_{\alpha\beta}=s_\alpha\in S_\alpha$, then

$$\begin{aligned}E &= \{(p_1,p_2)\}\cup\{(p_1,p_2)*s_\alpha\mid s_\alpha\in S_\alpha\}\\ &= \{(p_1,p_2)\}\cup\{(p_1,p_2)*(e_1,s)\mid s\in S_2\}\\ &= \{(p_1,p_2)\}\cup\{(p_1,p_2\bullet s)\mid s\in S_2\}\\ &= \big\{\{p_1\}\times\{\{p_2\}\cup\{p_2\bullet s\mid s\in S_2\}\}\big\}=\{\{p_1\}\times F\},\end{aligned}$$

where $F=\{p_2\}\cup\{p_2\bullet s\mid s\in S_2\}\in E(\Gamma_2)$. Therefore $E\in E(\Gamma_1\,\square\,\Gamma_2)$.

Case 2. If $s_{\alpha\beta}=s_\beta\in S_\beta$, then

$$\begin{aligned}E &= \{(p_1,p_2)\}\cup\{(p_1,p_2)*s_\beta\mid s_\beta\in S_\beta\}\\ &= \{(p_1,p_2)\}\cup\{(p_1,p_2)*(s,e_2)\mid s\in S_1\}\\ &= \{(p_1,p_2)\}\cup\{(p_1\circ s,p_2)\mid s\in S_1\}\\ &= \big\{\{\{p_1\}\cup\{p_1\circ s\mid s\in S_1\}\}\times\{p_2\}\big\}=\{G\times\{p_2\}\},\end{aligned}$$

where $G=\{\{p_1\}\cup\{p_1\circ s\mid s\in S_1\}\}\in E(\Gamma_1)$. Therefore $E\in E(\Gamma_1\,\square\,\Gamma_2)$. □

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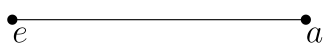
Example 4.4. *Consider* $\mathbf{P_1} = \{e, a\}$, $\mathbf{P_2} = \{0, 1, 2\}$ *and* $\mathbf{P} = \mathbf{P_1} \times \mathbf{P_2}$ *as three polygroups, where*

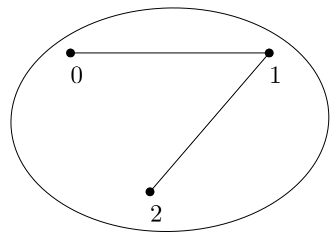
$\circ$	e	a
e	e	a
a	a	$\{e, a\}$

$\bullet$	0	1	2
0	0	1	2
1	1	$\{0, 2\}$	$\{1, 2\}$
2	2	$\{1, 2\}$	$\{0, 1\}$

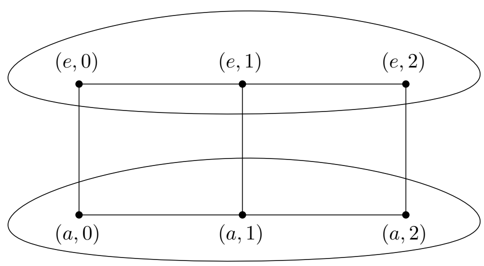
Let $S_1 = \{a\}$, $S_2 = \{1\}$ *and* $S = \{(e, 1), (a, 0)\}$ *where* $S_\alpha = \{(e, 1)\}$ *and* $S_\beta = \{(a, 0)\}$. *Then we have the Cayley hypergraphs in Figure 3.*



(a) $CH(\mathbf{P_1}, S_1)$



(b) $CH(\mathbf{P_2}, S_2)$



(c) $\Gamma_1 \,\square\, \Gamma_2 = CH(\mathbf{P}, S)$

FIGURE 3. The Cayley hypergraphs defined in Example 4.4

Note that Theorem 4.3 can be generalized as follows.

Theorem 4.5. *Let* $\Gamma_1 = CH(P_1, S_1)$ *and* $\Gamma_2 = CH(P_2, S_2)$ *be Cayley hypergraphs where*

$$S_1 = \{S_{11}, S_{12}, \ldots, S_{1k}\} \text{ and } S_2 = \{S_{21}, S_{22}, \ldots, S_{2l}\}.$$

Then $\Gamma_1 \,\square\, \Gamma_2 = CH(P, S)$ *such that* $P = P_1 \times P_2$ *and*

$$S = \Big\{ \underbrace{\{\{e_1\} \times S_{21}\}}_{S_{1\alpha}}, \ldots, \underbrace{\{\{e_1\} \times S_{2l}\}}_{S_{l\alpha}}, \underbrace{\{S_{11} \times \{e_2\}\}}_{S_{1\beta}}, \ldots, \underbrace{\{S_{1k} \times \{e_2\}\}}_{S_{k\beta}} \Big\}.$$

Definition 4.6. [4, 6] *Suppose that* $\mathbf{P_1} =< P_1, \circ, e,^{-1} >$ *and* $\mathbf{P_2} =< P_2, \bullet, e,^{-1} >$ *are two polygroups whose elements have been renamed so that* $P_1 \cap P_2 = \{e\}$. *A new system* $\mathbf{P_1}[\mathbf{P_2}] =< M, *, e,^I >$ *called*

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the extension of $\mathbf{P_1}$ *by* $\mathbf{P_2}$ *is formed in the following way: Set* $M = P_1 \cup P_2$ *and let* $e^I = e$, $x^I = x^{-1}$, $e * x = x * e = x$ *for all* $x \in M$, *and for all* $x, y \in M \setminus \{e\}$*:*

$$x * y = \begin{cases} x \circ y & \text{if } x, y \in P_1 \\ x & \text{if } x \in P_2,\ y \in P_1 \\ y & \text{if } x \in P_1,\ y \in P_2 \\ x \bullet y & \text{if } x, y \in P_2,\ y \neq x^{-1} \\ x \bullet y \cup P_1 & \text{if } x, y \in P_2,\ y = x^{-1}. \end{cases}$$

In this case $\mathbf{P_1}[\mathbf{P_2}]$ *is a polygroup which is called the extension of* $\mathbf{P_1}$ *by* $\mathbf{P_2}$*.*

Theorem 4.7. *Let* $\mathbf{P_1} =< P_1, \circ, e, ^{-1} >$ *and* $\mathbf{P_2} =< P_2, \bullet, e, ^{-I} >$ *be two polygroups,* $\langle S_1 \rangle = P_1$ *and* $\langle S_2 \rangle = P_2$*. Then*

(1) *Cayley hypergraph* $CH(\mathbf{P_1}[\mathbf{P_2}], S_2)$ *is connected;*
(2) *Cayley hypergraph* $CH(\mathbf{P_2}, S_2)$ *is a subhypergraph of* $CH(\mathbf{P_1}[\mathbf{P_2}], S_2)$*.*

Proof. (1) According to Theorem 3.4, it is sufficient to show that S_2 generates $\mathbf{P_1}[\mathbf{P_2}]$. We have

$$\begin{aligned} \langle S_2 \rangle &= \{x \mid x \in y_1 * y_2 * \cdots * y_k,\ y_i \in S_2\} \\ &= \{x \mid x \in \big(\bigcup_{y_i \in S_2} y_1 \bullet y_2 \bullet \cdots \bullet y_k \big) \cup P_1\} \\ &= \{x \mid x \in P_2 \cup P_1\} = P_2 \cup P_1. \end{aligned}$$

(2) Let $(x, y) \in P_2^2$ such that $x \sim y$ in $CH(\mathbf{P_2}, S_2)$. Then $x \in E(y)$ or $y \in E(x)$ or there exists $z \in P_2$ such that $\{x, y\} \subset E(z)$, by Lemma 3.3.

If $x \in E(y)$, then there exists $s_{12} \in S_2$ such that $x \in y \bullet s_{12}$. So $x \in y * s_{12}$. Therefore $x \sim y$ in $CH(\mathbf{P_1}[\mathbf{P_2}], S_2)$.

If $y \in E(x)$, then there exists $s_{22} \in S_2$ such that $y \in x \bullet s_{22}$. So $y \in x * s_{22}$. Therefore $y \sim x$ in $CH(\mathbf{P_1}[\mathbf{P_2}], S_2)$.

If $\{x, y\} \subset E(z)$, then there exists $(s_{32}, s_{42} \in S_2^2$ such that $x \in z \bullet s_{32}$ and $y \in z \bullet s_{42}$. So $x \in z * s_{32}$ and $y \in z * s_{42}$. Therefore $x \sim y$ in $CH(\mathbf{P_1}[\mathbf{P_2}], S_2)$. □

Example 4.8. *Let* $(\mathbf{P_1}, \circ)$, $(\mathbf{P_2}, \bullet)$ *and* $(\mathbf{P_1}[\mathbf{P_2}], *)$ *be polygroups whose tables are given below:*

$\circ$	e	a	b
e	e	a	b
a	a	$\{e, b\}$	$\{a, b\}$
b	b	$\{a, b\}$	$\{e, a\}$

$\bullet$	e	1	2	3
e	e	1	2	3
1	1	$\{e, 2\}$	$\{1, 3\}$	$\{2, 3\}$
2	2	$\{1, 3\}$	$\{e, 3\}$	$\{1, 2\}$
3	3	$\{2, 3\}$	$\{1, 2\}$	$\{e, 1\}$

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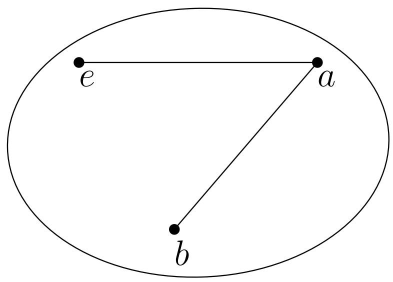
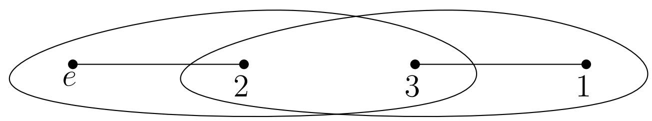
$*$	e	a	b	1	2	3
e	e	a	b	1	2	3
a	a	$\{e,b\}$	$\{a,b\}$	1	2	3
b	b	$\{a,b\}$	$\{e,a\}$	1	2	3
1	1	1	1	$\{e,a,b,2\}$	$\{1,3\}$	$\{2,3\}$
2	2	2	2	$\{1,3\}$	$\{e,a,b,3\}$	$\{1,2\}$
3	3	3	3	$\{2,3\}$	$\{1,2\}$	$\{e,a,b,1\}$

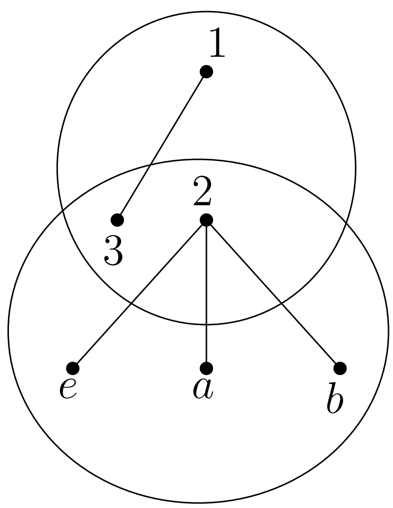
If $S_1 = \{a\}$ and $S_2 = \{2\}$, then we have the Cayley hypergraphs in Figure 4.



(a) $CH(\mathbf{P_1}, S_1)$



(b) $CH(\mathbf{P_2}, S_2)$



(c) $CH(\mathbf{P_1}[\mathbf{P_2}], S_2)$

FIGURE 4. The Cayley hypergraphs defined in Example 4.8

5. Conclusion

Algebraic hyperstructure theory is one of the important branches of modern algebra. A hyperstructure consist a non-empty set H and a hyperoperation $\circ$ on H. Thus to each ordered pair (a, b) of elements $a, b \in H$ is assigned a non-empty subset of H denoted by $a \circ b$. Polygroups are a special subclass of hyperstructures which are completely regular and reversible. In this paper, we investigated the Cayley hypergraphs over polygroups. We described the connection between Cayley hypergraphs over polygroups extension and Cartesian product of two Cayley hypergraphs. In particular, we obtained a necessary and sufficient for a Cayley hypergraph over a polygroup to be connected.

For further research, it will be interesting to consider Cayley hypergraphs over join spaces.

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References

[1] C. Berge, *Graphs and Hypergraphs*, **6**, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[2] A. Bretto, *Hypergraph Theory. An introduction*, Mathematical Engineering, Springer, Cham, 2013.

[3] M. Buratti, Cayley, Marty and Schreier hypergraphs, *Abh. Math. Sem. Univ. Hamburg*, **64** (1994) 151–161.

[4] S. D. Comer, Polygroups derived from cogroups, *J. Algebra*, **89** no. 2 (1984) 397–405.

[5] P. Corsini, Hypergraphs and hypergroups, *Algebra Universalis*, **35** no. 4 (1996) 548–555.

[6] B. Davvaz, *Polygroup theory and related systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[7] M. Farshi, B. Davvaz and S. Mirvakili, Degree hyoergroupoids associated with hypergraphs, *Filomat*, **28** no. 1 (2014) 119–129.

[8] A. Iranmanesh and M. N. Iradmusa, The combinatorial and algebraic structure of the hypergroup associated to a hypergraph, *. Mult.-Valued Logic Soft Comput.*, **11** no. 1-2 (2005) 127–136.

[9] J. Lee and Y. S. Kwon, Cayley hypergraphs and Cayley hypermaps, *Discrete Math.*, **313** no. 4 (2013) 540–549.

[10] V. Leoreanu and L. Leoreanu, Hypergroups associated with hypergraphs, *Ital. J. Pure Appl. Math.*, **4** (1998) 119–126.

[11] F. Marty, Sur une generalization de la notion de groupe, *In: 8th congres Math. Scandinaves, Stockholm*, (1934) 45–49.

[12] J. Meier, *Groups Graphs and Trees: An introduction to the Geometry of infinite groups*, Cambridge University Press, Cambridge, 2008.

[13] A. Nikkhah, B. Davvaz and S. Mirvakili, Hypergroups constructed from hypergraphs, *Filomat*, **32** no. 10 (2018) 3487–3494.

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