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THE DEGREE-ASSOCIATED RECONSTRUCTION NUMBER OF AN UNICENTROIDAL TREE

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ABSTRACT. As we know, by deleting one vertex of a graph G , we have a subgraph of G called a card of G . Also, investigation of that each graph with at least three vertices is determined by its multiset of cards, is called the reconstruction conjecture and the minimum number of dacards that determine G is denoted the degree-associated reconstruction number $drn(G)$. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees. A tree is unicentroidal or bicentroidal when it has one or two centroids, respectively. An unicentroidal tree T with centroid v is symmetrical if for two neighbours of u and u' of v , there exists an automorphism on T mapping u to u' . In [10], Shadravan and Borzooei proved that the conjecture is true for any non-symmetrical unicentroidal tree. In this paper, we proved that for any symmetrical unicentroidal tree T , $drn(T) \leq 2$. So, we concluded that the conjecture is true for any unicentroidal tree.

1. Introduction

In this paper, any graph is simple and any subgraph is a vertex induced subgraph. The size of a graph is the number of vertices of the graph. For a graph G , the vertex set and edge set are $V(G)$ and $E(G)$. The degree of vertex v is denoted by $d(v)$. The induced subgraph obtained by the deletion of a vertex v of G is denoted by $G - v$. Moreover, we denote by $G + v$, the graph which is created by adding a new vertex v and joining it to some vertices of G . The distance between two vertices v and u is the

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number of edges in a shortest path connecting them. An automorphism is just a permutation α of its vertex set which preserves adjacency: if uv is an edge then so is $\alpha(u)\alpha(v)$. A connected component of G , or simply a component, of G is a subgraph in which each pair of vertices is connected with each other via a path. A vertex v in tree T is called leaf if $d(v) = 1$. The neighbour of v is denoted by $p(l)$.

Let G be a graph. For any vertex v of G , the card C_v is the subgraph of G obtained by deleting v . The well-known graph reconstruction conjecture [6, 11] asserts that any graph of order at least three can be reconstructed from its deck of cards. Here the deck of graph G is the multiset of cards. For the surveys of results on this conjecture see [3, 4].

Usually, a graph is determined by less than its full deck. Introduced by Harary and Plantholt [5], the reconstruction number of a graph G , denoted $rn(G)$, is the minimum number of cards from the deck of G that suffice to determine G , in the sense that no graph not isomorphic to G has this multiset in its deck

A degree associated card or dacard dC_v of the graph G is the ordered pair $(C_v, d_G(v))$ where $d_G(v)$ is the degree of v in G . The dadeck of graph G , denoted by $dD(G)$, is the multiset of dacards of G . Ramachandran [9] defined the degree-associated reconstruction number $drn(G)$ of a graph G to be the size of the smallest submultiset of $dD(G)$ which is not contained in the dadeck of any other graph. In other words, $drn(G)$ is the minimum number of dacards from the dadeck of G that suffice to determine G . Ramachandran studied it for complete graphs, edgeless graphs, cycles, complete bipartite graphs, and disjoint unions of identical graphs. Barrus studied vertex-transitive graphs. They proved that $drn(G) \leq 3$ when G is not complete or edgeless. Barrus and West conjectured that $drn(G) \leq 2$ for all but finitely many trees.

The weight $\omega(u)$ of a vertex u in a tree T is the maximum number of vertices in a single component of $T - u$. A centroid of a tree is a vertex of minimum weight. A tree is unicentroidal or bicentroidal when it has one or two centroids, respectively. There exists the following Lemma about trees.

Lemma 1.1. [2] *An n -vertex tree T has one centroid or two adjacent centroids. It has one when the minimum vertex weight is less than $\frac{n}{2}$, two when it equals $\frac{n}{2}$.*

If u and v are two vertices of a simple graph G , and if there is an automorphism f which maps u to v , then u and v are alike in the graph, and are referred to as similar vertices and we write $v \sim u$. In addition, two dissimilar vertices u and v of a graph G are called pseudosimilar if the vertex-deleted subgraphs $G - u$ and $G - v$ are isomorphic. Let T be a tree. We call near-leaves of T , the leaves of the tree obtained by the deletion of all of the leaves of T .

Lemma 1.2. [7] *Let T be a tree with two leaves or two near-leaves u and v such that $T - u \cong T - v$. Then u and v are similar.*

An unicentroidal tree T with centroid v is symmetrical if for two neighbours of u and u' of v , there exists an automorphism on T mapping u to u' . The main theorem of this paper is stated as follows.

Theorem 1.3. *Let T be a symmetrical unicentroidal tree. Then $drn(T) \leq 2$.*

In this paper, we can conclude that the conjecture of Barrus and West is true for any unicentroidal tree by applying the following theorem, which was proved in [10].

Theorem 1.4. [10] *Let T be a non-symmetrical unicentroidal tree. Then $drn(T) \leq 2$.*

2. Basic Lemmas

In this section, our goal is to prove Lemma 2.11 for applying in the next section. Let T be an unicentroidal tree with a vertex v . All of the components of $T - v$ are called components of v in T . Now, let T be a biocentroidal tree with centroids v and u . The removal of edge vu in T splits the tree into two subtrees of exactly $\frac{n}{2}$ vertices which are called centroidal components of T . Moreover, the edge vu is called the centroidal edge of T .

Definition 2.1. *The vertex v_1 of T and v_2 of T' are called similar, written $v_1(T) \sim v_2(T')$ or briefly $v_1 \sim v_2$ if there exists an isomorphism from T to T' mapping v_1 to v_2 .*

For any two similar vertices v and u in a tree T , if $d(v, u)$ is even, then there exists a vertex w on the path between v and u such that $d(v, w) = d(u, w)$. We call the vertex w , the middle vertex of v and u . Also, if the $d(v, u)$ is odd, then there exists an edge e on the path between v and u such that the removal of e in T splits the tree into two subtrees of exactly $\frac{n}{2}$ vertices. We call the the edge e , the middle edge of v and u . Note that any tree with a middle edge is a biocentroidal tree.

Lemma 2.2. *Any two similar vertices v and u in a tree T have a middle vertex or a middle edge. They have middle vertex when $d(v, u)$ is even, middle edge when $d(v, u)$ is odd.*

Proof. The proof is straightforward. □

Note that if T is a tree with two similar vertices u and u' whose middle vertex is v , then the two components v in T containing u and u' , respectively, are of the same size.

Let T be a tree with a vertex v and let f be an automorphism on T . The vertex v is called a fix point in f , if $f(v) = v$.

Lemma 2.3. *Let G_1 and G_2 be two trees of sizes of k and $k+2$. Suppose that a tree T is obtained from G_1 and G_2 by joining vertex $u_1 \in V(G_1)$ to vertex $u_2 \in V(G_2)$. If there exists an automorphism f on T mapping v to v' where $v, v' \in V(G_1)$. Then the following statements hold.*

- 1) *The vertex u_1 is a fix point in f .*
- 2) *There is an automorphism on G_1 mapping v to v' and u_1 to u_1 .*

Proof. The proof is straightforward. □

Let H be a component of the vertex v in T . The minimal subtree contains v and H is denoted by H^+ .

Lemma 2.4. *Let T be a tree with vertex v which G_1 is the only smallest component of v and let $u_1, u_2 \in V(G_1)$. If there exists an automorphism f on T mapping u_1 to u_2 . Then the following statements hold.*

- 1) *The vertex v is a fix point in f .*
- 2) *There is an automorphism on G_1^+ mapping u_1 to u_2 and v to v .*

Proof. The proof is straightforward. □

Lemma 2.5. *Let T be a tree with a vertex v such that G_1 and G_2 are two components of v and let G_1 be the only smallest component of v . If $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$, then there is no automorphism on T mapping u_1 to u_2 and v to v .*

Proof. The proof is straightforward. □

Lemma 2.6. *Let T be a tree on n vertices with a vertex v . Then there exists in T a leaf $l \neq v$ such that for any automorphism f on $T - l$ in which $p(l)$ is not fix point, v is not also fix point in f .*

Proof. We prove the lemma by induction on n .

If the vertex v is not a leaf in T , then denote by H a smallest component of v in T . Consider H^+ and the vertex v in H^+ . By the induction hypothesis, there is a leaf l in H^+ such that for any automorphism f on $H^+ - l$ in which $p(l)$ is not fix point, v is not also fix point in f . Now, if $p(l) = v$, then clearly l has the desired condition in T (see T_7 in Figure 1). Also, if $p(l) \neq v$, then Lemmas 2.4 and 2.5 imply that l has the desired condition, since $H - l$ is the only smallest component v in $T - l$ (see T_6 in Figure 1).

If the vertex v is a leaf in T and T is a path tree. Then another leaf of T has the desired condition. Also, if T is not a path tree, then denote by v' the vertex of degree at least 3 in T which has minimum distance of v and by F the subtree which is obtained by deleting v and all the internal vertices on path between v and v' (see T_8 in Figure 1). By applying the induction hypothesis for F and v' , we have there is a leaf l in F such that for any automorphism f on $F - l$ in which $p(l)$ is not fix point, v' is not also fix point in f . Now, one can easily prove that l has also the desired condition for T and v . □

Lemma 2.7. *Let T be a tree with two distinct vertices v and u which are similar. Then for any two vertices v' and u' on the path between v and u which $d(v, v') = d(u, u')$, we have $d(v') = d(u')$.*

Proof. The proof is straightforward. □

A biocentroidal tree T with centroidal edge $e = vu$ is called symmetrical if there exists an automorphism on T mapping v to u .

A line 1 tree is a tree on $n + 1 \geq 5$ vertices which is obtained from a path $P : x_1, x_2, \dots, x_n$ by joining a leaf l to the vertex $x_{\frac{n}{2}}$ on P , where n is even (see L_4 and L_6 in Figure 1). This tree is denoted by L_n .

A line 2 tree is a tree on $n + k$ vertices which is obtained from a path $P : x_1, x_2, \dots, x_n$ by joining k leaves to k distinct consecutive vertices $x_{\frac{n-k+1}{2}}, x_{\frac{n-k+3}{2}}, \dots, x_{\frac{n+k-1}{2}}$ on P , where n is even and $k < n$ is odd (see T_4 and T_5 in Figure 1) or n is odd and $k < n$ is even (see T_3 in Figure 1).

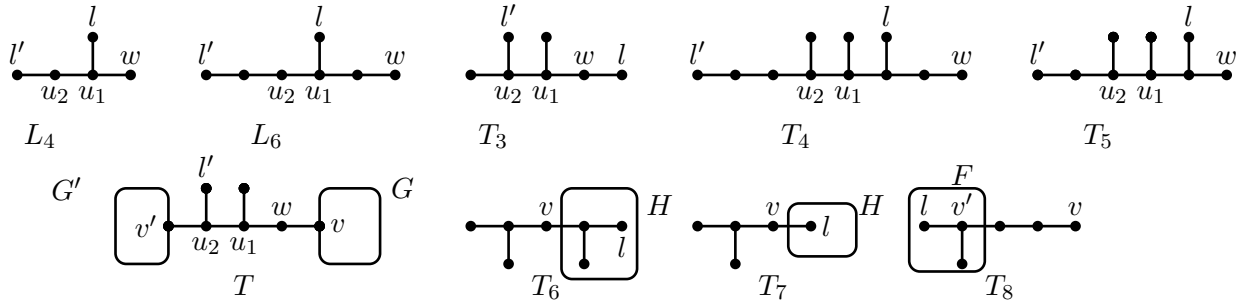


FIGURE 1. Some line trees and some trees in proof of Lemmas 2.6 and 2.9.

Any tree which is line 1 or line 2 is called a line tree.

Lemma 2.8. *Let T be a tree and let T' be a tree obtained by joining a new leaf l to a vertex v of T . If there exists an isomorphism from $T' - l'$ to T mapping l to v where $l' \neq l$ is a leaf in T' . Then T is a path tree with one end in v .*

Proof. The proof is straightforward. □

Lemma 2.9. *Let T be a tree with a leaf l and let $T - l$ be a symmetrical biocentroidal tree with centroidal components H_1 and H_2 such that $P(l) \in V(H_1)$. If l' is a leaf of T contained in H_2 and there exists a vertex w in $H_1 + l$ so that $p(l')(T - l') \sim w(T - l')$, then T is a path or a line tree.*

Proof. Denote by u_1u_2 the centroidal edge of $T - l$ such that $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$. Denote by P the path between w and u_1 and denote by u_3 , the vertex on p which is adjacent to u_1 . If $d_T(u_1) = 2$, then one can easily prove that T is a path. So, we may suppose that $d_T(u_1) \neq 2$. Now, we distinguish the following two cases.

Case 1. Suppose that $d_T(u_1) = 3$ and u_1 is adjacent to a leaf.

If l is a leaf adjacent to u_1 . Then we have $d_T(u_2) = 2$, since $T - l$ is symmetrical biocentroidal tree. Moreover, neither the edge u_1u_3 nor no other edge can splits $T - l'$ into two subtress of the same size, since, $d_T(u_1) = 3$ and u_1 is adjacent to a leaf. Hence, the middle vertex of $p(l')$ and w is the vertex u_1 . Now, consider the path P between w and u_1 and the path P' between $p(l')$ and u_1 . Since, $d_T(u_2) = 2$, we have $d_T(u_3) = 2$ by Lemma 2.7. On the other hand, $T - l$ is a symmetrical biocentroidal tree with centroidal edge u_1u_2 . Thus the vertex adjacent to u_2 which is contained in H_2 is also of degree 2. Now, one can easily prove that all internal vertices of both P and P' are of degree 2. Also, we can see that $d_T(p(l')) = 2$, since $T - l$ is a symmetrical biocentroidal tree. So, $d_{T-l'}(p(l')) = 1$ and hence, $d_T(w) = 1$ (see T_1 and T_2 in Figure 1). Therefore, T is a line1 tree.

If l is not the leaf adjacent to u_1 . Then we have $d_T(u_2) = 3$ and u_2 is adjacent to a leaf, since $T - l$ is symmetriacl biocentroidal. Moreover, the edge u_1u_3 can not splits $T - l'$ into two subtress of the same size, since, $d_T(u_1) = 3$ and u_1 is adjacent to a leaf. So, the middle vertex of $p(l')$ and w is the vertex u_1 . Denote by P the path on k vertices between w and u_1 and by P' the path on k vertices between $p(l')$

and u_1 . Also, denote by x_1, x_2, \dots, x_k , the vertices on P , where $x_1 = u_1$ and $x_k = w$. Similarly, denote by y_1, y_2, \dots, y_k , the vertices on P' , where $y_1 = u_1$ and $y_k = p(l')$.

Now, denote by i the largest value $2 \leq t \leq k$ for which the vertex x_{t-1} is not on the path between l and w . One can easily prove that for any $2 \leq j < i$, $d_T(y_j) = 3$ and y_j is adjacent to a leaf, since $T - l$ is a symmetrical biocentroidal tree and hence, $d_T(x_j) = 3$ and x_j is adjacent to a leaf, since $p(l')(T - l') \sim w(T - l')$. Now, we have two cases.

If $i \leq k - 1$, then $d_T(y_i) = 3$ and y_i is adjacent to a leaf, since $T - l$ is symmetrical biocentroidal tree. So, $d_T(x_i) = 3$ and x_i is adjacent to a leaf, and l is a leaf adjacent to x_i . Moreover, since, $T - l$ is a symmetrical biocentroidal tree, we have $d_T(y_{i+1}) = 2$. Now, one can easily prove that for any vertex x_j and y_j , where $k > j > i$, we have $d_T(x_j) = 2$ and $d_T(y_j) = 2$. Moreover, $d_T(p(l')) = 2$, since $T - l$ is a symmetrical biocentroidal tree and hence, $d_T(w) = 1$. Therefore, T is a line 2 tree (T_4 and T_5 in Figure 1).

If $i = k$, then since, $T - l$ is symmetrical biocentroidal tree and $d(x_{k-1}) = 3$ and x_{k-1} is adjacent to a leaf, we have $d_T(p(l')) = 3$ and $p(l')$ is adjacent to a leaf. So, $d_T(w) = 2$, since $w(T - l') \sim p(l')(T - l')$. The vertex x_{k-1} is a neighbour of w , denote by v the second neighbour. Also, the vertices l' and y_{k-1} are two neighbours of $p(l')$, denote by v' , the third neighbour. Now, denote by G , the component of $T - w$ containing v and denote by G' , the component of $T - p(l')$ containing v' (see T in Figure 1). Since $p(l')(T - l') \sim w(T - l')$, we have $v(G) \sim v'(G')$. Denote by G^* , the minimal subtree of T containing G and w . Now, if there is a leaf l of T contained in G such that $T - l$ is symmetrical biocentroidal tree, then $w(G - l) \sim v'(G')$. So, we can conclude that $w(G^* - l) \sim v(G)$. Now, one can easily prove that G is a path with one end in v by Lemma 2.8. Moreover, since, $T - l$ is symmetrical biocentroidal, G' is a path with one end in v' . Therefore, we have T is a line 2 tree (see T_3 in Figure 1).

Case 2. Suppose that $d_T(u_1) = 3$ such that u_1 is not adjacent to a leaf or $d_T(u_1) \geq 4$. In this case, the removal of u_3u_1 does not split the tree $T - l'$ into two subtrees of the same size. On the other hand, the two components of u_1 in $T - l'$ which respectively contain $p(l')$ and w do not have the same size. Hence, the edge u_3u_1 and the vertex u_1 can not be the middle edge and the middle vertex of $p(l')$ and w , respectively, a contradiction. □

Lemma 2.10. *Let T be a tree with vertex v and let H_1, H_2 and H_k be all of the components v in T where $k \geq 2$. Suppose that H_1 and H_k have the smallest and largest size, respectively, such that H_1 is the only smallest component of v . If $u \in V(H_1)$ and the following condition holds:*

$$|H_k| < \sum_{i=1}^{k-1} (|H_i|) + 1.$$

Then there is no vertex w contained in $T - H_1$ so that $w(T) \sim u(T)$.

Proof. For a contradiction, suppose that there is a vertex $w \in V(T - H_1)$ so that $w \sim u$. Clearly $w \neq v$. Now, let $w \in V(H_i)$ be a similar vertex to u where $H_i \neq H_1$. Clearly, the middle vertex or middle edge

of u and w must be contained in $(H_i)^+$. It follows that H_i must be larger than the sum of sizes of other components of v , a contradiction. \square

Two vertices v and u in a graph G are called similar if there exists an automorphism on G mapping v to u .

Lemma 2.11. *Let T be a tree, except L_4 , L_6 and P_n . Then T contains a leaf l such that there is no vertex $v \neq p(l)$ for which $v(T - l) \sim p(l)(T - l)$.*

Proof. If T be a line tree, except L_4 and L_6 , then one can easily show that there exists in T a leaf with the desired property. So, we may suppose that T is a tree, except a path or line tree. Choose an arbitrary leaf l_1 from T . Suppose that l_1 dose not have the desired condition. So, there is an automrohism on $T - l_1$ and a vertex $v \neq p(l_1)$ in $T - l_1$ so that $f(p(l_1)) = v$. Now, we have two cases :

Case 1. Suppose that $d(v, p(l_1))$ is odd. In this case, since $T - l_1$ is a biocentroidal tree, T is a tree with an edge $e = u_1u_2$ such that the removal of e splits T into two subtrees of exactly $\frac{n-1}{2}$ and $\frac{n+1}{2}$ vertices. Denote the subtree contains l_1 and u_1 by H_1 and the second subtree by H_2 . Clearly, $|H_1| = \frac{n+1}{2}$ and $|H_2| = \frac{n-1}{2}$. Now, consider the subtree H_2 and the vertex u_2 of H_2 . It follows from Lemma 2.6 that H_2 contains a leaf l such that there is no automorphism f on $H_2 - l$ in which u_2 is fix point and $p(l)$ is not fix point in f (see T_1 in Figure 2). Now, we prove that l has the desired condition of this lemma. For a contradiction, suppose that there is an automorphism f on $T - l$ and a vertex $w \neq p(l)$ so that $f(p(l)) = w$. Since, T is not path or a line tree, w is not contained in H_1 by Lemma 2.9. Thus, w is contained in $H_2 - l$. Now, since, $|H_2 - l| < |H_1|$, there is an automorphism on $H_2 - l$ mapping $p(l)$ to w and u_2 to u_2 by Lemma 2.3, a contradiction.

Case 2. Suppose that $d(v, p(l_1))$ is even. In this case, suppose that u is the vertex on the path between v and $p(l_1)$ which $d(v, u) = d(u, p(l_1))$. Denote by H_1 the component of u in T containing l_1 . Also, denote by F_1 , the component of u in T containing v . Clearly, $|H_1| = |F_1| + 1$. Now, denote by G , the smallest component of u in T . By considering G^+ and the vertex u , Lemma 2.6 implies that there exists in G^+ a leaf $l_2 \neq u$ such that for any automorphism f on $G^+ - l_2$ in which $p(l_2)$ is not fix point, u is not also fix point in f . Denote a largest component of u in $T - l_2$ by M . Also, denote by G_1, G_2, \dots, G_k all of the components of u in $T - l_2$, except M . Now, with respect to the following condition, we have two cases.

$$|M| \geq \sum_{i=1}^k (|G_i|) + 1. \tag{1}$$

Case 2.1. Suppose that the condition (1) does not hold. In this case, we prove that l_2 has the desired condition of the lemma. If $p(l_2) = u$, that is $|G| = 1$ (see T_2 in Figure 2), then the statement is obviously true, since the condition (1) does not hold. Now, if $p(l_2) \neq u$ (see T_3 in Figure 2), then for a contradiction, suppose that there is an automorphism f on $T - l_2$ and a vertex $w \neq p(l_2)$ so that $f(p(l_2)) = w$. The vertex w is contained in $G - l_2$ by Lemma 2.10. Now, since, $G - l_2$ is the smallest

component u in $T - l_2$, there is an automorphism on $G - l_2$ mapping $p(l)$ to w and u to u by Lemma 2.4, a contradiction.

Case 2.2. Suppose that the condition (1) holds. In this case, if there is no vertex $w \neq p(l_2)$ in $T - l_2$ such that $w(T - l_2) \sim p(l_2)(T - l_2)$, then l_2 has the desired condition of the lemma. So, we may suppose that there is a vertex $w \neq p(l_2)$ in $T - l_2$ such that $w(T - l_2) \sim p(l_2)(T - l_2)$. Clearly, w is contained in M . Moreover, the middle edge or middle vertex of w and $p(l_2)$ must be contained in M^+ . Now, we have three cases.

If M is H_1 and G is F_1 , then since the condition (1) holds and $|H_1| = |F_1| + 1$, we have if $|F_1| > 1$, then $d_T(u) = 2$, since F_1 is a smallest component of u in T . Now, denote by u_f and u_h , the two neighbours of u contained in F_1 and H_1 , respectively. We have $p(l_2)(T - l_2) \sim w(T - l_2)$ and u_h must be the middle vertex of $p(l_2)$ and w in $T - l_2$. On the other hand, $u_h(T - l) \sim u_f(T - l)$. So, one can easily prove that T is a path, a contradiction. Also, if $|F_1| = 1$, then it is easy to see that T is line 1 tree or a path, a contradiction.

If M is H_1 and G is not F_1 , then since the condition (1) holds and $|H_1| = |F_1| + 1$, we have $d_T(u) = 3$ and $|G| = 1$. In other words, l_2 is a leaf adjacent to u . Denote by u_h the neighbour of u which is contained in H_1 . We have $u(T - l_2) \sim u_h(T - l_2)$. Now, one can easily prove that T is a line 1 tree, a contradiction.

If M is not H_1 , then we find the leaf l which has the desired condition by using the following algorithm.

- 1: set $T := T, l_1 := l_2, v := w$
- 2: if $d(v, p(l_1))$ is odd, then go to Case 1
- 3: if $d(v, p(l_1))$ is even, then go to Case 2

The procedure is finished when we go to Case 1 or Case 2.1 in a step. Note that there is one step k that M has not the condition (1), since M becomes smaller and F_1 and H_1 become larger in each step. Thus, we go to Case 1 or Case 2.1 in step k and the procedure terminates with a desired leaf l .

See T_4 or T_5 in Figure 2. If we choose the leaf l_1 in step 1, then l_1 has not the desired condition. Next we choose l_2 . The leaf l_2 also has not the desired condition. But in the second step, we find the leaf l which has the desired property. □

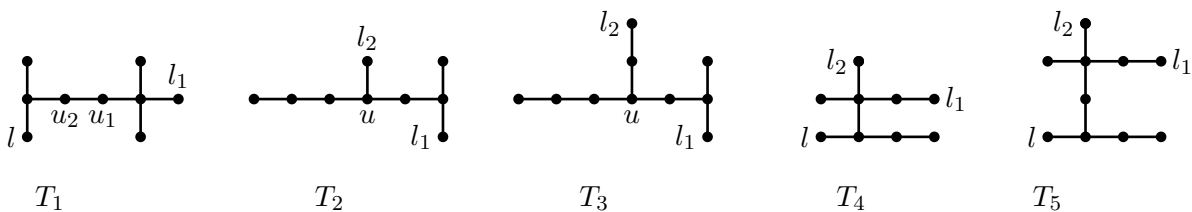


FIGURE 2. Some trees in proof of Lemma 2.11.

3. Proof of Theorem 1.3

Lemma 3.1. *Let T be an unacentroidal tree with centroid v all of whose components are of size m and let $d(v) \geq 3$. Suppose that $u \in V(H)$ is a neighbour of v where H is a component of v . Then the tree obtained from T by deleting the edge vu and joining u to a vertex of $T - H$, does not contain a vertex all of whose components are of size m .*

Proof. The proof is straightforward (see T_1 in Figure 3, all of the components of v are of size 5, note that the tree T' has no vertex all of whose componets are of size 5). □

Lemma 3.2. *Let T be an unacentroidal tree with centroid v . Suppose that we join a new leaf l to T such that $T + l$ is unacentroidal. Then the centroid of $T + l$ is also v .*

Proof. The proof is straightforward. □

Theorem 3.3. *Let T be an unacentroidal tree on n vertices with centroid v all of whose components are of the same size m and let $d(v) = 3$. Suppose that there exists at least one component of v which is not L_4, L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Denote by H the component of v which is not L_4, L_6 or a path. Let u be the neighbour of v contained in H . We have two cases.

Case 1. Suppose that $d_T(u) > 2$. Lemma 2.11 implies that H contains a leaf l such that there is no vertex $w \neq P(l)$ in $T - l$ for which $w(T - l) \sim p(l)(T - l)$. Moreover, since $d_T(u) > 2$, the leaf l is not adjacent to v . We prove that T is determined by two dacards dC_v and dC_l . Since l is a leaf, any reconstruction of the two dacards is a tree. Now, let T' be a tree shares dC_v and dC_l . The dacard dC_v implies that T' is an unacentroidal tree whose centroid is of degree more than 2 and all of the components of the centroid in T' are of the same size m , since C_v contains at least 3 components are of the same size m . Now, consider dC_l . If we join a leaf l to C_l such that $C_l + l$ is uncentroidal, then the centroids of C_l and $C_l + l$ are the same by Lemma 3.2. Now, by comparing components of C_v and components of the centroid of C_l , we find the component of the centroid in C_l to which we must join the leaf l . Denote it by H' . So, we must join l to H' such that $H' + l \cong H$. Moreover, there exists exactly one vertex in H' to which we can join l , since there is no vertex $w \neq P(l)$ in H' for which $w(H') \sim p(l)(H')$. Hence, $T' \cong T$ and T is determined.

Case 2. Suppose that $d_T(u) = 2$. We prove that T is determined by dC_v and dC_u . First, one can easily prove that any graph with the two dacards is a tree. So, let T' be a tree shares dC_v and dC_u . The dacard dC_v implies that T' is an unacentroidal tree whose centroid is of degree more than 2 and all of the components of the centroid in T' are of the same size m , since C_v contains at least 3 components which have the same size m . Now, we want to join a vertex u to two vertices of C_u such that $dC_v \in dD(C_u + u)$. The card C_u has two components of sizes of $n - m$ and $m - 1$, denoted by H_1 and H_2 , respectively. Since, T' is a tree, we must join u to a vertex H_1 and a vertex of H_2 . Lemma 3.1 implies that u must

be joined to v in H_1 . Now, by comparing all of components C_v and all of the components of vertex v in H_1 , we obtain the subtree F which u must be joined to a vertex H_2 such that $H_2 + u \cong F$. On the other hand, since there is no two pseudosimilar leaves in a tree, $T' \cong T$ and hence, T is determined (see $T_2, C_v(T_2)$ and $C_u(T_2)$ in Figure 3). □

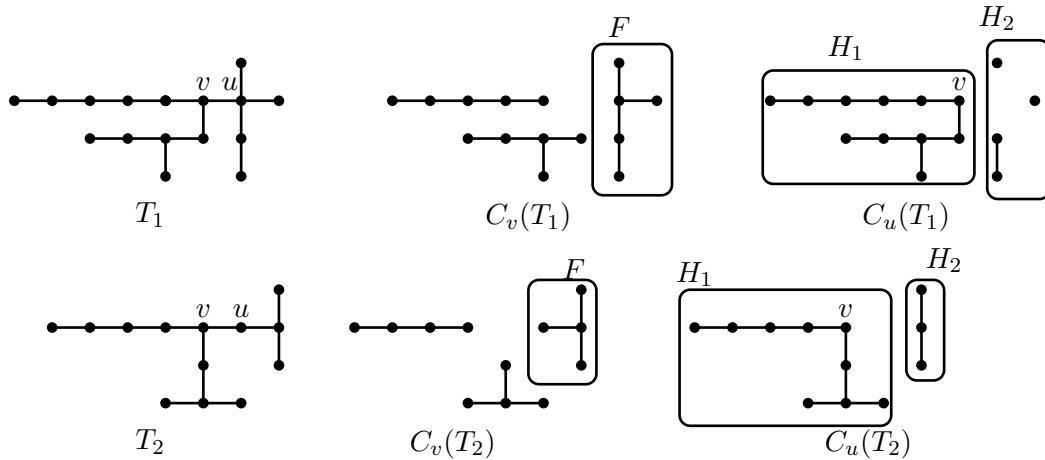


FIGURE 3. Some trees in proof of Theorems 3.3 and 3.5.

Lemma 3.4. *Let T be L_4, L_6 or a path. Then there is no two pseudosimilar vertices in T .*

Proof. The proof is straightforward. □

Theorem 3.5. *Let T be an unicycroidal tree on n vertices with centroid v and let $d(v) = 3$. Suppose that all of the components of v are L_4, L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Let H be a component of v . Suppose that $u \in V(H)$ is adjacent to v . Set $k = d(u)$. We prove that T is determined by dC_v and dC_u . Firstly, one can easily prove that any reconstruction of the two dacards is a tree. Now, let T' be a tree shares the two dacards. The card C_u contains k components. We want to join a vertex u to k vertices of C_u such that $dC_v \in dD(C_u + u)$. Since T' is a tree, we must join u to a vertex of all of the k components in C_u . One of the components is of the size $n - m$. Denote the component of size $n - m$ by H_1 . Consider the other components except H_1 , as a graph, denoted by H_2 . Lemma 3.1 implies that the vertex v in C_u is the centroid of T' . In addition, all of the components of v in H_1 are the components of v in T' . Now, we need to obtain the last component of v in T' . By comparing all of components C_v to all of the components of vertex v in H_1 , we find the last component of v in T' , denoted by F . Now, the vertex v in C_u must be joined to the vertex of F whose deletion results in the graph isomorphic to H_2 . Moreover, since, F is L_4, L_6 or a path, there are no two pseudosimilar vetices in F by Lemma 3.4. So, $T' \cong T$ and hence, T is determined (see $T_1, C_v(T_1)$ and $C_u(T_1)$ in Figure 3). □

Theorem 3.6. *Let T be a symmetrical unacentroidal tree with centroid v on n vertices such that $d(v) = 2$ and none of the two components of v are L_4 , L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Let u_1 and u_2 be the two neighbours of v . Denote by H_1 and H_2 the two components of v containing u_1 and u_2 , respectively.

Case 1. Suppose that $d_T(u_1) \geq 3$ and $d_T(u_2) \geq 3$. In this case, H_1 contains a leaf l such that there is no vertex $w \neq p(l)$ for which $w(H_1 - l) \sim p(l)(H_1 - l)$ by Lemma 2.11. Note that since $d_T(u_1) \geq 3$, we have $l \neq u_1$. Now, we prove that T is determined by dC_v and dC_l . Let T be a tree shares the two dacards. The dacard dC_v implies that T' is an unacentroidal tree whose centroid is of degree 2, since C_v has exactly the two components H_1 and H_2 of size $\frac{n-1}{2}$. Now, consider the card C_l . We want to join a leaf l to C_l such that $dC_v \in dD(C_l + l)$. Denote by F_1 and F_2 the two components of v in C_l containing u_1 and u_2 , respectively. Clearly, $|F_1| = \frac{n-1}{2}$ and $|F_2| = \frac{n+1}{2}$. Now, we must join the leaf l to one vertex of F_1 to obtain an unacentroidal tree whose centroid is of degree 2, since joining a leaf to a vertex of F_2 results in an unacentroidal tree whose centroid is u_2 and $d_T(u_2) = 3$. On the other hand, by comparing $\{F_1, F_2\}$ and $\{H_1, H_2\}$, we get that l must be joined to C_l such that $F_1 + l \cong H_1$. Now, Lemma 2.11 implies that the vertex $p(l)$ in F_1 is the only vertex to which we can join l . Hence, $T' \cong T$ and so T is determined (see T , $C_v(T)$ and $C_l(T)$ in Figure 4).

Case 2. Suppose that $d_T(u_1) = 2$ and $d_T(u_2) = 2$ (see T' , $C_v(T')$ and $C_{u_1}(T')$ in Figure 4). In this case, we prove that T is determined by dC_v and dC_{u_1} . Firstly, one can easily prove that every reconstruction of the two dacards is a tree. Now, let T' be a tree shares dC_v and dC_{u_1} . The card C_v has exactly two components H_1 and H_2 such that $H_1 \cong H_2$ and hence, $|H_1| = |H_2|$. So, dC_v implies that T' is an unacentroidal tree whose centroid is of degree 2. Assume that H is tree which $H \cong H_1 \cong H_2$. The card C_{u_1} has exactly two components of sizes $\frac{n-1}{2} - 1$ and $\frac{n-1}{2} + 1$, denoted by F_1 and F_2 , respectively. The dacard dC_{u_1} implies that the centorid of T' has a neighbour of degree 2. To distinguish the centroid of T' in F_2 , we must find a leaf whose deletion results in a tree isomorphic to H . Note that Lemma 1.2 implies that the centroid is distinguishable in F_2 . Denote the vertex by u in F_2 . On the other hand, we find a leaf u' in H whose deletion results in a tree isomorphic to F_1 . Now, we have to join u in F_2 to u' in H to obtain T' . Hence, we see that $T' \cong T$ and so T is determined. □

Theorem 3.7. *Let T be a symmetrical unacentroidal tree with centroid v on n vertices such that $d(v) = 2$ and both of the two components of v are L_4 , L_6 or a path. Then $drn(T) \leq 2$.*

Proof. Clearly, there are no two pseudosimilar vertices in a path. In addition, it is easy to see there is no two pseudosimilar vertices in T when T is a tree with $n \leq 8$ vertices. Now, denote by u_1 one the of neighbours of v . One can prove that T is determined by dC_v and dC_{u_1} by a proof similar to the proof of Case 2 in Theorem 3.6. □

Proof of Theorem 1.3. If T is an unacentroidal tree with centroid v such that $d(v) \geq 3$, then Theorems 3.3 and 3.6 imply that $drn(T) \leq 2$. And if $d(v) = 2$, then Theorems 3.6 and 3.7 imply that $drn(T) \leq 2$. □

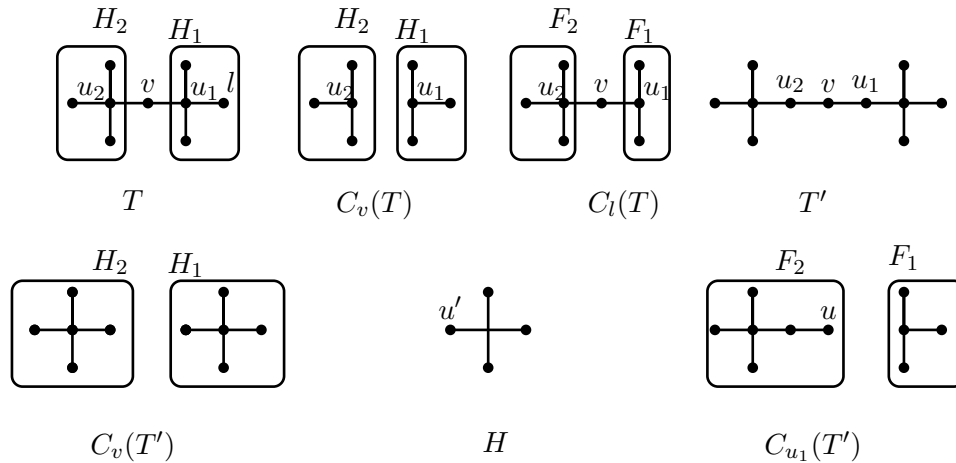


FIGURE 4. Some trees in proof of Theorem 3.6.

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