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COMMUTATIVE RINGS INTRODUCE A CLASS OF IDENTIFIABLE GRAPHS

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Abstract. Let R be a commutative ring with identity, and $\mathrm{A}(R)$ be the set of ideals with non-zero annihilator. The annihilating-ideal graph of R is defined as the graph $AG(R)$ with the vertex set $\mathrm{A}(R)^* = \mathrm{A}(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. In this paper, we characterize all positive integers n for which $AG(\mathbb{Z}_n)$ is identifiable.

1. **Introduction**

The notion of identifying codes was first introduced and studied by Karpovsky, Chakrabarty and Levitin in 1998 ([13]) for diagnosis of faults in multiprocessor systems. Since 1998 the study of identifying codes of graphs has grown a lot. Many researchers have been attracted by this branch, for instance, the identifying codes of trees and planar graphs in [3], graphs of girth at least 5 in [4], chains and cycles in [7], degree 4 Cayley graphs over Abelian groups in [8], graphs differing by one vertex in [9], graphs differing by one edge in [10], graphs that admit upper bound in [11], Cartesian product of a path and a complete graph in [12], nonzero component graph of a finite dimensional vector space in [14] and graphs that admit lower bound in [15] were studied. This paper is devoted to study identifying code of a graph associated with a ring. First let us recall some necessary notation and terminology from graph theory and ring theory.

Let $G = (V, E)$ be a graph. If $v \in V(G)$, then the closed neighborhood of v in G is denoted by

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$N_G[v]$. If $\{u,v\} \in E$, then we write $uv \in E$. Let n be a positive integer. By P_n and K_n, we mean the path and complete graph of order n. A subset $C \subseteq V(G)$ is called *identifying code*, if the sets $N_G[v] \cap C$, $v \in V(G)$, are all nonempty and different. If G admits an identifying code, we say that G is *identifiable* and denote by $i(G)$ the minimum cardinality of an identifying code of G. It is well-known that minimizing the size of an identifying code in a graph is NP-hard. We note that G is identifiable if and only if the sets $N_G[v]$ are distinct for all $v \in V(G)$. For any undefined notation or terminology in graph theory, we refer the reader to [6].

Throughout this paper, all rings are assumed to be commutative with identity. The set of all ideals of a ring R is denoted by $\mathrm{I}(R)$. For a subset T of a ring R we let $T^* = T \setminus \{0\}$. Furthermore, if I is an ideal of R, by $Ann(I)$, we mean the annihilator of I. An ideal I of R is called an *annihilating-ideal* if there exists a non-zero ideal J such that $IJ = 0$. The notation $A(R)$ is used to denote the set of all annihilating-ideals of R. For any undefined notation or terminology in ring theory, we refer the reader to [2].

Let R be a ring. The *annihilating-ideal graph* of R, denoted by $AG(R)$, is a graph with the vertex set $A(R)^*$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. The annihilating-ideal graph was first introduced in [5], and some of the properties of the annihilating-ideal graph have been studied. Further results on annihilating-ideal graphs may be found in [1, 16] and [18]. This paper is devoted to study the identifying code of $AG(\mathbb{Z}_n)$.

The following remark will be used frequently in this paper.

Remark 1.1. Consider $\mathbb{Z}_n$, the ring of integers modulo n. Throughout the paper, without loss of generality, we assume that $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$, where p_i's are distinct primes and n_i's are natural numbers. Then

(i) $\mathbb{Z}_n$ is Artinian. Thus it follows from [5, Proposition 1.3], that every non-trivial ideal of $\mathbb{Z}_n$ is a vertex of $\mathrm{AG}(\mathbb{Z}_n)$.

(ii) It follows from Chinese Remainder Theorem that

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}}.$$

(iii) $I \in I(\mathbb{Z}_n)$ if and only if $I = I_1 \times \cdots \times I_m$, where $I_i \in I(\mathbb{Z}_{p_i^{n_i}})$.

(iv) It is not hard to see that $|\mathrm{A}(\mathbb{Z}_n)^*| = \prod_{i=1}^{m}(n_i+1) - 2$.

2. Identifying code of $AG(\mathbb{Z}_n)$

First we study identifiable $AG(\mathbb{Z}_n)$.

Theorem 2.1. *The graph $AG(\mathbb{Z}_n)$ is not identifiable if and only if n is one of the following:*
(i) $n = p_1^{n_1}$, *where* n_1 *is odd.*
(ii) $n = p_1^2$.
(iii) $n = p_1 p_2$.

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Proof. We consider the following cases:

Case 1. $m = 1$.

If n_1 is odd, then it is not hard to check that $N[\langle p_1^{\lfloor n_1/2 \rfloor} \rangle] = N[\langle p_1^{\lfloor n_1/2 \rfloor + 1} \rangle]$. Hence $AG(\mathbb{Z}_n)$ is not identifiable. Assume that n_1 is even. If $n_1 = 2$, then $AG(\mathbb{Z}_n) = K_1$ and thus it is not identifiable. We show that $AG(\mathbb{Z}_n)$ is identifiable, if $n_1 \neq 2$. Suppose to the contrary, there exist α and β such that $N[\langle p_1^{\alpha} \rangle] = N[\langle p_1^{\beta} \rangle]$, where $1 \leq \alpha, \beta \leq n_1 - 1$. With no loss of generality, assume that $\beta > \alpha$. If $n_1 - \beta \neq \alpha$, then $\langle p_1^{n_1 - \beta} \rangle \in N[\langle p_1^{\beta} \rangle] \setminus N[\langle p_1^{\alpha} \rangle]$, which is impossible. If $n_1 - \beta = \alpha$, then $\langle p_1^{n_1/2} \rangle \in N[\langle p_1^{\beta} \rangle] \setminus N[\langle p_1^{\alpha} \rangle]$, a contradiction.

Case 2. $m = 2$.

if $n_1 = n_2 = 1$, then $AG(\mathbb{Z}_n) = K_2$, which is not identifiable. Let $n_1 \geq 2$ and there exist two vertices I and J such that $N[I] = N[J]$. We try to find a contradiction. Consider the following subcases:

Subcase 1. Let $I = (\mathbb{Z}_{p_1}, 0)$.

Then $J = (0, \mathbb{Z}_{p_2})$ or $J = (0, \langle p_2^{\beta} \rangle)$, where $1 \leq \beta \leq n_2 - 1$ (Note that if $n_2 = 1$, the latter case does not occure). If $J = (0, \mathbb{Z}_{p_2})$, then $(\langle p_1^{\alpha} \rangle, 0) \in N[J] \setminus N[I]$. If $J = (0, \langle p_2^{\beta} \rangle)$, then $(0, \mathbb{Z}_{p_2}) \in N[I] \setminus N[J]$, a contradiction.

Subcase 2. Let $I = (\mathbb{Z}_{p_1}, \langle p_2^{\beta} \rangle)$, where $1 \leq \beta \leq n_2 - 1$.

Then $J = (0, \langle p_2^{\gamma} \rangle)$, where $\beta + \gamma \geq n_2$. Hence $(\mathbb{Z}_{p_1}, 0) \in N[J] \setminus N[I]$.

Subcase 3. Let $I = (\langle p_1^{\alpha} \rangle, \langle p_2^{\beta} \rangle)$, where $1 \leq \alpha, \beta$.

Then $J \in \{(\langle p_1^{\gamma} \rangle, \langle p_2^{\zeta} \rangle), (0, \langle p_2^{\zeta} \rangle), (\langle p_1^{\gamma} \rangle, 0) | \, 1 \leq \gamma \leq n_1 - 1, 1 \leq \zeta \leq n_2 - 1, \alpha + \gamma \geq n_1, \beta + \zeta \geq n_2\}$. If $J = (0, \langle p_2^{\zeta} \rangle)$ or $J = (\langle p_1^{\gamma} \rangle, 0)$, then obviously $N[I] \neq N[J]$. Let $J = (\langle p_1^{\gamma} \rangle, \langle p_2^{\zeta} \rangle)$. Then $\alpha \neq \gamma$ or $\beta \neq \zeta$. Suppose that $\alpha \neq \gamma$ and without loss of generality we may assume that $\alpha > \gamma$. Thus $(\langle p_1^{n_1 - \alpha} \rangle, 0) \in N[I] \setminus N[J]$. Suppose $\langle p_1^{\alpha} \rangle = 0$ or $\langle p_2^{\beta} \rangle = 0$. Without loss of generality we may assume that $\langle p_1^{\alpha} \rangle = 0$, then $J \in \{(\langle p_1^{\gamma} \rangle, \langle p_2^{\zeta} \rangle), (0, \langle p_2^{\zeta} \rangle), (\langle p_1^{\gamma} \rangle, 0), (\mathbb{Z}_{p_1^{n_1}}, 0), (\mathbb{Z}_{p_1^{n_1}}, \langle p_2^{\zeta} \rangle) | \, 1 \leq \gamma \leq n_1 - 1, 1 \leq \zeta \leq n_2 - 1, \beta + \zeta \geq n_2\}$ which is similar to the above argument.

Case 3. $m \geq 3$.

Let $I = (I_1, \ldots, I_i, \ldots, I_m)$ be a vertex of $AG(\mathbb{Z}_n)$, where $I_i \in \{0, \mathbb{Z}_{p_i^{n_i}}, \langle p_i^{\alpha} \rangle | 1 \leq \alpha \leq n_i - 1\}$ and $N[I] = N[J]$, for some $J = (J_1, \ldots, J_m) \in V(AG(\mathbb{Z}_n))$. Consider the following subcases:

Subcase 1. Let $I_i = 0$.

Since $N[I] = N[J]$ and $I \neq J$, $J_i = 0$ and there exist I_j and J_j such that $I_j \neq J_j$. If $I_j = 0$, then $J_j = \langle p_j^{\alpha} \rangle$ or $J_j = \mathbb{Z}_{p_j^{n_j}}$. Thus it is easy to see that $N[I] \neq N[J]$. If $I_j = \langle p_j^{\alpha} \rangle$, then $J_j = \langle p_j^{\beta} \rangle$, where $\alpha \neq \beta$. By Case 1, if n_j is even then clearly $N[I] \neq N[J]$. If n_j is odd and $\beta > \alpha$, then $(0, \ldots, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, \langle p_j^{\lfloor n_j/2 \rfloor} \rangle, 0, \ldots, 0) \in N[J] \setminus N[I]$.

Subcase 2. Let $I_i = \mathbb{Z}_{p_i^{n_i}}$.

Then $J_i = 0$. Thus $(0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0) \in N[J] \setminus N[I]$. Also, if $I = (0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0)$, then $(\mathbb{Z}_{p_1^{n_1}}, \mathbb{Z}_{p_2^{n_2}}, \ldots, \mathbb{Z}_{p_{i-1}^{n_{i-1}}}, 0, \mathbb{Z}_{p_{i+1}^{n_{i+1}}}, \ldots, \mathbb{Z}_{p_m^{n_m}}) \in N[I] \setminus N[J]$.

Subcase 3. Let $I_i = \langle p_i^{\alpha} \rangle$, where $1 \leq \alpha \leq n_i - 1$.

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If $J_i = \mathbb{Z}_{p_i^{n_i}}$ or $J_i = \langle p_i^{\beta} \rangle$, where $\alpha + \beta \leq n$, then $J \notin N[I]$. Hence $J_i = \langle p_i^{\beta} \rangle$, where $\alpha + \beta \geq n$ or $J_i = 0$. If $\alpha \neq \beta$, then $N[I] \neq N[J]$. If $\alpha = \beta$, then there exist I_j and J_j such that $I_j \neq J_j$. It is not hard to check that $N[I] \neq N[J]$. Suppose that $J_i = 0$. Therefore $(0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0) \in N[J] \setminus N[I]$, a contradiction. □

Next we study $i(AG(\mathbb{Z}_n))$, if $m < 3$. First we need the following remark.

Remark 2.2. *Consider the graph $AG(\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2})$. It is easy to see that $AG(\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2}) = P_4$ and hence $i(AG(\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2})) = 3$. Moreover, one may easily see that $i(AG(\mathbb{Z}_{p_1^3} \times \mathbb{Z}_{p_2})) = 4$. Also, $i(AG(\mathbb{Z}_{p_1^3} \times \mathbb{Z}_{p_2^2})) = 5$ and $i(AG(\mathbb{Z}_{p_1^3} \times \mathbb{Z}_{p_2^3})) = 6$.*

Theorem 2.3. *Let $m < 3$. Then the following statements hold:*
(1) *If $n = p_1^{n_1}$ and $n_1 \neq 2$ is even, then $i(AG(\mathbb{Z}_n)) = n_1/2$.*
(2) *If $n = p_1^{n_1} p_2$ where $n_1 \geq 4$, then $i(AG(\mathbb{Z}_n)) \leq n_1$.*
(3) *Let $n = p_1^{n_1} p_2^{n_2}$, where $2 \leq n_1, n_2$. If n_1, n_2 are even, then $i(AG(\mathbb{Z}_n)) \leq n_1 + n_2$. Otherwise, $i(AG(\mathbb{Z}_n)) \leq n_1 + n_2 + 1$.*

Proof. (1) Let $C = \{\langle p_1^{\alpha} \rangle \mid 1 \leq \alpha \leq n_1/2\}$. We claim that C is an identifying code of $AG(\mathbb{Z}_n)$. If $I, J \in C$, then $I = N[I] \cap C \neq N[J] \cap C = J$. Suppose that $I, J \notin C$. Then there exist $(n_1/2) + 1 \leq \alpha, \beta \leq n_1 - 1$ such that $I = \langle p_1^{\alpha} \rangle$ and $J = \langle p_1^{\beta} \rangle$. Without loss of generality, assume that $\beta > \alpha$. Thus $\langle p_1^{n_1 - \beta} \rangle \in N[J] \cap C \setminus N[I] \cap C$. If $I \in C$ and $J \notin C$, then $|N[I] \cap C| = 1$ and $|N[J] \cap C| \geq 2$ and so the claim is proved. Next, we prove that C is a minimum identifying code of $AG(\mathbb{Z}_n)$. We show that every vertex in C is needed to C be an identifying code. Let C' be an arbitrary identifying code of $AG(\mathbb{Z}_n)$. One can check that, if $\langle p_1^k \rangle \notin C'$, for some $1 \leq k \leq (n_1/2) - 1$, then $N[\langle p_1^{n_1 - k} \rangle] \cap C' = N[\langle p_1^{n_1 - (k+1)} \rangle] \cap C'$. If $\langle p_1^{n_1/2} \rangle \notin C'$, then there exists $(n_1/2) + 1 \leq l \leq n_1 - 1$ such that $\langle p_1^l \rangle \in C'$. Thus $N[\langle p_1^{(n_1/2)+1} \rangle] \cap C' = N[\langle p_1^{(n_1/2)-1} \rangle] \cap C'$, as desired.

(2) Let $a = \lfloor \frac{n_1}{2} \rfloor$. Suppose that

$$C = \{(\langle p_1^{\alpha} \rangle, 0) | 1 \leq \alpha \leq a\} \cup \{(\langle p_1^{\alpha} \rangle, \mathbb{Z}_{p_2}) | 1 \leq \alpha \leq a - 1\} \cup \{(0, \mathbb{Z}_{p_2})\}.$$

If n_1 is odd, then we add the vertex $\{(\langle p_1^a \rangle, \mathbb{Z}_{p_2})\}$ to C. We claim that C is an identifying code of $AG(\mathbb{Z}_n)$. It is easy to see that $N[I] \cap C \neq \varnothing$, for every vertex I of $AG(\mathbb{Z}_n)$. Consider the following sets:

$$A = \{(\mathbb{Z}_{P_1^{n_1}}, 0), (0, \mathbb{Z}_{p_2})\}$$

and

$$B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly $\{A, B\}$ is partition of $V(AG(\mathbb{Z}_n))$. Consider the following cases:

Case 1. Let $I, J \in A$. With no loss of generality, assume that $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$ and $J = (0, \mathbb{Z}_{p_2})$. Then

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$(\langle p_1\rangle, 0) \in N[J]\cap C \setminus N[I]\cap C$.

Case 2. Let $I \in A$ and $J \in B$.

If $I = (0, \mathbb{Z}_{p_2})$, then $(\mathbb{Z}_{p_2}, 0) \in N[I]\cap C \setminus N[J]\cap C$.

Let $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$. Then $J \in \{(\langle p_1^{\alpha}\rangle, \mathbb{Z}_{p_2}), (\langle P_1^{\beta}\rangle, 0), (\langle p_1^{\gamma}\rangle, 0)\}$, where $1 \le \alpha \le n_1 - 1$, $1 \le \beta \le a$ and $a < \gamma \le n_1 - 1$.

If $J = (\langle p_1^{\alpha}\rangle, \mathbb{Z}_{p_2})$, then $(0, \mathbb{Z}_{p_2}) \in N[I]\cap C \setminus N[J]\cap C$.

If $J = (\langle p_1^{\beta}\rangle, 0)$, then $(\langle p_1^{\beta}\rangle, 0) \in N[J]\cap C \setminus N[I]\cap C$.

If $J = (\langle p_1^{\gamma}\rangle, 0)$, then $(\langle p_1^{a}\rangle, 0) \in N[J]\cap C \setminus N[I]\cap C$.

Case 3. Let $I, J \in B$.

Subcase 1. With no loss of generality, assume that $I = (\langle p_1^{\alpha}\rangle, 0)$, $J = (\langle p_1^{\beta}\rangle, \mathbb{Z}_{p_2})$ and $\alpha < \beta$. Then $(0, \mathbb{Z}_{p_2}) \in N[I]\cap C \setminus N[J]\cap C$.

Subcase 2. Suppose $I = (\langle p_1^{\alpha}\rangle, 0)$, $J = (\langle p_1^{\beta}\rangle, 0)$ and $\alpha < \beta$. If $\alpha < \beta \le a$, then $(\langle p_1^{\alpha}\rangle, 0) \in N[I]\cap C \setminus N[J]\cap C$. If $a \le \alpha < \beta$ or $\alpha < a < \beta$, then $(\langle p_1^{n_1-\beta}\rangle, \mathbb{Z}_{p_2}) \in N[J]\cap C \setminus N[I]\cap C$.

Subcase 3. Suppose $I = (\langle p_1^{\alpha}\rangle, \mathbb{Z}_{p_2})$, $J = (\langle p_1^{\beta}\rangle, \mathbb{Z}_{p_2})$ and $\alpha < \beta$. Then if $\alpha < \beta \le a$, then $(\langle p_1^{\alpha}\rangle, \mathbb{Z}_{p_2}) \in N[I]\cap C \setminus N[J]\cap C$. If $a \le \alpha < \beta$ or $\alpha < a < \beta$, then $(\langle p_1^{n_1-\beta}\rangle, 0) \in N[J]\cap C \setminus N[I]\cap C$.

(3) Let $A_1 = \{(\langle p_1^{i}\rangle, 0) | 1 \le i \le n_1 - 1\}$, $A_2 = \{(0, \langle p_2^{j}\rangle) | 1 \le j \le n_2 - 1\}$, $A_3 = \{(0, \mathbb{Z}_{p_2^{n_2}}), (\mathbb{Z}_{p_1^{n_1}}, 0)\}$ and $C = A_1 \cup A_2 \cup A_3$. If n_1 is odd, then we add the vertex $\{(\langle p_1^{\lfloor n_1/2\rfloor}\rangle, \mathbb{Z}_{p_2^{n_2}})\}$ to C. If n_2 is odd, then we add the vertex $\{(\mathbb{Z}_{p_1^{n_1}}, \langle p_2^{\lfloor n_2/2\rfloor}\rangle)\}$ to C. If n_1 and n_2 are odd, then we add the vertex $\{(\langle p_1^{\lfloor n_1/2\rfloor}\rangle, \langle p_2^{\lfloor n_2/2\rfloor}\rangle)\}$ to C. We claim that C is an identifying code of $AG(\mathbb{Z}_n)$. It is easy to see that $N[I]\cap C \neq \varnothing$, for every vertex I of $AG(\mathbb{Z}_n)$. Consider the following sets:

$$A = \{(\mathbb{Z}_{P_1^{n_1}}, 0), (0, \mathbb{Z}_{p_2^{n_2}})\}$$

and

$$B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly $\{A, B\}$ is a partition of $V(AG(\mathbb{Z}_n))$. Consider the following cases:

Case 1. Let $I, J \in A$. With no loss of generality, assume that $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$ and $J = (0, \mathbb{Z}_{p_2^{n_2}})$. Then $(\langle p_1\rangle, 0) \in N[J]\cap C \setminus N[I]\cap C$.

Case 2. Let $I \in A$ and $J \in B$.

Subcase 1. If $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$ and $J = (\langle p_1^{\alpha}\rangle, J_2)$, where $1 \le \alpha \le n_1 - 1$ and $J_2 \in \{0, \mathbb{Z}_{p_2^{n_2}}, \langle p_2^{\beta}\rangle | 1 \le \beta \le n_2 - 1\}$, then $(\langle p_1^{n_1-1}\rangle, 0) \in N[J]\cap C \setminus N[I]\cap C$.

Subcase 2. If $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$ and $J = (J_1, \langle p_2^{\beta}\rangle)$, where $1 \le \beta \le n_2 - 1$ and $J_1 \in \{0, \mathbb{Z}_{p_1^{n_1}}, \langle p_1^{\alpha}\rangle | 1 \le \alpha \le n_1 - 1\}$, then $(0, \mathbb{Z}_{p_2^{n_2}}) \in N[I]\cap C \setminus N[J]\cap C$.

When $I = (0, \mathbb{Z}_{p_2^{n_2}})$, the proof is similar.

Case 3. Let $I, J \in B$.

Subcase 1. With no loss of generality, assume that $I = (\langle p_1^{\alpha}\rangle, 0)$ and $J = (\langle p_1^{\beta}\rangle, \mathbb{Z}_{p_2^{n_2}})$, where

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$1 \leq \alpha, \beta \leq n_1 - 1$. Then $(0, \mathbb{Z}_{p_2^{n_2}}) \in N[I] \cap C \setminus N[J] \cap C$.

Subcase 2. Suppose $I = (\langle p_1^{\alpha} \rangle, \mathbb{Z}_{p_2^{n_2}})$ and $J = (\langle p_1^{\beta} \rangle, \mathbb{Z}_{p_2^{n_2}})$, where $1 \leq \alpha, \beta \leq n_1 - 1$. Since $I \neq J$, $\alpha \neq \beta$. With no loss of generality, assume than $\alpha < \beta$. Then $(\langle p_1^{n_1 - \beta} \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$.

Subcase 3. Suppose $I = (\langle p_1^{\alpha} \rangle, \langle p_2^{\gamma} \rangle)$ and $J = (\langle p_1^{\beta} \rangle, \langle p_2^{\zeta} \rangle)$, where $1 \leq \alpha, \beta \leq n_1 - 1$ and $1 \leq \gamma, \zeta \leq n_2 - 1$. Then the proof is similar to Subcase 2.

Subcase 4. Suppose $I = (\langle p_1^{\alpha} \rangle, 0)$ and $J = (\langle p_1^{\beta} \rangle, 0)$, where $1 \leq \alpha, \beta \leq n_1 - 1$ and $n_1 - \beta \neq \alpha$. Then the proof is similar to the Subcase 2. Suppose $n_1 - \beta = \alpha$. If n_1 is even, then $(\langle p_1^{n_1/2} \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$. If n_1 is odd and n_2 is even, then $(\langle p_1^{\lfloor n_1/2 \rfloor} \rangle, \mathbb{Z}_{p_2^{n_2}}) \in N[J] \cap C \setminus N[I] \cap C$. If n_1, n_2 are odd, then $(\langle p_1^{\lfloor n_1/2 \rfloor} \rangle, \langle p_2^{\lfloor n_2/2 \rfloor} \rangle) \in N[J] \cap C \setminus N[I] \cap C$.

Remaining cases are similar. □

In the rest of this section, we study $i(AG(\mathbb{Z}_n))$, if $m \geq 3$.

Remark 2.4. *Suppose that $n = p_1 p_2 p_3$. Then it is easy to see that $i(AG(\mathbb{Z}_n)) = 4$.*

Theorem 2.5. *Let $m \geq 3$. Then the following statements hold:*

(1) *Let $n = p_1 \cdots p_m$ and $d \geq 1$ be a positive integer. Then $i(AG(\mathbb{Z}_n)) \leq m + d + 1$, for every $2^d + 1 \leq m \leq 2^{d+1}$.*

(2) *Let $n = p_1^{n_1} \cdots p_k^{n_k} p_{k+1}^{n_{k+1}} \cdots p_m^{n_m}$, where $n_{k+1} = \cdots = n_m = 1$, $d \geq 1$ be a positive integers and $n_i > 1$ for every $1 \leq i \leq k$.*

(i) If $m - k = 1$ and n_i is even, for every i, $1 \leq i \leq k$, then $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 1$. Otherwise, $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 2$.

(ii) If $m - k = 2$, then $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 3$.

(iii) If $m - k = 3$ and n_i is even, for every i, $1 \leq i \leq k$, then $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 4$. Otherwise, $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 5$.

(iv) If $m - k \geq 4$ and n_i is even, for every i, $1 \leq i \leq k$, then $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + m - k + d + 1$, where $2^d + 1 \leq m - k \leq 2^{d+1}$. Otherwise, $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + m - k + d + 2$, where $2^d + 1 \leq m - k - 1 \leq 2^{d+1}$.

(3) *Let $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$. If n_i is even, for every $1 \leq i \leq m$, then $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_m$. Otherwise, $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_m + 1$.*

Proof. (1) Let

$$A = \{v_1 = (\mathbb{Z}_{p_1}, 0, \ldots, 0), \ldots, v_m = (0, \ldots, 0, \mathbb{Z}_{p_m})\}, \quad B = V(AG(\mathbb{Z}_n)) \setminus A,$$

$$A' = \{w_1, \ldots, w_{d+1}\} \subseteq B \quad (d < m),$$

where

$\{w_i\} \subseteq N[v_i]$, for every i, where $1 \leq i \leq \binom{d+1}{1}$.

For every $1 \leq i_1 < i_2 \leq d+1$, $\{w_{i_1}, w_{i_2}\} \subseteq N[v_i]$, for exactly one of v_i, where $\binom{d+1}{1} < i \leq \Sigma_{l=1}^{2}\binom{d+1}{l}$. Similarly, For every $1 \leq i_1 < i_2 < \cdots < i_j \leq d+1$, $\{w_{i_1}, w_{i_2}, \ldots, w_{i_j}\} \subseteq N[v_i]$, for exactly one of v_i, where $\Sigma_{l=1}^{j-1}\binom{d+1}{l} < i \leq \Sigma_{l=1}^{j}\binom{d+1}{l}$. Finally, $\{w_1, \ldots, w_{d+1}\} \subseteq N[v_i]$, for exactly one of v_i, where $\Sigma_{l=1}^{d}\binom{d+1}{l} < i \leq \Sigma_{l=1}^{d+1}\binom{d+1}{l}$.

Let $C = A \cup A'$. We show that C is an identifying code of $V(AG(\mathbb{Z}_n))$. Obviously,

$A \cup \{w_i\} = N[v_i] \cap C$, for every i, where $1 \leq i \leq \binom{d+1}{1}$.

For every i_1, i_2, $1 \leq i_1 < i_2 \leq d+1$, we have $A \cup \{w_{i_1}, w_{i_2}\} = N[v_i] \cap C$, for exactly one of v_i, where $\binom{d+1}{1} < i \leq \Sigma_{l=1}^{2}\binom{d+1}{l}$. Similarly, For every $i_1, i_2, \ldots, i_j$, $1 \leq i_1 < i_2 < \cdots < i_j \leq d+1$, we have $A \cup \{w_{i_1}, w_{i_2}, \ldots, w_{i_j}\} = N[v_i] \cap C$, for exactly one of v_i, where $\Sigma_{l=1}^{j-1}\binom{d+1}{l} < i \leq \Sigma_{l=1}^{j}\binom{d+1}{l}$ and $A \cup \{w_1, \ldots, w_{d+1}\} = N[v_i] \cap C$, for exactly one of v_i, where $\Sigma_{l=1}^{d}\binom{d+1}{l} < i \leq \Sigma_{l=1}^{d+1}\binom{d+1}{l}$.

If $m = \Sigma_{l=1}^{d+1}\binom{d+1}{l} + 1$, then $A = N[v_m] \cap C$.

Suppose $A_1 = \{v_1, \ldots, v_{\binom{d+1}{1}}\}$, $A_2 = \{v_{\binom{d+1}{1}+1}, \ldots, v_{\Sigma_{l=1}^{2}\binom{d+1}{l}}\}$,..., $A_{d+1} = \{v_{\Sigma_{l=1}^{d}\binom{d+1}{l}+1},$ $\ldots, v_{\Sigma_{l=1}^{d+1}\binom{d+1}{l}}\}$. If $m = \Sigma_{l=1}^{d+1}\binom{d+1}{l} + 1$, then $A_{d+2} = \{v_m\}$.

Consider the following cases:

Case 1. Let $I, J \in B$, $I = (I_1, \ldots, I_m)$ and $J = (J_1, \ldots, J_m)$. Then there exists an i, $1 \leq i \leq m$, such that $I_i \neq J_i$. Without loss of generality we may assume that $I_i = 0$ and $J_i = \mathbb{Z}_{p_i}$. Then $(0, \ldots, 0, \mathbb{Z}_{p_i}, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$.

Case 2. Let $I, J \in A$. If $I, J \in A_i$, for some $1 \leq i \leq d+2$, then there exists $w_t \in N[I] \cap C \setminus N[J] \cap C$. If $I \in A_i$ and $J \in A_j$, where $1 \leq i < j \leq d+2$, then $|N[I] \cap C| \neq |N[J] \cap C|$.

Case 3. Let $I \in A$ and $J \in B$. If $I \in A_i$, for some $1 \leq i \leq d+1$, then there exists $w_t \in N[I] \cap C \setminus N[J] \cap C$, where $w_t \in A'$. If $I \in A_{d+2} = \{v_m = (0, \ldots, 0, \mathbb{Z}_{p_m})\}$ and $J \in B$, then there exists an i, $1 \leq i \leq m-1$, such that $I_i \neq J_i = \mathbb{Z}_{p_i}$. So $(0, \ldots, 0, \mathbb{Z}_{p_i}, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$. Thus C is an identifying code of $AG(\mathbb{Z}_n)$.

(2) Let

$C = \{(0, \ldots, \langle p_i^{\alpha}\rangle, 0, \ldots, 0) | \forall 1 \leq i \leq n_k, \quad 1 \leq \alpha \leq n_i - 1\} \cup \{(\mathbb{Z}_{p_1^{n_1}}, 0, \ldots, 0), \ldots, (0, \ldots,$ $0, \mathbb{Z}_{p_m^{n_m}})\}$. If at least one of n_i is odd, $1 \leq i \leq k$, then we have to add the vertex

$$H = \{(I_1, \ldots, I_k, \mathbb{Z}_{p_{k+1}^{n_{k+1}}}, \ldots, \mathbb{Z}_{p_{m-1}^{n_{m-1}}}, 0)\}$$

to C, where every I_j is defined as follows:

$$\begin{cases} \langle p_j^{\lfloor n_j/2 \rfloor}\rangle, \text{if } n_j \text{ is odd}, \\ I_j = \mathbb{Z}_{{p_j}^{n_j}}, \text{if } n_j \text{ is even}. \end{cases}$$

It is not hard to check that $N[I] \cap C \neq \varnothing$, for every vertex I of $V(AG(\mathbb{Z}_n))$. We claim that C is an identifying code of $V(AG(\mathbb{Z}_n))$. Consider the following sets:

$$A = \{(I_1, I_2, \ldots, I_m) | I_i \in \{0, \mathbb{Z}_{p_i^{n_i}}\}, 1 \leq i \leq m\} \subseteq V(AG(\mathbb{Z}_n)) \text{ and } B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly $\{A, B\}$ is a partition of $V(AG(\mathbb{Z}_n))$. Consider the following cases:

Case 1. Let $I, J \in A$, $I = (I_1, \ldots, I_m)$ and $J = (J_1, \ldots, J_m)$. Then there exists an i, $1 \leq i \leq m$, such that $I_i \neq J_i$. Without loss of generality we may assume that $I_i = 0$ and $J_i = \mathbb{Z}_{p_i^{n_i}}$. Then $(0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$. Let n_i is even, for every i, $1 \leq i \leq k$. If $J = (0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0)$ and $i > k+1$, then if $m - k = 2$, we add the vertex $(\mathbb{Z}_{p_1^{n_1}}, \ldots, \mathbb{Z}_{p_{m-1}^{n_{m-1}}}, 0)$ to C. If $m - k = 3$, then by a similar argument in proof of Remark 2.4, the proof is complete. If $m - k \geq 4$, then by a similar argument in proof (1) we have to add $\{w_1, \ldots, w_{d+1}\}$ to C, where $2^d + 1 \leq m - k \leq 2^{d+1}$, and so there exists $w_t \in N[J] \cap C \setminus N[I] \cap C$. Let n_i is odd, for some i, $1 \leq i \leq k$. If $J = (0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0)$ and $i > k+1$, then if $m - k \geq 4$, then by a similar argument in proof (1) we have to add $\{w_1, \ldots, w_{d+1}\}$ to C, where $2^d + 1 \leq m - k - 1 \leq 2^{d+1}$, and so there exists $w_t \in N[J] \cap C \setminus N[I] \cap C$. If $i < k+1$, then $(0, \ldots, 0, \langle p_i \rangle, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$.

Case 2. Let $I = (I_1, \ldots, I_m) \in A$ and $J = (J_1, \ldots, J_m) \in B$. Then there exists an i, $1 \leq i \leq m$, such that $I_i \neq J_i$. Suppose that $J_i = \langle p_i^{\alpha} \rangle$, where $1 \leq \alpha \leq n_i - 1$. If $I_i = 0$, then $(0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$. If $I_i = \mathbb{Z}_{p_i^{n_i}}$, then $(0, \ldots, 0, \langle p_i^{n_i - 1} \rangle, 0, \ldots, 0) \in N[J] \cap C \setminus N[I] \cap C$.

Case 3. Let $I, J \in B$ and suppose that $I = (I_1, \ldots, I_m)$ and $J = (J_1, \ldots, J_m)$. Then there exists an i, $1 \leq i \leq m$, such that $I_i \neq J_i$. Without loss of generality we may assume that

Subcase 1. $I_i = 0$ and $J_i = \mathbb{Z}_{p_i^{n_i}}$. Then $(0, \ldots, 0, \mathbb{Z}_{P_i^{n_i}}, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$.

Subcase 2. $I_i = \langle p_i^{\alpha} \rangle$, where $1 \leq \alpha \leq n_i - 1$. If $J_i = 0$, then $(0, \ldots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \ldots, 0) \in N[J] \cap C \setminus N[I] \cap C$. If $J_i = \mathbb{Z}_{p_i^{n_i}}$, then $(0, \ldots, 0, \langle p_i^{n_i - 1} \rangle, 0, \ldots, 0) \in N[I] \cap C \setminus N[J] \cap C$. If $J_i = \langle p_i^{\beta} \rangle$, where $1 \leq \beta \leq n_i - 1$, then without loss of generality we may assume that $\beta > \alpha$ and hence $(0, \ldots, 0, \langle p_i^{n_i - \beta} \rangle, 0, \ldots, 0) \in N[J] \cap C \setminus N[I] \cap C$. In a special case, let $I_j = J_j = 0$, for every $1 \leq j \neq i \leq m$ and $n_i - \beta = \alpha$. If n_i is even, then $(0, \ldots, 0, \langle p_i^{n_i/2} \rangle, 0, \ldots, 0) \in N[J] \cap C \setminus N[I] \cap C$. If n_i is odd, then $H \in N[J] \cap C \setminus N[I] \cap C$.

Thus the claim is proved and hence C is an identifying code of $AG(\mathbb{Z}_n)$.

(3) It is enough to consider C as follows and repeat the proof in case (2). Let

$$C = \{(0, \ldots, \langle p_i^{\alpha} \rangle, 0, \ldots, 0) | \forall 1 \leq i \leq m, \quad 1 \leq \alpha \leq n_i - 1\} \cup \{(\mathbb{Z}_{p_1^{n_1}}, 0, \ldots, 0), \ldots, (0, \ldots, 0, \mathbb{Z}_{p_m^{n_m}})\}.$$

If at least one of n_i is odd, $1 \leq i \leq m$, then add the vertex $H = \{(I_1, \ldots, I_m)\}$ to C, where every I_j is defined as follows:

$$\begin{cases} \langle p_j^{\lfloor n_j/2 \rfloor} \rangle, \text{if } n_j \text{ is odd,} \\ I_j = \mathbb{Z}_{p_j{}^{n_j}}, \text{if } n_j \text{ is even.} \end{cases}$$

It is not hard to check that $N[I] \cap C \neq \varnothing$ for every vertex I. Consider the following sets:

$$A = \{(I_1, I_2, \ldots, I_m) | I_i \in \{0, \mathbb{Z}_{p_i^{n_i}}\}, \forall 1 \leq i \leq m\}$$

and

$$B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly $\{A, B\}$ is a partition of $V(AG(\mathbb{Z}_n))$.

Thus C is an identifying code of $AG(\mathbb{Z}_n)$. □

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