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## COMMUTATIVE RINGS INTRODUCE A CLASS OF IDENTIFIABLE GRAPHS

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**ABSTRACT.** Let  $R$  be a commutative ring with identity, and  $A(R)$  be the set of ideals with non-zero annihilator. The annihilating-ideal graph of  $R$  is defined as the graph  $AG(R)$  with the vertex set  $A(R)^* = A(R) \setminus \{0\}$  and two distinct vertices  $I$  and  $J$  are adjacent if and only if  $IJ = 0$ . In this paper, we characterize all positive integers  $n$  for which  $AG(\mathbb{Z}_n)$  is identifiable.

### 1. Introduction

The notion of identifying codes was first introduced and studied by Karpovsky, Chakrabarty and Levitin in 1998 ([13]) for diagnosis of faults in multiprocessor systems. Since 1998 the study of identifying codes of graphs has grown a lot. Many researchers have been attracted by this branch, for instance, the identifying codes of trees and planar graphs in [3], graphs of girth at least 5 in [4], chains and cycles in [7], degree 4 Cayley graphs over Abelian groups in [8], graphs differing by one vertex in [9], graphs differing by one edge in [10], graphs that admit upper bound in [11], Cartesian product of a path and a complete graph in [12], nonzero component graph of a finite dimensional vector space in [14] and graphs that admit lower bound in [15] were studied. This paper is devoted to study identifying code of a graph associated with a ring. First let us recall some necessary notation and terminology from graph theory and ring theory.

Let  $G = (V, E)$  be a graph. If  $v \in V(G)$ , then the closed neighborhood of  $v$  in  $G$  is denoted by

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$N_G[v]$ . If  $\{u, v\} \in E$ , then we write  $uv \in E$ . Let  $n$  be a positive integer. By  $P_n$  and  $K_n$ , we mean the path and complete graph of order  $n$ . A subset  $C \subseteq V(G)$  is called *identifying code*, if the sets  $N_G[v] \cap C$ ,  $v \in V(G)$ , are all nonempty and different. If  $G$  admits an identifying code, we say that  $G$  is *identifiable* and denote by  $i(G)$  the minimum cardinality of an identifying code of  $G$ . It is well-known that minimizing the size of an identifying code in a graph is NP-hard. We note that  $G$  is identifiable if and only if the sets  $N_G[v]$  are distinct for all  $v \in V(G)$ . For any undefined notation or terminology in graph theory, we refer the reader to [6].

Throughout this paper, all rings are assumed to be commutative with identity. The set of all ideals of a ring  $R$  is denoted by  $I(R)$ . For a subset  $T$  of a ring  $R$  we let  $T^* = T \setminus \{0\}$ . Furthermore, if  $I$  is an ideal of  $R$ , by  $Ann(I)$ , we mean the annihilator of  $I$ . An ideal  $I$  of  $R$  is called an *annihilating-ideal* if there exists a non-zero ideal  $J$  such that  $IJ = 0$ . The notation  $A(R)$  is used to denote the set of all annihilating-ideals of  $R$ . For any undefined notation or terminology in ring theory, we refer the reader to [2].

Let  $R$  be a ring. The *annihilating-ideal graph* of  $R$ , denoted by  $AG(R)$ , is a graph with the vertex set  $A(R)^*$  and two distinct vertices  $I$  and  $J$  are adjacent if and only if  $IJ = 0$ . The annihilating-ideal graph was first introduced in [5], and some of the properties of the annihilating-ideal graph have been studied. Further results on annihilating-ideal graphs may be found in [1, 16] and [18]. This paper is devoted to study the identifying code of  $AG(\mathbb{Z}_n)$ .

The following remark will be used frequently in this paper.

**Remark 1.1.** Consider  $\mathbb{Z}_n$ , the ring of integers modulo  $n$ . Throughout the paper, without loss of generality, we assume that  $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$ , where  $p_i$ 's are distinct primes and  $n_i$ 's are natural numbers. Then

(i)  $\mathbb{Z}_n$  is Artinian. Thus it follows from [5, Proposition 1.3], that every non-trivial ideal of  $\mathbb{Z}_n$  is a vertex of  $AG(\mathbb{Z}_n)$ .

(ii) It follows from Chinese Remainder Theorem that

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_m^{n_m}}.$$

(iii)  $I \in I(\mathbb{Z}_n)$  if and only if  $I = I_1 \times \cdots \times I_m$ , where  $I_i \in I(\mathbb{Z}_{p_i^{n_i}})$ .

(iv) It is not hard to see that  $|A(\mathbb{Z}_n)^*| = \prod_{i=1}^m (n_i + 1) - 2$ .

## 2. Identifying code of $AG(\mathbb{Z}_n)$

First we study identifiable  $AG(\mathbb{Z}_n)$ .

**Theorem 2.1.** *The graph  $AG(\mathbb{Z}_n)$  is not identifiable if and only if  $n$  is one of the following:*

(i)  $n = p_1^{n_1}$ , where  $n_1$  is odd.

(ii)  $n = p_1^2$ .

(iii)  $n = p_1 p_2$ .

*Proof.* We consider the following cases:

**Case 1.**  $m = 1$ .

If  $n_1$  is odd, then it is not hard to check that  $N[\langle p_1^{\lfloor n_1/2 \rfloor} \rangle] = N[\langle p_1^{\lfloor n_1/2 \rfloor + 1} \rangle]$ . Hence  $AG(\mathbb{Z}_n)$  is not identifiable. Assume that  $n_1$  is even. If  $n_1 = 2$ , then  $AG(\mathbb{Z}_n) = K_1$  and thus it is not identifiable. We show that  $AG(\mathbb{Z}_n)$  is identifiable, if  $n_1 \neq 2$ . Suppose to the contrary, there exist  $\alpha$  and  $\beta$  such that  $N[\langle p_1^\alpha \rangle] = N[\langle p_1^\beta \rangle]$ , where  $1 \leq \alpha, \beta \leq n_1 - 1$ . With no loss of generality, assume that  $\beta > \alpha$ . If  $n_1 - \beta \neq \alpha$ , then  $\langle p_1^{n_1 - \beta} \rangle \in N[\langle p_1^\beta \rangle] \setminus N[\langle p_1^\alpha \rangle]$ , which is impossible. If  $n_1 - \beta = \alpha$ , then  $\langle p_1^{n_1/2} \rangle \in N[\langle p_1^\beta \rangle] \setminus N[\langle p_1^\alpha \rangle]$ , a contradiction.

**Case 2.**  $m = 2$ .

if  $n_1 = n_2 = 1$ , then  $AG(\mathbb{Z}_n) = K_2$ , which is not identifiable. Let  $n_1 \geq 2$  and there exist two vertices  $I$  and  $J$  such that  $N[I] = N[J]$ . We try to find a contradiction. Consider the following subcases:

**Subcase 1.** Let  $I = (\mathbb{Z}_{p_1}, 0)$ .

Then  $J = (0, \mathbb{Z}_{p_2})$  or  $J = (0, \langle p_2^\beta \rangle)$ , where  $1 \leq \beta \leq n_2 - 1$  (Note that if  $n_2 = 1$ , the latter case does not occur). If  $J = (0, \mathbb{Z}_{p_2})$ , then  $(\langle p_1^\alpha \rangle, 0) \in N[J] \setminus N[I]$ . If  $J = (0, \langle p_2^\beta \rangle)$ , then  $(0, \mathbb{Z}_{p_2}) \in N[I] \setminus N[J]$ , a contradiction.

**Subcase 2.** Let  $I = (\mathbb{Z}_{p_1}, \langle p_2^\beta \rangle)$ , where  $1 \leq \beta \leq n_2 - 1$ .

Then  $J = (0, \langle p_2^\gamma \rangle)$ , where  $\beta + \gamma \geq n_2$ . Hence  $(\mathbb{Z}_{p_1}, 0) \in N[J] \setminus N[I]$ .

**Subcase 3.** Let  $I = (\langle p_1^\alpha \rangle, \langle p_2^\beta \rangle)$ , where  $1 \leq \alpha, \beta$ .

Then  $J \in \{(\langle p_1^\gamma \rangle, \langle p_2^\zeta \rangle), (0, \langle p_2^\zeta \rangle), (\langle p_1^\gamma \rangle, 0) \mid 1 \leq \gamma \leq n_1 - 1, 1 \leq \zeta \leq n_2 - 1, \alpha + \gamma \geq n_1, \beta + \zeta \geq n_2\}$ .

If  $J = (0, \langle p_2^\zeta \rangle)$  or  $J = (\langle p_1^\gamma \rangle, 0)$ , then obviously  $N[I] \neq N[J]$ . Let  $J = (\langle p_1^\gamma \rangle, \langle p_2^\zeta \rangle)$ . Then  $\alpha \neq \gamma$  or  $\beta \neq \zeta$ . Suppose that  $\alpha \neq \gamma$  and without loss of generality we may assume that  $\alpha > \gamma$ . Thus  $(\langle p_1^{n_1 - \alpha} \rangle, 0) \in N[I] \setminus N[J]$ . Suppose  $\langle p_1^\alpha \rangle = 0$  or  $\langle p_2^\beta \rangle = 0$ . Without loss of generality we may assume that  $\langle p_1^\alpha \rangle = 0$ , then  $J \in \{(\langle p_1^\gamma \rangle, \langle p_2^\zeta \rangle), (0, \langle p_2^\zeta \rangle), (\langle p_1^\gamma \rangle, 0), (\mathbb{Z}_{p_1}^{n_1}, 0), (\mathbb{Z}_{p_1}^{n_1}, \langle p_2^\zeta \rangle) \mid 1 \leq \gamma \leq n_1 - 1, 1 \leq \zeta \leq n_2 - 1, \beta + \zeta \geq n_2\}$  which is similar to the above argument.

**Case 3.**  $m \geq 3$ .

Let  $I = (I_1, \dots, I_i, \dots, I_m)$  be a vertex of  $AG(\mathbb{Z}_n)$ , where  $I_i \in \{0, \mathbb{Z}_{p_i}^{n_i}, \langle p_i^\alpha \rangle \mid 1 \leq \alpha \leq n_i - 1\}$  and  $N[I] = N[J]$ , for some  $J = (J_1, \dots, J_m) \in V(AG(\mathbb{Z}_n))$ . Consider the following subcases:

**Subcase 1.** Let  $I_i = 0$ .

Since  $N[I] = N[J]$  and  $I \neq J$ ,  $J_i = 0$  and there exist  $I_j$  and  $J_j$  such that  $I_j \neq J_j$ . If  $I_j = 0$ , then  $J_j = \langle p_j^\alpha \rangle$  or  $J_j = \mathbb{Z}_{p_j}^{n_j}$ . Thus it is easy to see that  $N[I] \neq N[J]$ . If  $I_j = \langle p_j^\alpha \rangle$ , then  $J_j = \langle p_j^\beta \rangle$ , where  $\alpha \neq \beta$ . By Case 1, if  $n_j$  is even then clearly  $N[I] \neq N[J]$ . If  $n_j$  is odd and  $\beta > \alpha$ , then  $(0, \dots, \mathbb{Z}_{p_i}^{n_i}, 0, \dots, \langle p_j^{\lfloor n_j/2 \rfloor} \rangle, 0, \dots, 0) \in N[J] \setminus N[I]$ .

**Subcase 2.** Let  $I_i = \mathbb{Z}_{p_i}^{n_i}$ .

Then  $J_i = 0$ . Thus  $(0, \dots, 0, \mathbb{Z}_{p_i}^{n_i}, 0, \dots, 0) \in N[J] \setminus N[I]$ . Also, if  $I = (0, \dots, 0, \mathbb{Z}_{p_i}^{n_i}, 0, \dots, 0)$ , then  $(\mathbb{Z}_{p_1}^{n_1}, \mathbb{Z}_{p_2}^{n_2}, \dots, \mathbb{Z}_{p_{i-1}}^{n_{i-1}}, 0, \mathbb{Z}_{p_{i+1}}^{n_{i+1}}, \dots, \mathbb{Z}_{p_m}^{n_m}) \in N[I] \setminus N[J]$ .

**Subcase 3.** Let  $I_i = \langle p_i^\alpha \rangle$ , where  $1 \leq \alpha \leq n_i - 1$ .

If  $J_i = \mathbb{Z}_{p_i}^{n_i}$  or  $J_i = \langle p_i^\beta \rangle$ , where  $\alpha + \beta \leq n$ , then  $J \notin N[I]$ . Hence  $J_i = \langle p_i^\beta \rangle$ , where  $\alpha + \beta \geq n$  or  $J_i = 0$ . If  $\alpha \neq \beta$ , then  $N[I] \neq N[J]$ . If  $\alpha = \beta$ , then there exist  $I_j$  and  $J_j$  such that  $I_j \neq J_j$ . It is not hard to check that  $N[I] \neq N[J]$ . Suppose that  $J_i = 0$ . Therefore  $(0, \dots, 0, \mathbb{Z}_{p_i}^{n_i}, 0, \dots, 0) \in N[J] \setminus N[I]$ , a contradiction.  $\square$

Next we study  $i(AG(\mathbb{Z}_n))$ , if  $m < 3$ . First we need the following remark.

**Remark 2.2.** Consider the graph  $AG(\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2})$ . It is easy to see that  $AG(\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2}) = P_4$  and hence  $i(AG(\mathbb{Z}_{p_1^2} \times \mathbb{Z}_{p_2})) = 3$ . Moreover, one may easily see that  $i(AG(\mathbb{Z}_{p_1^3} \times \mathbb{Z}_{p_2})) = 4$ . Also,  $i(AG(\mathbb{Z}_{p_1^3} \times \mathbb{Z}_{p_2^2})) = 5$  and  $i(AG(\mathbb{Z}_{p_1^3} \times \mathbb{Z}_{p_2^3})) = 6$ .

**Theorem 2.3.** Let  $m < 3$ . Then the following statements hold:

- (1) If  $n = p_1^{n_1}$  and  $n_1 \neq 2$  is even, then  $i(AG(\mathbb{Z}_n)) = n_1/2$ .
- (2) If  $n = p_1^{n_1} p_2$  where  $n_1 \geq 4$ , then  $i(AG(\mathbb{Z}_n)) \leq n_1$ .
- (3) Let  $n = p_1^{n_1} p_2^{n_2}$ , where  $2 \leq n_1, n_2$ . If  $n_1, n_2$  are even, then  $i(AG(\mathbb{Z}_n)) \leq n_1 + n_2$ . Otherwise,  $i(AG(\mathbb{Z}_n)) \leq n_1 + n_2 + 1$ .

*Proof.* (1) Let  $C = \{\langle p_1^\alpha \rangle \mid 1 \leq \alpha \leq n_1/2\}$ . We claim that  $C$  is an identifying code of  $AG(\mathbb{Z}_n)$ . If  $I, J \in C$ , then  $I = N[I] \cap C \neq N[J] \cap C = J$ . Suppose that  $I, J \notin C$ . Then there exist  $(n_1/2) + 1 \leq \alpha, \beta \leq n_1 - 1$  such that  $I = \langle p_1^\alpha \rangle$  and  $J = \langle p_1^\beta \rangle$ . Without loss of generality, assume that  $\beta > \alpha$ . Thus  $\langle p_1^{n_1-\beta} \rangle \in N[J] \cap C \setminus N[I] \cap C$ . If  $I \in C$  and  $J \notin C$ , then  $|N[I] \cap C| = 1$  and  $|N[J] \cap C| \geq 2$  and so the claim is proved. Next, we prove that  $C$  is a minimum identifying code of  $AG(\mathbb{Z}_n)$ . We show that every vertex in  $C$  is needed to  $C$  be an identifying code. Let  $C'$  be an arbitrary identifying code of  $AG(\mathbb{Z}_n)$ . One can check that, if  $\langle p_1^k \rangle \notin C'$ , for some  $1 \leq k \leq (n_1/2) - 1$ , then  $N[\langle p_1^{n_1-k} \rangle] \cap C' = N[\langle p_1^{n_1-(k+1)} \rangle] \cap C'$ . If  $\langle p_1^{n_1/2} \rangle \notin C'$ , then there exists  $(n_1/2) + 1 \leq l \leq n_1 - 1$  such that  $\langle p_1^l \rangle \in C'$ . Thus  $N[\langle p_1^{(n_1/2)+1} \rangle] \cap C' = N[\langle p_1^{(n_1/2)-1} \rangle] \cap C'$ , as desired.

- (2) Let  $a = \lfloor \frac{n_1}{2} \rfloor$ . Suppose that

$$C = \{(\langle p_1^\alpha \rangle, 0) \mid 1 \leq \alpha \leq a\} \cup \{(\langle p_1^\alpha \rangle, \mathbb{Z}_{p_2}) \mid 1 \leq \alpha \leq a - 1\} \cup \{(0, \mathbb{Z}_{p_2})\}.$$

If  $n_1$  is odd, then we add the vertex  $\{(\langle p_1^a \rangle, \mathbb{Z}_{p_2})\}$  to  $C$ . We claim that  $C$  is an identifying code of  $AG(\mathbb{Z}_n)$ . It is easy to see that  $N[I] \cap C \neq \emptyset$ , for every vertex  $I$  of  $AG(\mathbb{Z}_n)$ . Consider the following sets:

$$A = \{(\mathbb{Z}_{p_1^{n_1}}, 0), (0, \mathbb{Z}_{p_2})\}$$

and

$$B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly  $\{A, B\}$  is partition of  $V(AG(\mathbb{Z}_n))$ . Consider the following cases:

**Case 1.** Let  $I, J \in A$ . With no loss of generality, assume that  $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$  and  $J = (0, \mathbb{Z}_{p_2})$ . Then

$(\langle p_1 \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

**Case 2.** Let  $I \in A$  and  $J \in B$ .

If  $I = (0, \mathbb{Z}_{p_2})$ , then  $(\mathbb{Z}_{p_2}, 0) \in N[I] \cap C \setminus N[J] \cap C$ .

Let  $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$ . Then  $J \in \{(\langle p_1^\alpha \rangle, \mathbb{Z}_{p_2}), (\langle P_1^\beta \rangle, 0), (\langle p_1^\gamma \rangle, 0)\}$ , where  $1 \leq \alpha \leq n_1 - 1$ ,  $1 \leq \beta \leq a$  and  $a < \gamma \leq n_1 - 1$ .

If  $J = (\langle p_1^\alpha \rangle, \mathbb{Z}_{p_2})$ , then  $(0, \mathbb{Z}_{p_2}) \in N[I] \cap C \setminus N[J] \cap C$ .

If  $J = (\langle p_1^\beta \rangle, 0)$ , then  $(\langle p_1^\beta \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

If  $J = (\langle p_1^\gamma \rangle, 0)$ , then  $(\langle p_1^\alpha \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

**Case 3.** Let  $I, J \in B$ .

**Subcase 1.** With no loss of generality, assume that  $I = (\langle p_1^\alpha \rangle, 0)$ ,  $J = (\langle p_1^\beta \rangle, \mathbb{Z}_{p_2})$  and  $\alpha < \beta$ . Then  $(0, \mathbb{Z}_{p_2}) \in N[I] \cap C \setminus N[J] \cap C$ .

**Subcase 2.** Suppose  $I = (\langle p_1^\alpha \rangle, 0)$ ,  $J = (\langle p_1^\beta \rangle, 0)$  and  $\alpha < \beta$ . If  $\alpha < \beta \leq a$ , then  $(\langle p_1^\alpha \rangle, 0) \in N[I] \cap C \setminus N[J] \cap C$ . If  $a \leq \alpha < \beta$  or  $\alpha < a < \beta$ , then  $(\langle p_1^{n_1-\beta} \rangle, \mathbb{Z}_{p_2}) \in N[J] \cap C \setminus N[I] \cap C$ .

**Subcase 3.** Suppose  $I = (\langle p_1^\alpha \rangle, \mathbb{Z}_{p_2})$ ,  $J = (\langle p_1^\beta \rangle, \mathbb{Z}_{p_2})$  and  $\alpha < \beta$ . Then if  $\alpha < \beta \leq a$ , then  $(\langle p_1^\alpha \rangle, \mathbb{Z}_{p_2}) \in N[I] \cap C \setminus N[J] \cap C$ . If  $a \leq \alpha < \beta$  or  $\alpha < a < \beta$ , then  $(\langle p_1^{n_1-\beta} \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

(3) Let  $A_1 = \{(\langle p_1^i \rangle, 0) | 1 \leq i \leq n_1 - 1\}$ ,  $A_2 = \{(0, \langle p_2^j \rangle) | 1 \leq j \leq n_2 - 1\}$ ,  $A_3 = \{(0, \mathbb{Z}_{p_2^{n_2}}), (\mathbb{Z}_{p_1^{n_1}}, 0)\}$  and  $C = A_1 \cup A_2 \cup A_3$ . If  $n_1$  is odd, then we add the vertex  $\{(\langle p_1^{\lfloor n_1/2 \rfloor} \rangle, \mathbb{Z}_{p_2^{n_2}})\}$  to  $C$ . If  $n_2$  is odd, then we add the vertex  $\{(\mathbb{Z}_{p_1^{n_1}}, \langle p_2^{\lfloor n_2/2 \rfloor} \rangle)\}$  to  $C$ . If  $n_1$  and  $n_2$  are odd, then we add the vertex  $\{(\langle p_1^{\lfloor n_1/2 \rfloor} \rangle, \langle p_2^{\lfloor n_2/2 \rfloor} \rangle)\}$  to  $C$ . We claim that  $C$  is an identifying code of  $AG(\mathbb{Z}_n)$ . It is easy to see that  $N[I] \cap C \neq \emptyset$ , for every vertex  $I$  of  $AG(\mathbb{Z}_n)$ . Consider the following sets:

$$A = \{(\mathbb{Z}_{p_1^{n_1}}, 0), (0, \mathbb{Z}_{p_2^{n_2}})\}$$

and

$$B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly  $\{A, B\}$  is a partition of  $V(AG(\mathbb{Z}_n))$ . Consider the following cases:

**Case 1.** Let  $I, J \in A$ . With no loss of generality, assume that  $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$  and  $J = (0, \mathbb{Z}_{p_2^{n_2}})$ . Then  $(\langle p_1 \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

**Case 2.** Let  $I \in A$  and  $J \in B$ .

**Subcase 1.** If  $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$  and  $J = (\langle p_1^\alpha \rangle, J_2)$ , where  $1 \leq \alpha \leq n_1 - 1$  and  $J_2 \in \{0, \mathbb{Z}_{p_2^{n_2}}, \langle p_2^\beta \rangle | 1 \leq \beta \leq n_2 - 1\}$ , then  $(\langle p_1^{n_1-1} \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

**Subcase 2.** If  $I = (\mathbb{Z}_{p_1^{n_1}}, 0)$  and  $J = (J_1, \langle p_2^\beta \rangle)$ , where  $1 \leq \beta \leq n_2 - 1$  and  $J_1 \in \{0, \mathbb{Z}_{p_1^{n_1}}, \langle p_1^\alpha \rangle | 1 \leq \alpha \leq n_1 - 1\}$ , then  $(0, \mathbb{Z}_{p_2^{n_2}}) \in N[I] \cap C \setminus N[J] \cap C$ .

When  $I = (0, \mathbb{Z}_{p_2^{n_2}})$ , the proof is similar.

**Case 3.** Let  $I, J \in B$ .

**Subcase 1.** With no loss of generality, assume that  $I = (\langle p_1^\alpha \rangle, 0)$  and  $J = (\langle p_1^\beta \rangle, \mathbb{Z}_{p_2^{n_2}})$ , where

$1 \leq \alpha, \beta \leq n_1 - 1$ . Then  $(0, \mathbb{Z}_{p_2^{n_2}}) \in N[I] \cap C \setminus N[J] \cap C$ .

**Subcase 2.** Suppose  $I = (\langle p_1^\alpha \rangle, \mathbb{Z}_{p_2^{n_2}})$  and  $J = (\langle p_1^\beta \rangle, \mathbb{Z}_{p_2^{n_2}})$ , where  $1 \leq \alpha, \beta \leq n_1 - 1$ . Since  $I \neq J$ ,  $\alpha \neq \beta$ . With no loss of generality, assume that  $\alpha < \beta$ . Then  $(\langle p_1^{n_1 - \beta} \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

**Subcase 3.** Suppose  $I = (\langle p_1^\alpha \rangle, \langle p_2^\gamma \rangle)$  and  $J = (\langle p_1^\beta \rangle, \langle p_2^\zeta \rangle)$ , where  $1 \leq \alpha, \beta \leq n_1 - 1$  and  $1 \leq \gamma, \zeta \leq n_2 - 1$ . Then the proof is similar to Subcase 2.

**Subcase 4.** Suppose  $I = (\langle p_1^\alpha \rangle, 0)$  and  $J = (\langle p_1^\beta \rangle, 0)$ , where  $1 \leq \alpha, \beta \leq n_1 - 1$  and  $n_1 - \beta \neq \alpha$ . Then the proof is similar to the Subcase 2. Suppose  $n_1 - \beta = \alpha$ . If  $n_1$  is even, then  $(\langle p_1^{n_1/2} \rangle, 0) \in N[J] \cap C \setminus N[I] \cap C$ . If  $n_1$  is odd and  $n_2$  is even, then  $(\langle p_1^{\lfloor n_1/2 \rfloor} \rangle, \mathbb{Z}_{p_2^{n_2}}) \in N[J] \cap C \setminus N[I] \cap C$ . If  $n_1, n_2$  are odd, then  $(\langle p_1^{\lfloor n_1/2 \rfloor} \rangle, \langle p_2^{\lfloor n_2/2 \rfloor} \rangle) \in N[J] \cap C \setminus N[I] \cap C$ .

Remaining cases are similar. □

In the rest of this section, we study  $i(AG(\mathbb{Z}_n))$ , if  $m \geq 3$ .

**Remark 2.4.** Suppose that  $n = p_1 p_2 p_3$ . Then it is easy to see that  $i(AG(\mathbb{Z}_n)) = 4$ .

**Theorem 2.5.** Let  $m \geq 3$ . Then the following statements hold:

(1) Let  $n = p_1 \cdots p_m$  and  $d \geq 1$  be a positive integer. Then  $i(AG(\mathbb{Z}_n)) \leq m + d + 1$ , for every  $2^d + 1 \leq m \leq 2^{d+1}$ .

(2) Let  $n = p_1^{n_1} \cdots p_k^{n_k} p_{k+1}^{n_{k+1}} \cdots p_m^{n_m}$ , where  $n_{k+1} = \cdots = n_m = 1$ ,  $d \geq 1$  be a positive integers and  $n_i > 1$  for every  $1 \leq i \leq k$ .

(i) If  $m - k = 1$  and  $n_i$  is even, for every  $i, 1 \leq i \leq k$ , then  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 1$ . Otherwise,  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 2$ .

(ii) If  $m - k = 2$ , then  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 3$ .

(iii) If  $m - k = 3$  and  $n_i$  is even, for every  $i, 1 \leq i \leq k$ , then  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 4$ . Otherwise,  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + 5$ .

(iv) If  $m - k \geq 4$  and  $n_i$  is even, for every  $i, 1 \leq i \leq k$ , then  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + m - k + d + 1$ , where  $2^d + 1 \leq m - k \leq 2^{d+1}$ . Otherwise,  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_k + m - k + d + 2$ , where  $2^d + 1 \leq m - k - 1 \leq 2^{d+1}$ .

(3) Let  $n = p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$ . If  $n_i$  is even, for every  $1 \leq i \leq m$ , then  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_m$ . Otherwise,  $i(AG(\mathbb{Z}_n)) \leq n_1 + \cdots + n_m + 1$ .

*Proof.* (1) Let

$$A = \{v_1 = (\mathbb{Z}_{p_1}, 0, \dots, 0), \dots, v_m = (0, \dots, 0, \mathbb{Z}_{p_m})\}, \quad B = V(AG(\mathbb{Z}_n)) \setminus A,$$

$$A' = \{w_1, \dots, w_{d+1}\} \subseteq B \quad (d < m),$$

where

$\{w_i\} \subseteq N[v_i]$ , for every  $i$ , where  $1 \leq i \leq \binom{d+1}{1}$ .

For every  $1 \leq i_1 < i_2 \leq d + 1$ ,  $\{w_{i_1}, w_{i_2}\} \subseteq N[v_i]$ , for exactly one of  $v_i$ , where  $\binom{d+1}{1} < i \leq \sum_{l=1}^2 \binom{d+1}{l}$ .

Similarly, For every  $1 \leq i_1 < i_2 < \dots < i_j \leq d + 1$ ,  $\{w_{i_1}, w_{i_2}, \dots, w_{i_j}\} \subseteq N[v_i]$ , for exactly one of  $v_i$ , where  $\sum_{l=1}^{j-1} \binom{d+1}{l} < i \leq \sum_{l=1}^j \binom{d+1}{l}$ . Finally,  $\{w_1, \dots, w_{d+1}\} \subseteq N[v_i]$ , for exactly one of  $v_i$ , where  $\sum_{l=1}^d \binom{d+1}{l} < i \leq \sum_{l=1}^{d+1} \binom{d+1}{l}$ .

Let  $C = A \cup A'$ . We show that  $C$  is an identifying code of  $V(AG(\mathbb{Z}_n))$ . Obviously,

$A \cup \{w_i\} = N[v_i] \cap C$ , for every  $i$ , where  $1 \leq i \leq \binom{d+1}{1}$ .

For every  $i_1, i_2$ ,  $1 \leq i_1 < i_2 \leq d + 1$ , we have  $A \cup \{w_{i_1}, w_{i_2}\} = N[v_i] \cap C$ , for exactly one of  $v_i$ , where  $\binom{d+1}{1} < i \leq \sum_{l=1}^2 \binom{d+1}{l}$ . Similarly, For every  $i_1, i_2, \dots, i_j$ ,  $1 \leq i_1 < i_2 < \dots < i_j \leq d + 1$ , we

have  $A \cup \{w_{i_1}, w_{i_2}, \dots, w_{i_j}\} = N[v_i] \cap C$ , for exactly one of  $v_i$ , where  $\sum_{l=1}^{j-1} \binom{d+1}{l} < i \leq \sum_{l=1}^j \binom{d+1}{l}$  and

$A \cup \{w_1, \dots, w_{d+1}\} = N[v_i] \cap C$ , for exactly one of  $v_i$ , where  $\sum_{l=1}^d \binom{d+1}{l} < i \leq \sum_{l=1}^{d+1} \binom{d+1}{l}$ .

If  $m = \sum_{l=1}^{d+1} \binom{d+1}{l} + 1$ , then  $A = N[v_m] \cap C$ .

Suppose  $A_1 = \{v_1, \dots, v_{\binom{d+1}{1}}\}$ ,  $A_2 = \{v_{\binom{d+1}{1}+1}, \dots, v_{\sum_{l=1}^2 \binom{d+1}{l}}\}, \dots, A_{d+1} = \{v_{\sum_{l=1}^d \binom{d+1}{l}+1},$

$\dots, v_{\sum_{l=1}^{d+1} \binom{d+1}{l}}\}$ . If  $m = \sum_{l=1}^{d+1} \binom{d+1}{l} + 1$ , then  $A_{d+2} = \{v_m\}$ .

Consider the following cases:

**Case 1.** Let  $I, J \in B$ ,  $I = (I_1, \dots, I_m)$  and  $J = (J_1, \dots, J_m)$ . Then there exists an  $i$ ,  $1 \leq i \leq m$ , such that  $I_i \neq J_i$ . Without loss of generality we may assume that  $I_i = 0$  and  $J_i = \mathbb{Z}_{p_i}$ . Then  $(0, \dots, 0, \mathbb{Z}_{p_i}, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ .

**Case 2.** Let  $I, J \in A$ . If  $I, J \in A_i$ , for some  $1 \leq i \leq d + 2$ , then there exists  $w_t \in N[I] \cap C \setminus N[J] \cap C$ .

If  $I \in A_i$  and  $J \in A_j$ , where  $1 \leq i < j \leq d + 2$ , then  $|N[I] \cap C| \neq |N[J] \cap C|$ .

**Case 3.** Let  $I \in A$  and  $J \in B$ . If  $I \in A_i$ , for some  $1 \leq i \leq d + 1$ , then there exists  $w_t \in N[I] \cap C \setminus N[J] \cap C$ , where  $w_t \in A'$ . If  $I \in A_{d+2} = \{v_m = (0, \dots, 0, \mathbb{Z}_{p_m})\}$  and  $J \in B$ , then there exists an  $i$ ,  $1 \leq i \leq m - 1$ , such that  $I_i \neq J_i = \mathbb{Z}_{p_i}$ . So  $(0, \dots, 0, \mathbb{Z}_{p_i}, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ .

Thus  $C$  is an identifying code of  $AG(\mathbb{Z}_n)$ .

(2) Let

$C = \{(0, \dots, \langle p_i^\alpha \rangle, 0, \dots, 0) | \forall 1 \leq i \leq n_k, 1 \leq \alpha \leq n_i - 1\} \cup \{(\mathbb{Z}_{p_1}^{n_1}, 0, \dots, 0), \dots, (0, \dots, 0, \mathbb{Z}_{p_m}^{n_m})\}$ . If at least one of  $n_i$  is odd,  $1 \leq i \leq k$ , then we have to add the vertex

$$H = \{(I_1, \dots, I_k, \mathbb{Z}_{p_{k+1}}^{n_{k+1}}, \dots, \mathbb{Z}_{p_{m-1}}^{n_{m-1}}, 0)\}$$

to  $C$ , where every  $I_j$  is defined as follows:

$$\begin{cases} \langle p_j^{\lfloor n_j/2 \rfloor} \rangle, & \text{if } n_j \text{ is odd,} \\ I_j = \mathbb{Z}_{p_j}^{n_j}, & \text{if } n_j \text{ is even.} \end{cases}$$

It is not hard to check that  $N[I] \cap C \neq \emptyset$ , for every vertex  $I$  of  $V(AG(\mathbb{Z}_n))$ . We claim that  $C$  is an identifying code of  $V(AG(\mathbb{Z}_n))$ . Consider the following sets:

$$A = \{(I_1, I_2, \dots, I_m) \mid I_i \in \{0, \mathbb{Z}_{p_i^{n_i}}\}, 1 \leq i \leq m\} \subseteq V(AG(\mathbb{Z}_n)) \text{ and } B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly  $\{A, B\}$  is a partition of  $V(AG(\mathbb{Z}_n))$ . Consider the following cases:

**Case 1.** Let  $I, J \in A$ ,  $I = (I_1, \dots, I_m)$  and  $J = (J_1, \dots, J_m)$ . Then there exists an  $i$ ,  $1 \leq i \leq m$ , such that  $I_i \neq J_i$ . Without loss of generality we may assume that  $I_i = 0$  and  $J_i = \mathbb{Z}_{p_i^{n_i}}$ . Then  $(0, \dots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ . Let  $n_i$  is even, for every  $i$ ,  $1 \leq i \leq k$ . If  $J = (0, \dots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \dots, 0)$  and  $i > k + 1$ , then if  $m - k = 2$ , we add the vertex  $(\mathbb{Z}_{p_1^{n_1}}, \dots, \mathbb{Z}_{p_{m-1}^{n_{m-1}}}, 0)$  to  $C$ . If  $m - k = 3$ , then by a similar argument in proof of Remark 2.4, the proof is complete. If  $m - k \geq 4$ , then by a similar argument in proof (1) we have to add  $\{w_1, \dots, w_{d+1}\}$  to  $C$ , where  $2^d + 1 \leq m - k \leq 2^{d+1}$ , and so there exists  $w_t \in N[J] \cap C \setminus N[I] \cap C$ . Let  $n_i$  is odd, for some  $i$ ,  $1 \leq i \leq k$ . If  $J = (0, \dots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \dots, 0)$  and  $i > k + 1$ , then if  $m - k \geq 4$ , then by a similar argument in proof (1) we have to add  $\{w_1, \dots, w_{d+1}\}$  to  $C$ , where  $2^d + 1 \leq m - k - 1 \leq 2^{d+1}$ , and so there exists  $w_t \in N[J] \cap C \setminus N[I] \cap C$ . If  $i < k + 1$ , then  $(0, \dots, 0, \langle p_i \rangle, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ .

**Case 2.** Let  $I = (I_1, \dots, I_m) \in A$  and  $J = (J_1, \dots, J_m) \in B$ . Then there exists an  $i$ ,  $1 \leq i \leq m$ , such that  $I_i \neq J_i$ . Suppose that  $J_i = \langle p_i^\alpha \rangle$ , where  $1 \leq \alpha \leq n_i - 1$ . If  $I_i = 0$ , then  $(0, \dots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ . If  $I_i = \mathbb{Z}_{p_i^{n_i}}$ , then  $(0, \dots, 0, \langle p_i^{n_i-1} \rangle, 0, \dots, 0) \in N[J] \cap C \setminus N[I] \cap C$ .

**Case 3.** Let  $I, J \in B$  and suppose that  $I = (I_1, \dots, I_m)$  and  $J = (J_1, \dots, J_m)$ . Then there exists an  $i$ ,  $1 \leq i \leq m$ , such that  $I_i \neq J_i$ . Without loss of generality we may assume that

**Subcase 1.**  $I_i = 0$  and  $J_i = \mathbb{Z}_{p_i^{n_i}}$ . Then  $(0, \dots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ .

**Subcase 2.**  $I_i = \langle p_i^\alpha \rangle$ , where  $1 \leq \alpha \leq n_i - 1$ . If  $J_i = 0$ , then  $(0, \dots, 0, \mathbb{Z}_{p_i^{n_i}}, 0, \dots, 0) \in N[J] \cap C \setminus N[I] \cap C$ . If  $J_i = \mathbb{Z}_{p_i^{n_i}}$ , then  $(0, \dots, 0, \langle p_i^{n_i-1} \rangle, 0, \dots, 0) \in N[I] \cap C \setminus N[J] \cap C$ . If  $J_i = \langle p_i^\beta \rangle$ , where  $1 \leq \beta \leq n_i - 1$ , then without loss of generality we may assume that  $\beta > \alpha$  and hence  $(0, \dots, 0, \langle p_i^{n_i-\beta} \rangle, 0, \dots, 0) \in N[J] \cap C \setminus N[I] \cap C$ . In a special case, let  $I_j = J_j = 0$ , for every  $1 \leq j \neq i \leq m$  and  $n_i - \beta = \alpha$ . If  $n_i$  is even, then  $(0, \dots, 0, \langle p_i^{n_i/2} \rangle, 0, \dots, 0) \in N[J] \cap C \setminus N[I] \cap C$ . If  $n_i$  is odd, then  $H \in N[J] \cap C \setminus N[I] \cap C$ .

Thus the claim is proved and hence  $C$  is an identifying code of  $AG(\mathbb{Z}_n)$ .

(3) It is enough to consider  $C$  as follows and repeat the proof in case (2). Let

$$C = \{(0, \dots, \langle p_i^\alpha \rangle, 0, \dots, 0) \mid \forall 1 \leq i \leq m, \quad 1 \leq \alpha \leq n_i - 1\} \cup \{(\mathbb{Z}_{p_1^{n_1}}, 0, \dots, 0), \dots, (0, \dots, 0, \mathbb{Z}_{p_m^{n_m}})\}.$$

If at least one of  $n_i$  is odd,  $1 \leq i \leq m$ , then add the vertex  $H = \{(I_1, \dots, I_m)\}$  to  $C$ , where every  $I_j$  is defined as follows:



$$\begin{cases} \langle p_j^{\lfloor n_j/2 \rfloor} \rangle, & \text{if } n_j \text{ is odd,} \\ I_j = \mathbb{Z}_{p_j^{n_j}}, & \text{if } n_j \text{ is even.} \end{cases}$$

It is not hard to check that  $N[I] \cap C \neq \emptyset$  for every vertex  $I$ . Consider the following sets:

$$A = \{(I_1, I_2, \dots, I_m) \mid I_i \in \{0, \mathbb{Z}_{p_i^{n_i}}\}, \forall 1 \leq i \leq m\}$$

and

$$B = V(AG(\mathbb{Z}_n)) \setminus A.$$

Clearly  $\{A, B\}$  is a partition of  $V(AG(\mathbb{Z}_n))$ .

Thus  $C$  is an identifying code of  $AG(\mathbb{Z}_n)$ . □

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