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SOME PROPERTIES OF THE GENERALIZED SIERPIŃSKI GASKET GRAPHS

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Abstract. The generalized Sierpiński gasket graphs $S[G,t]$ are introduced as the graphs obtained from the Sierpiński graphs $S(G,t)$ by contracting single edges between copies of previous phases. The family $S[G,t]$ is a generalization of a previously studied class of generalized Sierpiński gasket graphs $S[n,t]$. In this paper, several properties of $S[G,t]$ are studied. In particular, adjacency of vertices, degree sequence, general first Zagreb index, hamiltonicity, and Eulerian.

1. Introduction

All graphs introduced in this paper are assumed to be simple and finite. Throughout this paper, $G = (V,E)$ is a non-empty graph of order n with the vertex set $V = \{1,2,\ldots,n\}$ and the edge set E. The degree of a vertex v of G is indicated by $\deg_G(v)$ which equals the size of the set of its neighborhood $N_G(v)$. Decomposition into special substructures which are inheriting considerable features is a significant method used for the investigation of some mathematical structures, especially when the regarded structures have self-similarity features. In these cases, we usually only need to study the substructures and the way that they are related to each other.

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Klavžar et al. introduced the Sierpiński graphs $S(n,t)$, for the first time and they showed that $S(3,t)$ is isomorphic to the graph of the Tower of Hanoi, see [16] and [17] for more details. The Hanoi graphs ([2], [8], [11]) are derived from the states of the Tower of Hanoi problem.

Sierpiński and Sierpiński-type Graphs are studied in fractal theory [28] and appear naturally in different areas of mathematics and in various scientific fields like in computer science (as a model for interconnection networks which are known as WK-recursive networks), topology, and mathematics of the Tower of Hanoi, see [4] and [25] for more details.

One of the most significant families of such graphs is constituted by the Sierpiński gasket graphs introduced by Scorer, Grundy, and Smith in [26] which plays a crucial role in dynamical systems, probability, and psychology, see [7], [13] and [19]. Figure 1 shows the Sierpiński gasket graph for S_3.

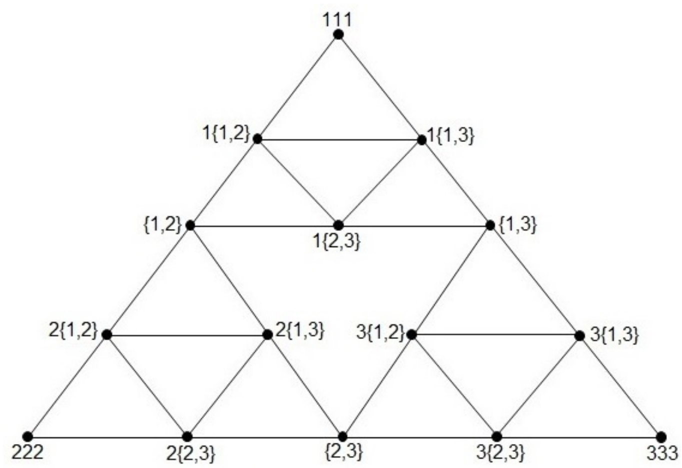

FIGURE 1. The Sierpiński gasket graph S_3.

Definition 1.1. *Let $G=(V,E)$ be a non-empty graph of order $n\geq 2$, and t be a positive integer. We denote by V^t the set of words of length t on alphabet V. The concatenation of two words $u=u_1u_2\cdots u_t$ and $v=v_1v_2\cdots v_t$ is denoted by uv. The Sierpiński graph of G of dimension t, denoted by $S(n,t)$, is the graph with vertex set V^t, where uv is an edge if and only if there exists $i\in\{1,\dots,t\}$ such that:*

$$\begin{aligned}(i)\ u_j &= v_j \text{ if } j<i,\\ (ii)\ u_i &\neq v_i,\\ (iii)\ u_j &= v_i \text{ and } v_j=u_i \text{ if } j>i.\end{aligned}$$

This construction was generalized in [5] for any graph G as follows.

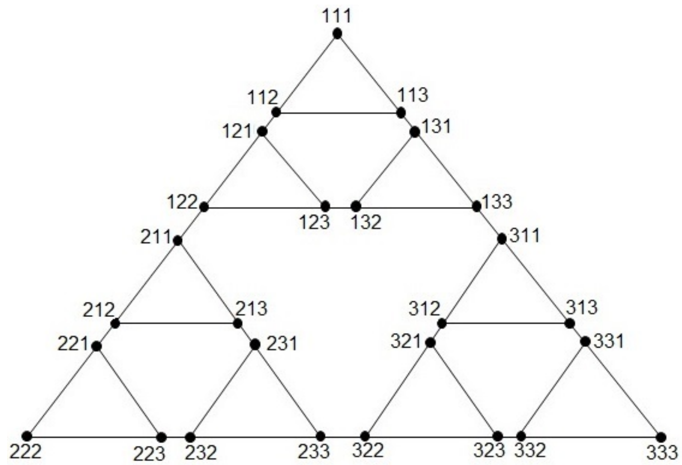

FIGURE 2. The Sierpiński graph $S(3,3)$.

Definition 1.2. *The t-th Sierpiński graph of G, denoted by $S(G,t)$, is the graph with vertex set V^t and two vertices u and v are adjacent if and only if there exists $i \in \{1,\ldots,t\}$ such that:*

$$\begin{aligned}(i)\ u_j &= v_j\ if\ j < i,\\ (ii)\ u_i &\neq v_i\ and\ u_iv_i \in E(G),\\ (iii)\ u_j &= v_i\ and\ v_j = u_i\ if\ j > i.\end{aligned}$$

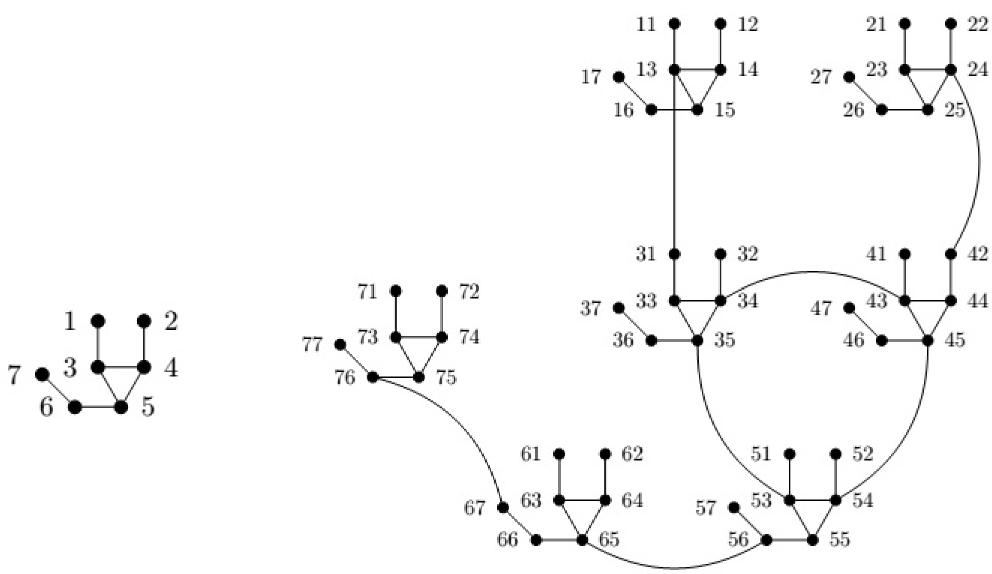

FIGURE 3. The generalized Sierpiński graph $S(G,2)$.

The Sierpiński gasket graph S_t is constructed from the Sierpiński graph $S(3,t)$ by contracting all of its edges that lie in no triangles K_3, see [15]. We can apply the same construction method for any

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Sierpiński graph $S(n,t)$ by contracting all of its edges that lie in no induced subgraph K_n and call it generalized Sierpiński gasket graph $S[n,t]$, see [13].

We already know the case for $n=3$, namely the Sierpiński gasket $S_t = S[3,t]$. An example of $S[4,2]$ is shown in Figure 4.

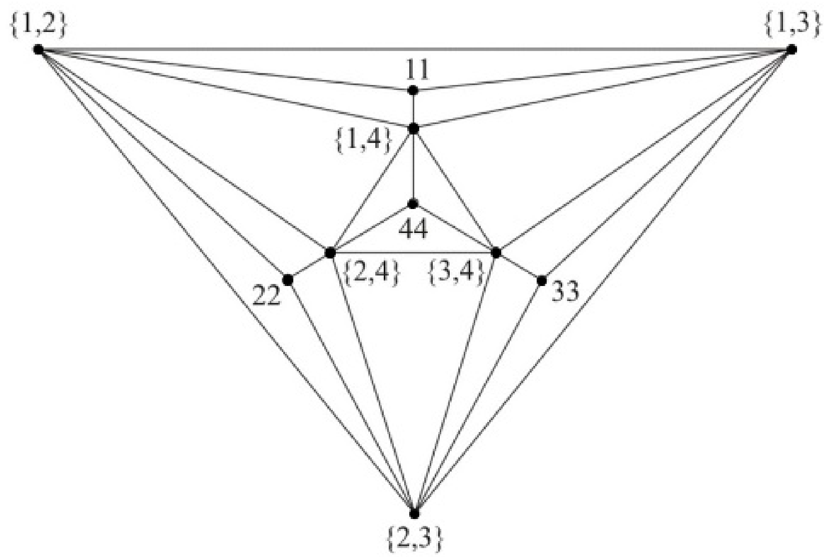

FIGURE 4. The generalized Sierpiński gasket graph $S[4,2]$.

In this paper, we generalize this construction for each graph G by defining the t-th generalized Sierpiński gasket graph of G, denoted by $S[G,t]$. Sierpiński, Sierpiński-type, and Sierpiński graphs have many interesting properties and were studied extensively in the literature. The reader is invited to read, for instance, the following recent papers [22], [18], [17], [9], [10], [6] and the references therein. Our purpose in this paper is to find some similar cases for the graph $S[G,t]$.

2. Generalized Sierpiński Gasket Graph S[G, t]

Since one of the most important families of Sierpiński graphs is the Sierpiński gasket S_t, we generalize this structure for each graph G by contracting all edges between copies.

Definition 2.1. *Suppose $G=(V,E)$ be a graph of order $n \geq 2$, t be a positive integer, the t-th generalized Sierpiński gasket graph of G, is the t-th Sierpiński graph $S(G,t)$, with the difference that two adjacent vertices $u = v_1v_2\cdots v_rv_jv_l\cdots v_l$ and $v = v_1v_2\ldots v_rv_lv_j\cdots v_j$, $0 \leq r \leq t-2$ of graph $S(G,t)$, are contracted in $S[G,t]$ when $v_lv_j \in E(G)$ and this vertex in generalized Sierpiński gasket graph is denoted by $v_1v_2\cdots v_r\{v_j,v_l\}_{t-r}$. See $S[C_4,3]$ on Figure 5.*

Remark 2.2. *Note that in $S[G,t]$ there are two types of vertices; contracted and non-contracted ones. Hereafter, we denote the non-contracted vertices of the form $v_1v_2\cdots v_t$, as type I, and the other vertices*

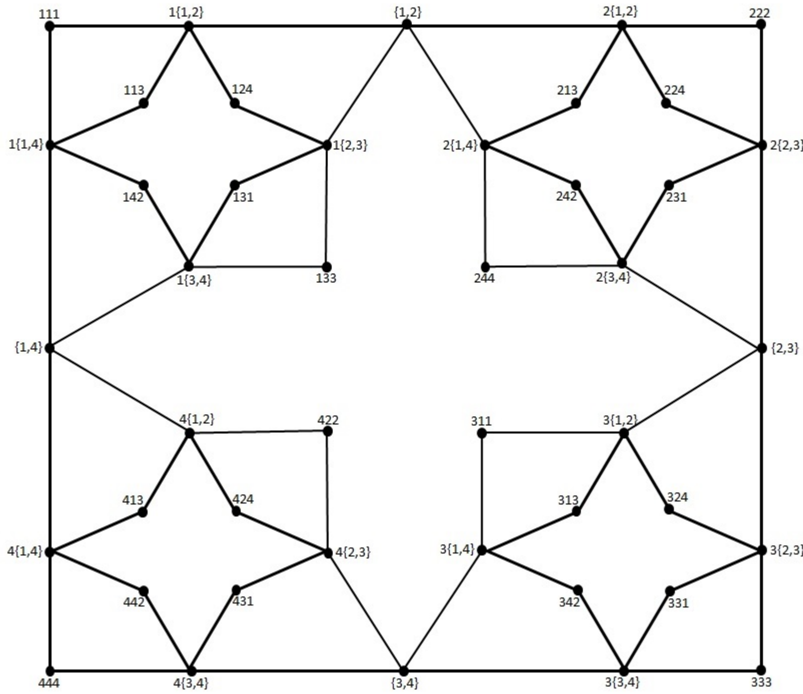

FIGURE 5. The generalized Sierpiński gasket graph $S[C_4, 3]$.

of the form $v_1v_2\cdots v_r\{v_i, v_j\}_{t-r}$ (of length t) as type II. By Definition 2.1, the vertices of $S[G, t]$ are contracted if and only if they satisfy in the following conditionens for some $i \in \{1, \ldots, t\}$,

$$\begin{aligned}(i)\ u_j &= v_j \ if \ j < i,\\ (ii)\ u_i &\neq v_i \ and \ u_iv_i \in E(G),\ i \neq t,\\ (iii)\ u_j &= v_i \ and \ v_j = u_i \ if \ j > i.\end{aligned}$$

Remark 2.3. *Similar to $S(G, t)$, the generalized Sierpiński gasket graph $S[G, t]$ can be constructed from $S[G, t-1]$ with all vertices of G replaced by copies of $S[G, t-1]$, and then amalgamating vertices $v_iv_jv_j\cdots v_j$ and $v_jv_iv_i\cdots v_i$ if and only if $v_iv_j \in E(G)$. We identify these two contracted vertices in $S[G, t]$ with one vertex in the form $\{v_i, v_j\}_t$.*

Let G be a graph with order and size n and m respectively. We obtain the order and size of the generalized Sierpiński gasket graph as follows.

Proposition 2.4. *Assume that $G = (V, E)$ is a graph with n vertices and m edges, and let t be a positive integer. Then, $S[G, t]$ have $n^t - m\,\frac{n^{t-1}-1}{n-1}$ vertices and mn^{t-1} edges.*

Proof. By Definition 2.1, for $t \geqslant 2$, we have $|V(S[G,t])| = n|V(S[G,t-1])| - m$ and since $|V(S[G,1])| = n$, hence $|V(S[G,t])| = n^t - m\,\frac{n^{t-1}-1}{n-1}$. On the other hand, $|E(S[G,t])| = n|E(S[G,t-1])|$ and since $|E(S[G,1])| = m$, thus $|E(S[G,t])| = mn^{t-1}$. □

Remark 2.5. *The generalized Sierpiński gasket graph $S[G,t]$ is connected if and only if G is connected.*

Corollary 2.6. *Let $G=(V,E)$ be a simple graph with n vertices and m edges, and t be a positive integer. Then the number of contracted vertices in the t-th generalized Sierpiński gasket graph of G is $m\frac{n^{t-1}-1}{n-1}$.*

Proof. For $t=2$, the $S[G,2]$ consists of n copy of G in which $|E(G)|$ edges between copies are contracted. Hence, $S[G,2]$ has m contracted vertices. According to Remark 2.3, since $S[G,t]$ consists of n copy of $S[G,t-1]$ with m new contracted vertices, thus by induction, the generalized Sierpiński gasket graph in step t has $n(m\frac{n^{t-2}-1}{n-1})+m=m\frac{n^{t-1}-1}{n-1}$ contracted vertices. □

Corollary 2.7. *Let T be a tree. For any positive integer t, $S[T,t]$ is also a tree.*

Proof. Let T be a tree with n vertices. Since T is connected, $S[T,t]$ is connected. On the other hand, by Proposition 2.4 $S[T,t]$ has $n^{t-1}(n-1)+1$ vertices and $n^{t-1}(n-1)$ edges. So, $S[T,t]$ is a tree. □

Remark 2.8. *In the light of Definition 2.1, it is clear that by Theorem 4 in [23] one has*

$$\varepsilon(S[T,t])=\varepsilon(S(T,t))=\frac{\varepsilon(T)(n^t-2n^{t-1}+1)}{n-1},$$

in which, $\varepsilon(T)$ is the number of leaves of T.

Adjacency of vertices in $S[G,t]$ is given in the next theorem.

Theorem 2.9. *Let $t\geq 2$, $x=v_1\cdots v_r\{v_i,v_j\}_{t-r}$ and $y=v_1v_2\cdots v_t$ be vertices in $S[G,t]$, $v_i,v_j\in V(G)$ and $v_i\neq v_j$.*

(i) *If $0\leq r\leq t-3$, then,*

$$\left\{v_1v_2\cdots v_rv_iv_j\cdots v_j\{v_j,v_l\}_2\right\}\in N_{S[G,t]}(x)\Longleftrightarrow v_l\in N_G(v_j)$$

$$\left\{v_1v_2\cdots v_rv_jv_i\cdots v_i\{v_i,v_l\}_2\right\}\in N_{S[G,t]}(x)\Longleftrightarrow v_l\in N_G(v_i).$$

(ii) *If $r=t-2$, then,*

$$\left\{v_1v_2\cdots v_{t-2}v_iv_l\right\}\in N_{S[G,t]}(x)\Longleftrightarrow v_l\in N_G(v_j),v_l\notin N_G(v_i),$$

$$\left\{v_1v_2\cdots v_{t-2}v_jv_l\right\}\in N_{S[G,t]}(x)\Longleftrightarrow v_l\in N_G(v_i),v_l\notin N_G(v_j),$$

$$\left\{v_1v_2\cdots v_{t-2}\{v_i,v_l\}_2\right\}\in N_{S[G,t]}(x)\Longleftrightarrow v_l\in N_G(v_j)\cap N_G(v_i),$$

$$\left\{v_1v_2\cdots v_{t-2}\{v_j,v_l\}_2\right\}\in N_{S[G,t]}(x)\Longleftrightarrow v_l\in N_G(v_j)\cap N_G(v_i).$$

(iii) *For $y=v_1v_2\cdots v_t$, we have,*

$$\left\{v_1v_2\ldots v_{t-1}v_l\right\}\in N_{S[G,t]}(y)\Longleftrightarrow v_l\in N_G(v_t),v_l\notin N_G(v_{t-1}),$$

$$\left\{v_1v_2\cdots v_{t-2}\{v_{t-1},v_l\}_2\right\}\in N_{S[G,t]}(y)\Longleftrightarrow v_l\in N_G(v_t)\cap N_G(v_{t-1}).$$

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Proof. **(i)** Let $r \le t-3$, and $x = v_1 \cdots v_r\{v_i, v_j\}_{t-r} \in V(S[G,t])$. By Definition 2.1, x is obtained from contraction of two vertices $\bar{x} = v_1v_2\cdots v_rv_iv_j\cdots v_jv_j$ and $\bar{\bar{x}} = v_1v_2\cdots v_rv_jv_i\cdots v_iv_i$ of $S(G,t)$. Let $v_l \in N_G(v_j)$. Then $\bar{x}$ is adjacent to $v_1v_2\cdots v_rv_iv_j\cdots v_jv_l = v_1v_2\cdots v_rv_iv_j\cdots v_j\{v_j, v_l\}_2$. Similarly, for $v_l \in N_G(v_i)$, $\bar{\bar{x}}$ is adjacent to $v_1v_2\cdots v_rv_jv_i\cdots v_i\{v_i, v_l\}_2$.

If $z = v_1v_2\cdots v_rv_iv_j\cdots v_j\{v_j, v_l\}_2 \in N_{S[G,t]}(x)$, then, $\bar{z} = v_1v_2\cdots v_rv_iv_j\cdots v_jv_l \in N_{S[G,t]}(\bar{x})$, thus, by Definition 1.2, $v_l \in N_G(v_j)$. Also, the argument applies to $r = 0$.

(ii) Now, let $r = t-2$. Then, we have $\bar{x} = v_1v_2\cdots v_{t-2}v_iv_j$ and $\bar{\bar{x}} = v_1v_2\cdots v_{t-2}v_jv_i$. Suppose that $v_l \in N_G(v_j)$. If v_l is not adjacent to v_i in G, then $\bar{x}$ will be adjacent to $v_1v_2\cdots v_{t-2}v_iv_l$ in $S[G,t]$. Also, if $v_l \in N_G(v_i)$, then according to Definition 2.1, $\bar{x}$ is adjacent to $v_1v_2\cdots v_{t-2}\{v_i, v_l\}_2$. Similarly, the argument applies to $\bar{\bar{x}}$. If $z = v_1v_2\cdots v_{t-2}v_iv_l \in N_{S[G,t]}(x)$, then it is obvious that $z \in N_{S[G,t]}(\bar{x})$, thus, $v_l \in N_G(v_j)$ and $v_l \notin N_G(v_i)$. Now, if $z_1 = v_1v_2\cdots v_{t-2}\{v_i, v_l\}_2 \in N_{S[G,t]}(x)$, then, $\bar{z_1} = v_1v_2\cdots v_{t-2}v_iv_l$ is adjacent to $\bar{x}$. Thus, $v_l \in N_G(v_j) \cap N_G(v_i)$.

(iii) Let $v_l \in N_G(v_t)$. Then, $y = v_1v_2\cdots v_t$ is adjacent to $v_1v_2\cdots v_{t-2}v_{t-1}v_l$. But, if v_l and v_{t-1} are adjacent to each other in G, then by Definition 2.1, y is adjacent to $v_1\cdots v_{t-2}\{v_{t-1}, v_l\}_2$ in $S[G,t]$. $\square$

Proposition 2.10. *Let $G = (V,E)$ be a simple graph and $t \ge 1$ be an integer. Then,*

$$\deg_{S[G,t]}(x) = \begin{cases} \deg_G(v_i) + \deg_G(v_j) & if\ x = v_1v_2\cdots v_r\{v_i, v_j\}_{t-r}, \\ \deg_G(v_t) & if\ x = v_1v_2\cdots v_t. \end{cases}$$

Proof. For $x = v_1v_2\cdots v_r\{v_i, v_j\}_{t-r}$, according to Theorem 2.9, it is clear that $\deg_{S[G,t]}(x) = \deg_G(v_i) + \deg_G(v_j)$. Let $x = v_1v_2\cdots v_t$. For $t = 1$, we have $\deg_{S[G,1]}(v_i) = \deg_G(v_i)$. Now, we assume that $t \ge 2$. Note that, if $N_G(v_t) = \emptyset$, then $x = v_1v_2\cdots v_t$, is an isolated vertex in $S[G,t]$. Therefore, if $\deg_G(v_t) = 0$, then $\deg_{S[G,t]}(v_1\cdots v_t) = 0$, and the result follows. Let $\deg_G(v_t) = l \ge 1$, and $N_G(v_t) = \{z_1, \ldots, z_l\}$. For any $i \in \{1, 2, \ldots, l\}$, x is adjacent to $p_i = v_1v_2\cdots v_{t-1}z_i$ in $S[G,t]$. Hence, $\deg_{S[G,t]}(v_1v_2\cdots v_t) \ge l = \deg_G(v_t)$. Now, suppose that $k = k_1k_2\cdots k_t$ be a neighbour of x in $S[G,t]$. Therefore, by part (iii) of previous theorem, $v_i = k_i$ for $i \le t-1$ and $v_t \ne k_t$. Also, v_t is adjacent to k_t in G. Then x is adjacent to $v_1\cdots v_{t-1}k_t$. Thus, $k_t \in N_G(v_t) = \{z_1, \cdots, z_l\}$, i.e. $k \in p_i$. Hence,

$$\deg_{S[G,t]}(v_1v_2\cdots v_t) = \deg_G(v_t).$$

$\square$

Remark 2.11. **(i)** *If G is an r-regular, then $S[G,t]$ is (r,2r)-regular.*

(ii) *In general, for every $e = uv \in E(G)$, we have $\deg(e) = \deg(u) + \deg(v)$. Now, suppose that w is a vertex of type II. Since w has been an edge in the generalized Sierpiński graph which is contracted in $S[G,t]$, by Proposition 2.10, we have $\deg(w) = \deg(e) = \deg(u) + \deg(v)$.*

Corollary 2.12. *Let $G = (V, E)$ be a graph of order $n \geq 2$ and $t \geq 1$ be a positive integer. Then,*

$$(i) \quad \delta(S[G,t]) = \delta(G),$$

$$(ii) \quad \Delta(S[G,t]) = max\{\deg(x) + \deg(y) \mid xy \in E(G)\}.$$

Also, we have $\Delta(S(G,t)) = 1 + \Delta(G) \leq \Delta(S[G,t]) \leq 2\Delta(G)$, when $E(G) \neq \emptyset$ and $t \geq 2$.

Theorem 2.13. *Suppose that $\delta(G) \leq k \leq \Delta(G)$, then the number of vertices with degree k in $S[G,t]$ is*

$$|V_k|n^{t-1} - \frac{n^{t-1} - 1}{n - 1}(k|V_k| - a_k).$$

where $V_k = \{v \in V : \deg_G(v) = k\}$ and $a_k = |\{e = xy \in E(G)| \deg_G(x) + \deg_G(y) = k\}|$.

Proof. The result follows for $t = 1$. Let $t \geq 2$ and $z \in V_k$. Set $\Theta_{k,z} = \{v_1 \cdots v_{t-1}z \in V(S[G,t]),$ $v_1, v_2, \ldots, v_{t-1} \in V(G)\}$. Let $\Theta_k = \cup_{z \in V_k} \Theta_{k,z}$. By Proposition 2.10, it is clear that the number of vertices of type I with degree k is $\mid \Theta_k \mid = \sum_{z \in V_k} \mid \Theta_{k,z} \mid = n^{t-1} \mid V_k \mid$.

On the other hand, let Γ be the set of all vertices of type II of form $w = v_1 v_2 \cdots v_r \{v_i, v_j\}_{t-r}$ in $S[G,t]$, such that $\deg(v_i) = p$, $\deg(v_j) = q$. By Remark 2.11 and Proposition 2.10, $\deg(w) = \deg(v_i) + \deg(v_j) = p + q$. Let a_k denote the number of edges of G in which the sum of degrees of ending vertices is k, hence, there are $\sum n^r a_k$ vertices of type II with degree k in $S[G,t]$. By Definition 2.1, contracting the vertices of degree k yields vertices of degree more than k.

Therefore, the number of these vertices in generalized Sierpiński gasket graph is equal to $\sum n^r (k|V_k|)$.

Hence, the number of vertices of degree k in $S[G,t]$ is

$$\mid V_k \mid n^{t-1} - \frac{n^{t-1} - 1}{n - 1}(k \mid V_k \mid - a_k).$$

□

3. Hamiltonicity and Eulerian

A graph G is called Eulerian if and only if has a spanning closed trail containing all edges of the graph. On the other hand, the graph G is Eulerian if and only if G is connected and all its vertices have even degree. Graph G is called Hamiltonian when it contains a cycle that travels exactly once over each vertex in a graph.

In [27], demonstrated by Tequia and Godbole that generalized Sierpiński gasket graphs are Hamiltonian, and this graph is Eulerian if and only if n is odd.

Also, in [13], Jakovac proved that generalized Sierpiński gasket graphs for $t \geq 1$ and $n \geq 3$ are Hamiltonian. In this section, we obtain two properties of Hamiltonian and Eulerian for generalized Sierpiński gasket graph.

Proposition 3.1. *Let G be a simple graph with $n \geq 4$ and $\delta(G) \leq 2$, then for $t \geq 2$, $S[G,t]$ is not Hamiltonian.*

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Proof. First, assume that G is a graph with a minimum degree of 2. Suppose that v_i is a vertex of G that $\deg(v_i) = 2$, and let v_j and v_k be its neighbors. Consider $[V_i^{'}]$ that $V_i^{'} = \{v_i^{t-1}v_l | v_l \in V(G)\} \bigcup v_i^{t-2}\{v_i, N(v_i)\}$, where $v_i^{t-2}\{v_i, N(v_i)\} = \{v_i^{t-2}\{v_i, v_j\}, v_i^{t-2}\{v_i, v_k\}\}$. It is evident that $v_i^{t-2}\{v_i, N(v_i)\}$ is a separating set for $[V_i^{'}]$. According to Proposition 2.10, $\deg(v_i^t) = \deg(v_i) = 2$. On the other hand, by Theorem 2.9, the vertex v_i^t is only adjacent to $v_i^{t-2}\{v_i, v_j\}$ and $v_i^{t-2}\{v_i, v_k\}$. Since G has at least 4 vertices, by removing $v_i^{t-2}\{v_i, N(v_i)\}$, the induced subgraph $[V_i^{'}]$ has at least two components, one of them is the isolated vertex v_i^t. By Definition 2.1, the vertices $v_i^{t-2}\{v_i, N(v_i)\}$, connect $[V_i^{'}]$ to other copies of $S[G, t]$. Every path between $v_i^{t-1}v_l$ and $v_i^{t-2}\{v_i, N(v_i)\}$, does not contain a vertex of type II, thus, deleting the vertices $v_i^{t-2}\{v_i, N(v_i)\}$, makes at least three components for $S[G, t]$. Therefore, $S[G, t]$ is not Hamiltonian.

Now, suppose that G has a vertex v_i with $\deg(v_i) = 1$, let $v_j \in N(v_i)$. Then, $v_i^{t-2}\{v_i, v_j\}$ is a cut vertex for $S[G, t]$. Hence, $S[G, t]$ is not Hamiltonian and the proof is complete. □

In Figure 5, we see that deleting two vertices $3\{2, 3\}$ and $3\{3, 4\}$ makes three components for the graph and hence $S[C_4, 3]$ is not Hamiltonian.

Remark 3.2. *If G is Hamiltonian, then, according to Proposition 3.1, we cannot guarantee that $S[G, t]$ is Hamiltonian. For example C_4 is Hamiltonian, but $S[C_4, t]$ for any $t \geq 2$ is not Hamiltonian.*

Theorem 3.3. *Let G be a simple graph having no isolated vertex, then G is Eulerian, if and only if $S[G, t]$ is Eulerian.*

Proof. First, let G be an Eulerian graph. Then G is connected and every vertex of this graph has an even degree. According to Remark 2.5 and Proposition 2.10, $S[G, t]$ is connected and even. Hence, $S[G, t]$ is Eulerian. Now, assume that $S[G, t]$ is Eulerian. By Remark 2.5, G is connected. If G contains a vertex $v_i \in V(G)$ with odd degree, then by Proposition 2.10 the degree of vertex $v_i v_i \cdots v_i$ in $S[G, t]$ is equal to $\deg_G(v_i)$ which is odd, a contradiction. Thus, each vertex of G has even degree and hence, G is Eulerian. □

4. General first Zagreb index

Chemical applications of graph theory determine a lot of properties such as thermodynamic and physicochemical properties, and chemical and biological activities. We can express these properties as topological indices. Researchers in [12], and [24] determined some of the topological indices of Sierpiński networks and Sierpiński graphs. The general first Zagreb index of a graph G is defined as follows [20]:

$$Z_\alpha(G) = \sum_{\{u,v\} \in E(G)} ((\deg_G(u))^{\alpha-1}) + (\deg_G(v))^{\alpha-1}) = \sum_{u \in V(G)} (\deg_G(u))^{\alpha}.$$

Where $\alpha \neq 0, 1$ is a real number. In particular, if $\alpha = 2$, then the general first Zagreb index, is called the first Zagreb index ($Z_2(G) = M_1(G)$). Also, if $\alpha = 3$, then the general first Zagreb index is called

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the forgotten topological index ($Z_3(G) = F(G)$), [1], [3], and [21]. In this section, we obtain a general Zagreb topological index (see [14]) for generalized Sierpiński gasket graphs.

Theorem 4.1. *Let G be a simple graph of order $n \geq 2$. For each integer $\alpha \geq 0$, the general first Zagreb index of the generalized Sierpiński gasket graph $S[G,t]$, $t \geq 1$ is given by*

$$Z_\alpha(S[G,t]) = n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{k=2}^{2\Delta(G)} a_k \cdot k^\alpha.$$

Proof. According to the definition of the general first Zagreb index and by Theorem 2.13, $Z_\alpha(S[G,t])$ is equal to

$$\begin{aligned} Z_\alpha(S[G,t]) &= \sum_{k=1}^{2\Delta(G)} \left(\mid V_k \mid n^{t-1} - \frac{n^{t-1}-1}{n-1}(k \mid V_k \mid - a_k)\right) k^\alpha \\ &= \sum_{k=1}^{2\Delta(G)} \mid V_k \mid n^{t-1}k^\alpha - \frac{n^{t-1}-1}{n-1}\sum_{k=1}^{2\Delta(G)} k \mid V_k \mid k^\alpha \\ &\quad + \frac{n^{t-1}-1}{n-1}\sum_{k=1}^{2\Delta(G)} a_k \cdot k^\alpha \\ &= n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}Z_{\alpha+1}(G) + \frac{n^{t-1}-1}{n-1}\sum_{k=2}^{2\Delta(G)} a_k \cdot k^\alpha. \end{aligned}$$

□

Corollary 4.2. *Let G be a k-regular graph. Then we have*

$$Z_\alpha(S[G,t]) = n^{t-1}Z_\alpha(G) - \frac{n^{t-1}-1}{n-1}(Z_{\alpha+1}(G) - m(2k)^\alpha).$$

Especially, if $G = C_n$ then,

$$Z_\alpha(S[C_n,t]) = n^{t-1}Z_\alpha(C_n) - \frac{n^{t-1}-1}{n-1}(Z_{\alpha+1}(C_n) - m(4)^\alpha).$$

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