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THE DEGREE-ASSOCIATED RECONSTRUCTION NUMBER OF AN UNICENTROIDAL TREE

RAJAB ALI BORZOOEI* AND MEHRNOOSH SHADRAVAN

Abstract. As we know, by deleting one vertex of a graph G, we have a subgraph of G called a card of G. Also, investigation of that each graph with at least three vertices is determined by its multiset of cards, is called the reconstruction conjecture and the minimum number of dacards that determine G is denoted the degree-associated reconstruction number $drn(G)$. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees. A tree is unicentroidal or bicentroidal when it has one or two centroids, respectively. An unicentroidal tree T with centroid v is symmetrical if for two neighbours of u and u' of v, there exists an automorphism on T mapping u to u'. In [10], Shadravan and Borzooei proved that the conjecture is true for any non-symmetrical unicentroidal tree. In this paper, we proved that for any symmetrical unicentroidal tree T, $drn(T) \leq 2$. So, we concluded that the conjecture is true for any unicentroidal tree.

1. Introduction

In this paper, any graph is simple and any subgraph is a vertex induced subgraph. The size of a graph is the number of vertices of the graph. For a graph G, the vertex set and edge set are $V(G)$ and $E(G)$. The degree of vertex v is denoted by $d(v)$. The induced subgraph obtained by the deletion of a vertex v of G is denoted by $G - v$. Moreover, we denote by $G + v$, the graph which is created by adding a new vertex v and joining it to some vertices of G. The distance between two vertices v and u is the

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*Corresponding author.

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number of edges in a shortest path connecting them. An automorphism is just a permutation α of its vertex set which preserves adjacency: if uv is an edge then so is $\alpha(u)\alpha(v)$. A connected component of G, or simply a component, of G is a subgraph in which each pair of vertices is connected with each other via a path. A vertex v in tree T is called leaf if $d(v) = 1$. The neighbour of v is denoted by $p(l)$.

Let G be a graph. For any vertex v of G, the card C_v is the subgraph of G obtained by deleting v. The well-known graph reconstruction conjecture [6, 11] asserts that any graph of order at least three can be reconstructed from its deck of cards. Here the deck of graph G is the multiset of cards. For the surveys of results on this conjecture see [3, 4].

Usually, a graph is determined by less than its full deck. Introduced by Harary and Plantholt [5], the reconstruction number of a graph G, denoted rn(G), is the minimum number of cards from the deck of G that suffice to determine G, in the sense that no graph not isomorphic to G has this multiset in its deck

A degree asssociated card or dacard dC_v of the graph G is the ordered pair $(C_v, d_G(v))$ where $d_G(v)$ is the degree of v in G. The dadeck of graph G, denoted by $dD(G)$, is the multiset of dacards of G. Ramachandran [9] defined the degree-associated reconstruction number $drn(G)$ of a graph G to be the size of the smallest submultiset of $dD(G)$ which is not contained in the dadeck of any other graph. In other words, $drn(G)$ is the minimum number of dacards from the dadeck of G that suffic to determine G. Ramachandran studied it for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs. Barrus studied vertex-transitive graphs. They proved that $drn(G) \leq 3$ when G is not complete or edgeless. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitetly many trees.

The weight $\omega(u)$ of a vertex u in a tree T is the maximum number of vertices in a single component of $T - u$. A centroid of a tree is a vertex of minimum weight. A tree is unicentroidal or bicentroidal when it has one or two centroids, respectively. There exists the following Lemma about trees.

Lemma 1.1. [2] *An n-vertex tree T has one centroid or two adjacent centroids. It has one when the minimum vertex weight is less than $\frac{n}{2}$, two when it equals $\frac{n}{2}$.*

If u and v are two vertices of a simple graph G, and if there is an automorphism f which maps u to v, then u and v are alike in the graph, and are referred to as similar vertices and we write $v \sim u$. In addition, two dissimilar vertices u and v of a graph G are called pseudosimilar if the vertex-deleted subgraphs $G - u$ and $G - v$ are isomorphic. Let T be a tree. We call near-leaves of T, the leaves of the tree obtained by the deletion of all of the leaves of T.

Lemma 1.2. [7] *Let T be a tree with two leaves or two near-leaves u and v such that $T - u \cong T - v$. Then u and v are similar.*

An unicentroidal tree T with centroid v is symmetrical if for two neighbours of u and u' of v, there exists an automorphism on T mapping u to u'. The main theorem of this paper is stated as follows.

Theorem 1.3. *Let T be a symmetrical unicentroidal tree. Then $drn(T) \leq 2$.*

In this paper, we can conclude that the conjecture of Barrus and West is true for any unicentroidal tree by applying the following theorem, which was proved in [10].

Theorem 1.4. [10] *Let T be a non-symmetrical unicentroidal tree. Then $drn(T) \leq 2$.*

2. Basic Lemmas

In this section, our goal is to prove Lemma 2.11 for applying in the next section. Let T be an unicentroidal tree with a vertex v. All of the components of $T - v$ are called components of v in T. Now, let T be a biocentroidal tree with centroids v and u. The removal of edge vu in T splits the tree into two subtrees of exactly $\frac{n}{2}$ vertices which are called centroidal components of T. Moreover, the edge vu is called the centroidal edge of T.

Definition 2.1. *The vertex v_1 of T and v_2 of T' are called similar, written $v_1(T) \sim v_2(T')$ or briefly $v_1 \sim v_2$ if there exists an isomorphism from T to T' mapping v_1 to v_2.*

For any two similar vertices v and u in a tree T, if $d(v, u)$ is even, then there exists a vertex w on the path between v and u such that $d(v, w) = d(u, w)$. We call the vertex w, the middle vertex of v and u. Also, if the $d(v, u)$ is odd, then there exists an edge e on the path between v and u such that the removel of e in T splits the tree into two subtrees of exactly $\frac{n}{2}$ vertices. We call the the edge e, the middle edge of v and u. Note that any tree with a middle edge is a biocentroidal tree.

Lemma 2.2. *Any two similar vertices v and u in a tree T have a middle vertex or a middle edge. They have middle vertex when $d(v, u)$ is even, middle edge when $d(v, u)$ is odd.*

Proof. The proof is straightforward. □

Note that if T is a tree with two similar vertices u and u' whose middle vetex is v, then the two components v in T containing u and u', respectively, are of the same size.

Let T be a tree with a vertex v and let f be an automorphism on T. The vertex v is called a fix point in f, if $f(v) = v$.

Lemma 2.3. *Let G_1 and G_2 be two trees of sizes of k and $k+2$. Suppose that a tree T is obtained from G_1 and G_2 by joining vertex $u_1 \in V(G_1)$ to vertex $u_2 \in V(G_2)$. If there exists an automorphism f on T mapping v to v' where $v, v' \in V(G_1)$. Then the following statements hold.*

1) *The vertex u_1 is a fix point in f.*
2) *There is an automorphism on G_1 mapping v to v' and u_1 to u_1.*

Proof. The proof is straightforward. □

Let H be a component of the vertex v in T. The minimal subtree contains v and H is denoted by H^+.

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Lemma 2.4. *Let T be a tree with vertex v which G_1 is the only smallest component of v and let $u_1, u_2 \in V(G_1)$. If there exists an automorphism f on T mapping u_1 to u_2. Then the following statements hold.*

1) *The vertex v is a fix point in f.*
2) *There is an automorphism on ${G_1}^+$ mapping u_1 to u_2 and v to v.*

Proof. The proof is straightforward. □

Lemma 2.5. *Let T be a tree with a vertex v such that G_1 and G_2 are two components of v and let G_1 be the only smallest component of v. If $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, then there is no automorphism on T mapping u_1 to u_2 and v to v.*

Proof. The proof is straightforward. □

Lemma 2.6. *Let T be a tree on n vertices with a vertex v. Then there exists in T a leaf $l \neq v$ such that for any automorphism f on $T - l$ in which $p(l)$ is not fix point, v is not also fix point in f.*

Proof. We prove the lemma by induction on n.

If the vertex v is not a leaf in T, then denote by H a smallest component of v in T. Consider H^+ and the vertex v in H^+. By the induction hypothesis, there is a leaf l in H^+ such that for any automorphism f on $H^+ - l$ in which $p(l)$ is not fix point, v is not also fix point in f. Now, if $p(l) = v$, then clearly l has the desired condition in T (see T_7 in Figure 1). Also, if $p(l) \neq v$, then Lemmas 2.4 and 2.5 imply that l has the desired condition, since $H - l$ is the only smallest component v in $T - l$ (see T_6 in Figure 1).

If the vertex v is a leaf in T and T is a path tree. Then another leaf of T has the desired condition. Also, if T is not a path tree, then denote by v' the vertex of degree at least 3 in T which has minimum distance of v and by F the subtree which is obtained by deleting v and all the internal vertices on path between v and v' (see T_8 in Figure 1). By applying the induction hypothesis for F and v', we have there is a leaf l in F such that for any automorphism f on $F - l$ in which $p(l)$ is not fix point, v' is not also fix point in f. Now, one can easily prove that l has also the desired condition for T and v. □

Lemma 2.7. *Let T be a tree with two distinct vertices v and u which are similar. Then for any two vertices v' and u' on the path between v and u which $d(v, v') = d(u, u')$, we have $d(v') = d(u')$.*

Proof. The proof is straightforward. □

A biocentroidal tree T with centroidal edge $e = vu$ is called symmetrical if there exists an automorphism on T mapping v to u.

A line 1 tree is a tree on $n+1 \geq 5$ vertices which is obtained from a path $P : x_1, x_2, \ldots, x_n$ by joining a leaf l to the vertex $x_{\frac{n}{2}}$ on P, where n is even (see L_4 and L_6 in Figure 1). This tree is denoted by L_n.

A line 2 tree is a tree on $n + k$ vertices which is obtained from a path $P : x_1, x_2, \ldots, x_n$ by joining k leaves to k distinct consecutive vertices $x_{\frac{n-k+1}{2}}, x_{\frac{n-k+3}{2}}, \ldots, x_{\frac{n+k-1}{2}}$ on P, where n is even and $k < n$ is odd (see T_4 and T_5 in Figure 1) or n is odd and $k < n$ is even (see T_3 in Figure 1).

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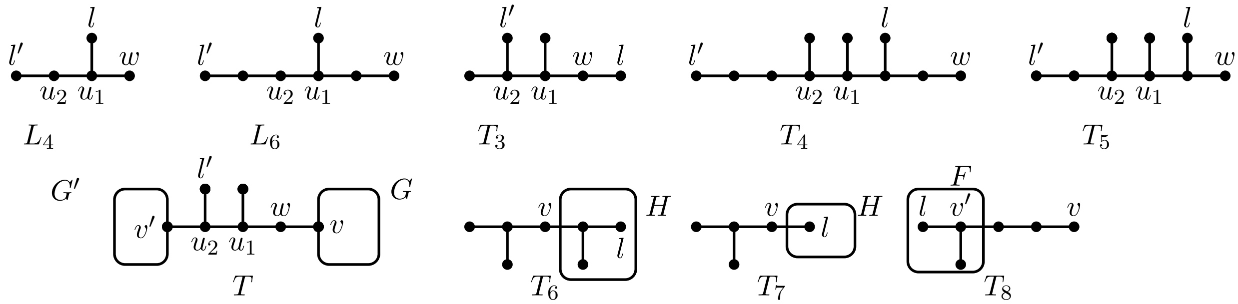

FIGURE 1. Some line trees and some trees in proof of Lemmas 2.6 and 2.9.

Any tree which is line 1 or line 2 is called a line tree.

Lemma 2.8. *Let T be a tree and let T' be a tree obtained by joining a new leaf l to a vertex v of T. If there exists an isomorphism from $T' - l'$ to T mapping l to v where $l' \neq l$ is a leaf in T'. Then T is a path tree with one end in v.*

Proof. The proof is straightforward. □

Lemma 2.9. *Let T be a tree with a leaf l and let $T-l$ be a symmetrical biocentroidal tree with centroidal components H_1 and H_2 such that $P(l) \in V(H_1)$. If l' is a leaf of T contained in H_2 and there exists a vertex w in $H_1 + l$ so that $p(l')(T-l') \sim w(T-l')$, then T is a path or a line tree.*

Proof. Denote by u_1u_2 the centroidal edge of $T - l$ such that $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$. Denote by P the path between w and u_1 and denote by u_3, the vertex on p which is adjacent to u_1. If $d_T(u_1) = 2$, then one can easily prove that T is a path. So, we may suppose that $d_T(u_1) \neq 2$. Now, we distinguish the following two cases.

Case 1. Suppose that $d_T(u_1) = 3$ and u_1 is adjacent to a leaf.

If l is a leaf adjacent to u_1. Then we have $d_T(u_2) = 2$, since $T - l$ is symmetrical biocentroidal tree. Moreover, neither the edge u_1u_3 nor no other edge can splits $T - l'$ into two subtress of the same size, since, $d_T(u_1) = 3$ and u_1 is adjacent to a leaf. Hence, the middle vertex of $p(l')$ and w is the vertex u_1. Now, consider the path P between w and u_1 and the path P' between $p(l')$ and u_1. Since, $d_T(u_2) = 2$, we have $d_T(u_3) = 2$ by Lemma 2.7. On the other hand, $T - l$ is a symmetrical biocentroidal tree with centroidal edge u_1u_2. Thus the vertex adjacent to u_2 which is contained in H_2 is also of degree 2. Now, one can easily prove that all internal vertices of both P and P' are of degree 2. Also, we can see that $d_T(p(l')) = 2$, since $T - l$ is a symmetrical biocentroidal tree. So, $d_{T-l'}(p(l')) = 1$ and hence, $d_T(w) = 1$ (see T_1 and T_2 in Figure 1). Therefore, T is a line1 tree.

If l is not the leaf adjacent to u_1. Then we have $d_T(u_2) = 3$ and u_2 is adjacent to a leaf, since $T - l$ is symmetriacl biocentroidal. Moreover, the edge u_1u_3 can not splits $T - l'$ into two subtress of the same size, since, $d_T(u_1) = 3$ and u_1 is adjacent to a leaf. So, the middle vertex of $p(l')$ and w is the vertex u_1. Denote by P the path on k vertices between w and u_1 and by P' the path on k vertices between $p(l')$

and u_1. Also, denote by x_1, x_2,…,x_k, the vertices on P, where $x_1 = u_1$ and $x_k = w$. Similarly, denote by y_1, y_2,…,y_k, the vertices on P', where $y_1 = u_1$ and $y_k = p(l')$.

Now, denote by i the largest value $2 \leq t \leq k$ for which the vertex x_{t-1} is not on the path between l and w. One can easily prove that for any $2 \leq j < i$, $d_T(y_j) = 3$ and y_j is adjacent to a leaf, since $T - l$ is a symmetrical biocentroidal tree and hence, $d_T(x_j) = 3$ and x_j is adjacent to a leaf, since $p(l')(T - l') \sim w(T - l')$. Now, we have two cases.

If $i \leq k - 1$, then $d_T(y_i) = 3$ and y_i is adjacent to a leaf, since $T - l$ is symmetrical bicentroidal tree. So, $d_T(x_i) = 3$ and x_i is adjacent to a leaf, and l is a leaf adjacent to x_i. Moreover, since, $T - l$ is a symmetrical biocentroidal tree, we have $d_T(y_{i+1}) = 2$. Now, one can easily prove that for any vertex x_j and y_j, where $k > j > i$, we have $d_T(x_j) = 2$ and $d_T(y_j) = 2$. Moreover, $d_T(p(l')) = 2$, since $T - l$ is a symmetrical biocentroidal tree and hence, $d_T(w) = 1$. Therefore, T is a line 2 tree (T_4 and T_5 in Figure 1).

If $i = k$, then since, $T - l$ is symmetrical biocentroidal tree and $d(x_{k-1}) = 3$ and x_{k-1} is adjacent to a leaf, we have $d_T(p(l')) = 3$ and $p(l')$ is adjacent to a leaf. So, $d_T(w) = 2$, since $w(T - l') \sim p(l')(T - l')$. The vertex x_{k-1} is a neighbour of w, denote by v the second neighbour. Also, the vertices l' and y_{k-1} are two neighbours of $p(l')$, denote by v', the third neighbour. Now, denote by G, the component of $T - w$ containing v and denote by G', the component of $T - p(l')$ containing v' (see T in Figure 1). Since $p(l')(T - l') \sim w(T - l')$, we have $v(G) \sim v'(G')$. Denote by G^*, the minimal subtree of T containing G and w. Now, if there is a leaf l of T contained in G such that $T - l$ is symmetrical biocentroidal tree, then $w(G - l) \sim v'(G')$. So, we can conclude that $w(G^* - l) \sim v(G)$. Now, one can easily prove that G is a path with one end in v by Lemma 2.8. Moreover, since, $T - l$ is symmetrical biocentroidal, G' is a path with one end in v'. Therefore, we have T is a line 2 tree (see T_3 in Figure 1).

Case 2. Suppose that $d_T(u_1) = 3$ such that u_1 is not adjacent to a leaf or $d_T(u_1) \geq 4$. In this case, the removal of u_3u_1 does not split the tree $T - l'$ into two subtrees of the same size. On the other hand, the two components of u_1 in $T - l'$ which repectively contain $p(l')$ and w do not have the same size. Hence, the edge u_3u_1 and the vertex u_1 can not be the middle edge and the middle vertex of $p(l')$ and w, respectively, a contradiction. □

Lemma 2.10. *Let T be a tree with vertex v and let H_1,H_2 and H_k be all of the components v in T where $k \geq 2$. Suppose that H_1 and H_k have the smallest and largest size, respectively, such that H_1 is the only smallest component of v. If $u \in V(H_1)$ and the following condition holds:*

$$|H_k| < \sum_{i=1}^{k-1}(|H_i|) + 1.$$

Then there is no vertex w contained in $T - H_1$ so that $w(T) \sim u(T)$.

Proof. For a contradiction, suppose that there is a vertex $w \in V(T - H_1)$ so that $w \sim u$. Clearly $w \neq v$. Now, let $w \in V(H_i)$ be a similar vertex to u where $H_i \neq H_1$. Clearly, the middle vertex or middle edge of u and w must be contained in $(H_i)^+$. It follows that H_i must be larger than the sum of sizes of other components of v, a contradiction. □

Two vertices v and u in a graph G are called similar if there exists an automorphism on G mapping v to u.

Lemma 2.11. *Let T be a tree, except L_4, L_6 and P_n. Then T contains a leaf l such that there is no vertex $v \neq p(l)$ for which $v(T-l) \sim p(l)(T-l)$.*

Proof. If T be a line tree, except L_4 and L_6, then one can easily show that there exists in T a leaf with the desired property. So, we may suppose that T is a tree, except a path or line tree. Choose an arbitrary leaf l_1 from T. Suppose that l_1 dose not have the desired condition. So, there is an automrohism on $T-l_1$ and a vertex $v \neq p(l_1)$ in $T-l_1$ so that $f(p(l_1)) = v$. Now, we have two cases :

Case 1. Suppose that $d(v, p(l_1))$ is odd. In this case, since $T-l_1$ is a biocentroidal tree, T is a tree with an edge $e = u_1u_2$ such that the removal of e splits T into two subtrees of exactly $\frac{n-1}{2}$ and $\frac{n+1}{2}$ vertices. Denote the subtree contains l_1 and u_1 by H_1 and the second subtree by H_2. Clearly, $|H_1| = \frac{n+1}{2}$ and $|H_2| = \frac{n-1}{2}$. Now, consider the subtree H_2 and the vertex u_2 of H_2. It follows from Lemma 2.6 that H_2 contains a leaf l such that there is no automorphism f on $H_2 - l$ in which u_2 is fix point and $p(l)$ is not fix point in f (see T_1 in Figure 2). Now, we prove that l has the desired condition of this lemma. For a contradiction, suppose that there is an automorphism f on $T-l$ and a vertex $w \neq p(l)$ so that $f(p(l)) = w$. Since, T is not path or a line tree, w is not contained in H_1 by Lemma 2.9. Thus, w is contained in $H_2 - l$. Now, since, $|H_2 - l| < |H_1|$, there is an automorphism on $H_2 - l$ mapping $p(l)$ to w and u_2 to u_2 by Lemma 2.3, a contradiction.

Case 2. Suppose that $d(v, p(l_1))$ is even. In this case, suppose that u is the vertex on the path between v and $p(l_1)$ which $d(v, u) = d(u, p(l_1))$. Denote by H_1 the component of u in T containing l_1. Also, denote by F_1, the component of u in T containing v. Clearly, $|H_1| = |F_1| + 1$. Now, denote by G, the smallest component of u in T. By considering G^+ and the vertex u, Lemma 2.6 implies that there exists in G^+ a leaf $l_2 \neq u$ such that for any automorphism $f$on $G^+ - l_2$ in which $p(l_2)$ is not fix point, u is not also fix point in f. Denote a largest component of u in $T - l_2$ by M. Also, denote by $G_1, G_2, \ldots, G_k$ all of the components of u in $T - l_2$, except M. Now, with respect to the following condition, we have two cases.

$$|M| \geq \sum_{i=1}^{k}(|G_i|) + 1. \qquad (1)$$

Case 2.1. Suppose that the condition (1) does not hold. In this case, we prove that l_2 has the desired condition of the lemma. If $p(l_2) = u$, that is $|G| = 1$ (see T_2 in Figure 2), then the statement is obviously true, since the condition (1) does not hold. Now, if $p(l_2) \neq u$ (see T_3 in Figure 2), then for a contradiction, suppose that there is an automorphism f on $T - l_2$ and a vertex $w \neq p(l_2)$ so that $f(p(l_2)) = w$. The vertex w is contained in $G - l_2$ by Lemma 2.10. Now, since, $G - l_2$ is the smallest

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component u in $T - l_2$, there is an automorphism on $G - l_2$ mapping $p(l)$ to w and u to u by Lemma 2.4, a contradiction.

Case 2.2. Suppose that the condition (1) holds. In this case, if there is no vertex $w \neq p(l_2)$ in $T - l_2$ such that $w(T-l_2) \sim p(l_2)(T-l_2)$, then l_2 has the desired condition of the lemma. So, we may suppose that there is a vertex $w \neq p(l_2)$ in $T - l_2$ such that $w(T-l_2) \sim p(l_2)(T-l_2)$. Clearly, w is contained in M. Moreover, the middle edge or middle vertex of w and $p(l_2)$ must be contained in M^+. Now, we have three cases.

If M is H_1 and G is F_1, then since the condition (1) holds and $|H_1| = |F_1|+1$, we have if $|F_1| > 1$, then $d_T(u) = 2$, since F_1 is a smallest component of u in T. Now, denote by u_f and u_h, the two neighbours of u contained in F_1 and H_1, respectively. We have $p(l_2)(T-l_2) \sim w(T-l_2)$ and u_h must be the middle vertex of $p(l_2)$ and w in $T - l_2$. On the other hand, $u_h(T-l) \sim u_f(T-l)$. So, one can easily prove that T is a path, a contradition. Also, if $|F_1| = 1$, then it is easy to see that T is line 1 tree or a path, a contradition.

If M is H_1 and G is not F_1, then since the condition (1) holds and $|H_1| = |F_1|+1$, we have $d_T(u) = 3$ and $|G| = 1$. In other words, l_2 is a leaf adjacent to u. Denote by u_h the neighbour of u which is contained in H_1. We have $u(T-l_2) \sim u_h(T-l_2)$. Now, one can easily prove that T is a line 1 tree, a contradiction.

If M is not H_1, then we find the leaf l which has the desired condition by using the following algorithm.

1: set $T := T, l_1 := l_2, v := w$
2: if $d(v, p(l_1))$ is odd, then go to Case 1
3: if $d(v, p(l_1))$ is even, then go to Case 2

The procedure is finished when we go to Case 1 or Case 2.1 in a step. Note that there is one step k that M has not the condition (1), since M becomes smaller and F_1 and H_1 become larger in each step. Thus, we go to Case 1 or Case 2.1 in step k and the procedure terminates with a desired leaf l.

See T_4 or T_5 in Figure 2. If we choose the leaf l_1 in step 1, then l_1 has not the desired condition. Next we choose l_2. The leaf l_2 also has not the desired condition. But in the second step, we find the leaf l which has the desired property. □

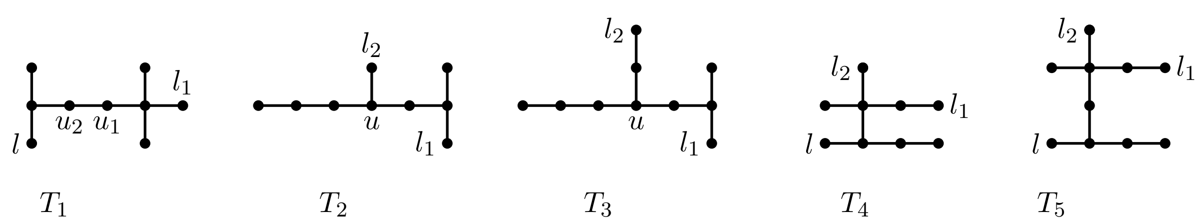

FIGURE 2. Some trees in proof of Lemma 2.11.

3. Proof of Theorem 1.3

Lemma 3.1. *Let T be an unicentroidal tree with centroid v all of whose components are of size m and let $d(v) \geq 3$. Suppose that $u \in V(H)$ is a neighbour of v where H is a component of v. Then the tree obtained from T by deleting the edge vu and joining u to a vertex of $T - H$, does not contain a vertex all of whose components are of size m.*

Proof. The proof is straightforward (see T_1 in Figure 3, all of the components of v are of size 5, note that the tree T' has no vertex all of whose componets are of size 5). □

Lemma 3.2. *Let T be an unicentroidal tree with centroid v. Suppose that we join a new leaf l to T such that $T + l$ is unicentroidal. Then the centroid of $T + l$ is also v.*

Proof. The proof is straightforward. □

Theorem 3.3. *Let T be an unicentroidal tree on n vertices with centroid v all of whose components are of the same size m and let $d(v) = 3$. Suppose that there exists at least one component of v which is not L_4, L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Denote by H the component of v which is not L_4, L_6 or a path. Let u be the neighbour of v contained in H. We have two cases.

Case 1. Suppose that $d_T(u) > 2$. Lemma 2.11 implies that H contains a leaf l such that there is no vertex $w \neq P(l)$ in $T - l$ for which $w(T - l) \sim p(l)(T - l)$. Moreover, since $d_T(u) > 2$, the leaf l is not adjacent to v. We prove that T is determined by two dacards dC_v and dC_l. Since l is a leaf, any reconstruction of the two dacards is a tree. Now, let T' be a tree shares dC_v and dC_l. The dacard dC_v implies that T' is an unicentroidal tree whose centroid is of degree more than 2 and all of the components of the centroid in T' are of the same size m, since C_v contains at least 3 components are of the same size m. Now, consider dC_l. If we join a leaf l to C_l such that $C_l + l$ is uncentroidal, then the centroids of C_l and $C_l + l$ are the same by Lemma 3.2. Now, by comparing components of C_v and components of the centroid of C_l, we find the component of the centroid in C_l to which we must join the leaf l. Denote it by H'. So, we must join l to H' such that $H' + l' \cong H$. Moreover, there exists exactly one vertex in H' to which we can join l, since there is no vertex $w \neq P(l)$ in H' for which $w(H') \sim p(l)(H')$. Hence, $T' \cong T$ and T is determined.

Case 2. Suppose that $d_T(u) = 2$. We prove that T is determined by dC_v and dC_u. First, one can easily prove that any graph with the two dacards is a tree. So, let T' be a tree shares dC_v and dC_u. The dacard dC_v implies that T' is an unicentroidal tree whose centroid is of degree more than 2 and all of the components of the centroid in T' are of the same size m, since C_v contains at least 3 components which have the same size m. Now, we want to join a vertex u to two vertices of C_u such that $dC_v \in dD(C_u + u)$. The card C_u has two components of sizes of $n - m$ and $m - 1$, denoted by H_1 and H_2, respectively. Since, T' is a tree, we must join u to a vertex H_1 and a vertex of H_2. Lemma 3.1 implies that u must be joined to v in H_1. Now, by comparing all of components C_v and all of the components of vertex v in H_1, we obtain the subtree F which u must be joined to a vertex H_2 such that $H_2 + u \cong F$. On the other hand, since there is no two pseudosimilar leaves in a tree, $T' \cong T$ and hence, T is determined (see T_2, $C_v(T_2)$ and $C_u(T_2)$ in Figure 3). □

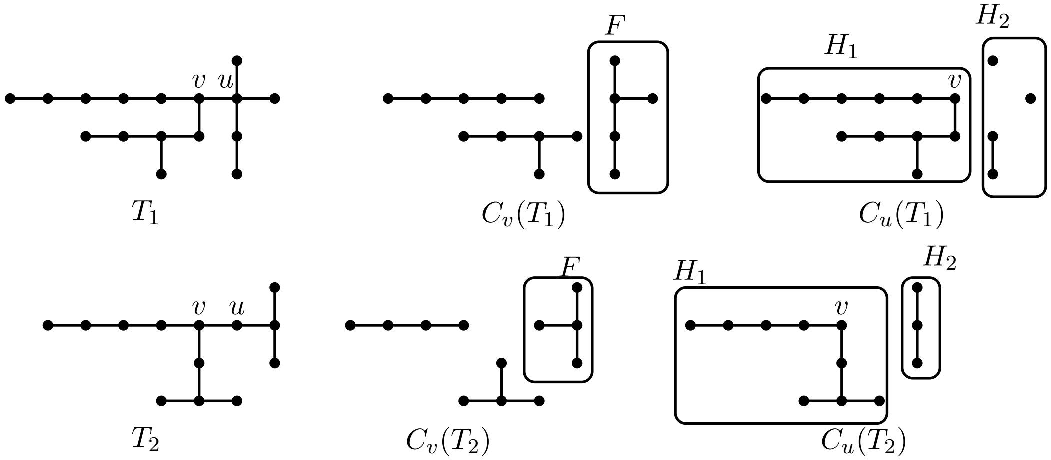

FIGURE 3. Some trees in proof of Theorems 3.3 and 3.5.

Lemma 3.4. *Let T be L_4, L_6 or a path. Then there is no two pseudosimilar vertices in T.*

Proof. The proof is straightforward. □

Theorem 3.5. *Let T be an unicentroidal tree on n vertices with centroid v and let $d(v) = 3$. Suppose that all of the components of v are L_4, L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Let H be a component of v. Suppose that $u \in V(H)$ is adjacent to v. Set $k = d(u)$. We prove that T is determined by dC_v and dC_u. Firstly, one can easily prove that any reconstruction of the two dacards is a tree. Now, let T' be a tree shares the two dacards. The card C_u contains k components. We want to join a vertex u to k vertices of C_u such that $dC_v \in dD(C_u + u)$. Since T' is a tree, we must join u to a vertex of all of the k components in C_u. One of the components is of the size $n - m$. Denote the component of size $n - m$ by H_1. Consider the other components except H_1, as a graph, denoted by H_2. Lemma 3.1 implies that the vertex v in C_u is the centroid of T'. In addition, all of the components of v in H_1 are the components of v in T'. Now, we need to obtain the last component of v in T'. By comparing all of components C_v to all of the components of vertex v in H_1, we find the last component of v in T', denoted by F. Now, the vertex v in C_u must be joined to the vertex of F whose deletion results in the graph isomorphic to H_2. Moreover, since, F is L_4, L_6 or a path, there are no two pseudosimilar vetices in F by Lemma 3.4. So, $T' \cong T$ and hence, T is determined (see T_1, $C_v(T_1)$ and $C_u(T_1)$ in Figure 3). □

Theorem 3.6. *Let T be a symmetrical unicentroidal tree with centroid v on n vertices such that $d(v) = 2$ and none of the two components of v are L_4, L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Let u_1 and u_2 be the two neighbours of v. Denote by H_1 and H_2 the two components of v containing u_1 and u_2, respectively.

Case 1. Suppose that $d_T(u_1) \geq 3$ and $d_T(u_2) \geq 3$. In this case, H_1 contains a leaf l such that there is no vertex $w \neq p(l)$ for which $w(H_1 - l) \sim p(l)(H_1 - l)$ by Lemma 2.11. Note that since $d_T(u_1) \geq 3$, we have $l \neq u_1$. Now, we prove that T is determined by dC_v and dC_l. Let T be a tree shares the two dacards. The dacard dC_v implies that T' is an unicentroidal tree whose centroid is of degree 2, since C_v has exactly the two components H_1 and H_2 of size $\frac{n-1}{2}$. Now, consider the card C_l. We want to join a leaf l to C_l such that $dC_v \in dD(C_l + l)$. Denote by F_1 and F_2 the two components of v in C_l containing u_1 and u_2, respectively. Clearly, $|F_1| = \frac{n-1}{2}$ and $|F_2| = \frac{n+1}{2}$. Now, we must join the leaf l to one vertex of F_1 to obtain an unicentroidal tree whose centroid is of degree 2, since joining a leaf to a vertex of F_2 results in an unicentroidal tree whose centroid is u_2 and $d_T(u_2) = 3$. On the other hand, by comparing $\{F_1, F_2\}$ and $\{H_1, H_2\}$, we get that l must be joined to C_l such that $F_1 + l \cong H_1$. Now, Lemma 2.11 implies that the vertex $p(l)$ in F_1 is the only vertex to which we can join l. Hence, $T' \cong T$ and so T is determined (see T, $C_v(T)$ and $C_l(T)$ in Figure 4).

Case 2. Suppose that $d_T(u_1) = 2$ and $d_T(u_2) = 2$ (see T', $C_v(T')$ and $C_{u_1}(T')$ in Figure 4). In this case, we prove that T is determined by dC_v and dC_{u_1}. Firstly, one can easily prove that every reconstruction of the two dacards is a tree. Now, let T' be a tree shares dC_v and dC_{u_1}. The card C_v has exactly two components H_1 and H_2 such that $H_1 \cong H_2$ and hence, $|H_1| = |H_2|$. So, dC_v implies that T' is an unicentroidal tree whose centroid is of degree 2. Assume that H is tree which $H \cong H_1 \cong H_2$. The card C_{u_1} has exactly two components of sizes $\frac{n-1}{2} - 1$ and $\frac{n-1}{2} + 1$, denoted by F_1 and F_2, respectively. The dacard dC_{u_1} implies that the centorid of T' has a neighbour of degree 2. To distinguish the centroid of T' in F_2, we must find a leaf whose deletion results in a tree isomorphic to H. Note that Lemma 1.2 implies that the centroid is distinguishable in F_2. Denote the vertex by u in F_2. On the other hand, we find a leaf u' in H whose deletion results in a tree isomorphic to F_1. Now, we have to join u in F_2 to u' in H to obtain T'. Hence, we see that $T' \cong T$ and so T is determined. □

Theorem 3.7. *Let T be a symmetrical unicentroidal tree with centroid v on n vertices such that $d(v) = 2$ and both of the two components of v are L_4, L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Clearly, there are no two pseudosimilar vertices in a path. In addition, it is easy to see there is no two pseudosimilar vertices in T when T is a tree with $n \leq 8$ vertices. Now, denote by u_1 one the of neighbours of v. One can prove that T is determined by dC_v and dC_{u_1} by a proof similar to the proof of Case 2 in Theorem 3.6. □

Proof of Theorem 1.3. If T is an unicentroidal tree with centroid v such that $d(v) \geq 3$, then Theorems 3.3 and 3.6 imply that $drn(T) \leq 2$. And if $d(v) = 2$, then Theorems 3.6 and 3.7 imply that $drn(T) \leq 2$. □

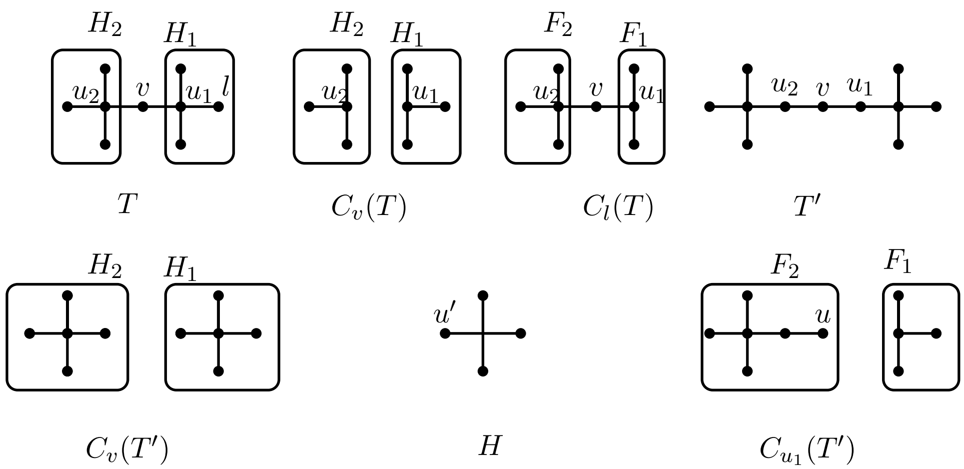

FIGURE 4. Some trees in proof of Theorem 3.6.

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Mehrnoosh Shadravan

Department of Mathematics, Shahid Beheshti University, Tehran, Iran

Email: m_shadravan@sbu.ac.ir

Rajab Ali Borzooei

(1)Department of Mathematics, Shahid Beheshti University, Tehran, Iran

(2)Department of Mathematics, Faculty of engineering and natural sciences, Istinye university, Istanbul, Turkiye

Email: borzooei@sbu.ac.ir

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