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## SOME PROPERTIES OF THE GENERALIZED SIERPIŃSKI GASKET GRAPHS

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**ABSTRACT.** The generalized Sierpiński gasket graphs  $S[G, t]$  are introduced as the graphs obtained from the Sierpiński graphs  $S(G, t)$  by contracting single edges between copies of previous phases. The family  $S[G, t]$  is a generalization of a previously studied class of generalized Sierpiński gasket graphs  $S[n, t]$ . In this paper, several properties of  $S[G, t]$  are studied. In particular, adjacency of vertices, degree sequence, general first Zagreb index, hamiltonicity, and Eulerian.

### 1. Introduction

All graphs introduced in this paper are assumed to be simple and finite. Throughout this paper,  $G = (V, E)$  is a non-empty graph of order  $n$  with the vertex set  $V = \{1, 2, \dots, n\}$  and the edge set  $E$ . The degree of a vertex  $v$  of  $G$  is indicated by  $\deg_G(v)$  which equals the size of the set of its neighborhood  $N_G(v)$ . Decomposition into special substructures which are inheriting considerable features is a significant method used for the investigation of some mathematical structures, especially when the regarded structures have self-similarity features. In these cases, we usually only need to study the substructures and the way that they are related to each other.

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Klavžar et al. introduced the Sierpiński graphs  $S(n, t)$ , for the first time and they showed that  $S(3, t)$  is isomorphic to the graph of the Tower of Hanoi, see [16] and [17] for more details. The Hanoi graphs ([2], [8], [11]) are derived from the states of the Tower of Hanoi problem.

Sierpiński and Sierpiński-type Graphs are studied in fractal theory [28] and appear naturally in different areas of mathematics and in various scientific fields like in computer science (as a model for interconnection networks which are known as WK-recursive networks), topology, and mathematics of the Tower of Hanoi, see [4] and [25] for more details.

One of the most significant families of such graphs is constituted by the Sierpiński gasket graphs introduced by Scorer, Grundy, and Smith in [26] which plays a crucial role in dynamical systems, probability, and psychology, see [7], [13] and [19]. Figure 1 shows the Sierpiński gasket graph for  $S_3$ .

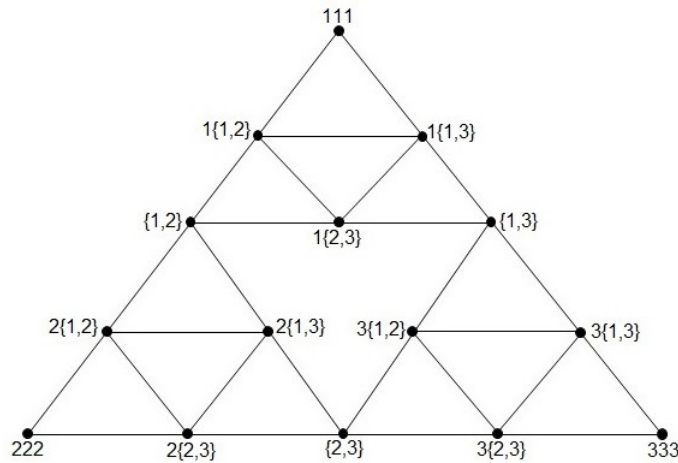


FIGURE 1. The Sierpiński gasket graph  $S_3$ .

**Definition 1.1.** Let  $G = (V, E)$  be a non-empty graph of order  $n \geq 2$ , and  $t$  be a positive integer. We denote by  $V^t$  the set of words of length  $t$  on alphabet  $V$ . The concatenation of two words  $u = u_1u_2 \cdots u_t$  and  $v = v_1v_2 \cdots v_t$  is denoted by  $uv$ . The Sierpiński graph of  $G$  of dimension  $t$ , denoted by  $S(n, t)$ , is the graph with vertex set  $V^t$ , where  $uv$  is an edge if and only if there exists  $i \in \{1, \dots, t\}$  such that:

- (i)  $u_j = v_j$  if  $j < i$ ,
- (ii)  $u_i \neq v_i$ ,
- (iii)  $u_j = v_i$  and  $v_j = u_i$  if  $j > i$ .

This construction was generalized in [5] for any graph  $G$  as follows.

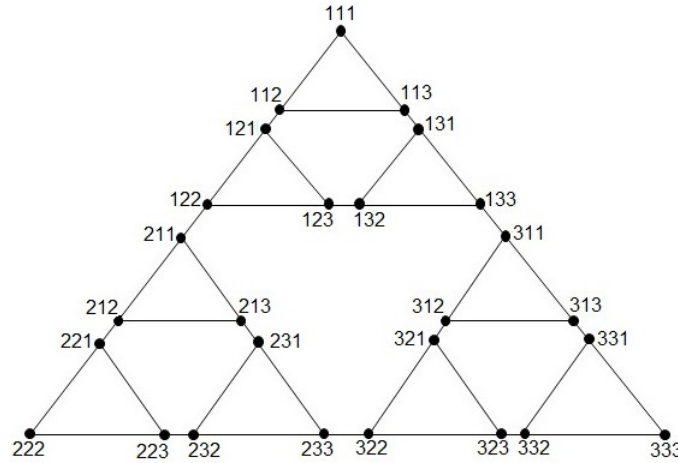


FIGURE 2. The Sierpiński graph  $S(3, 3)$ .

**Definition 1.2.** The  $t$ -th Sierpiński graph of  $G$ , denoted by  $S(G, t)$ , is the graph with vertex set  $V^t$  and two vertices  $u$  and  $v$  are adjacent if and only if there exists  $i \in \{1, \dots, t\}$  such that:

- (i)  $u_j = v_j$  if  $j < i$ ,
- (ii)  $u_i \neq v_i$  and  $u_i v_i \in E(G)$ ,
- (iii)  $u_j = v_i$  and  $v_j = u_i$  if  $j > i$ .

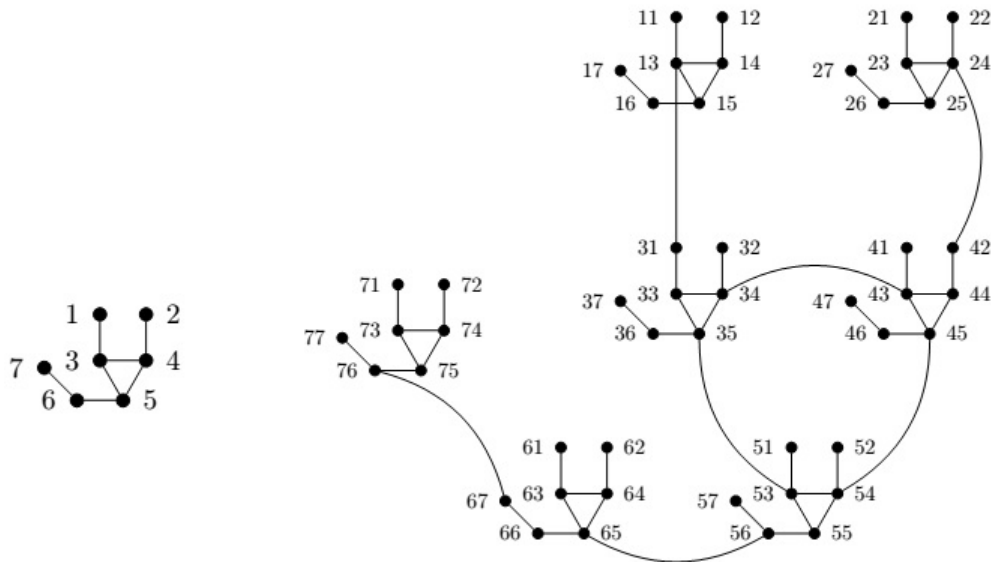


FIGURE 3. The generalized Sierpiński graph  $S(G, 2)$ .

The Sierpiński gasket graph  $S_t$  is constructed from the Sierpiński graph  $S(3, t)$  by contracting all of its edges that lie in no triangles  $K_3$ , see [15]. We can apply the same construction method for any

Sierpiński graph  $S(n, t)$  by contracting all of its edges that lie in no induced subgraph  $K_n$  and call it generalized Sierpiński gasket graph  $S[n, t]$ , see [13].

We already know the case for  $n = 3$ , namely the Sierpiński gasket  $S_t = S[3, t]$ . An example of  $S[4, 2]$  is shown in Figure 4.

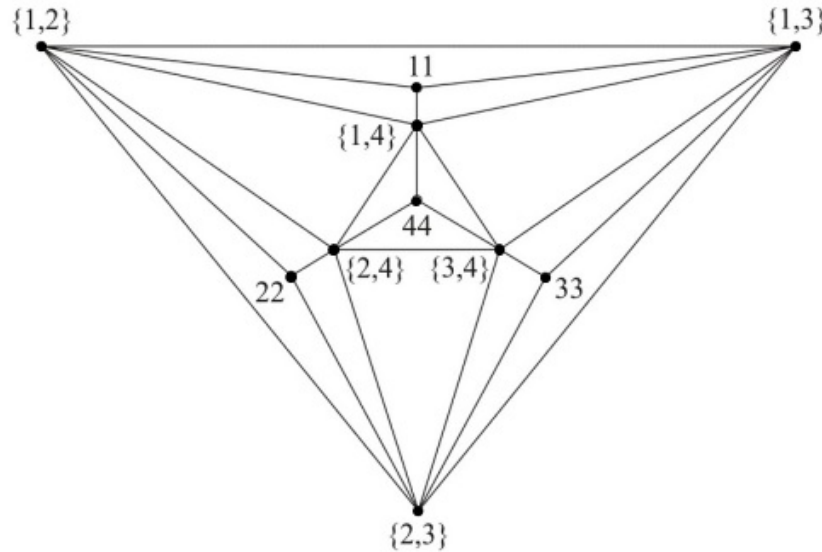


FIGURE 4. The generalized Sierpiński gasket graph  $S[4, 2]$ .

In this paper, we generalize this construction for each graph  $G$  by defining the  $t$ -th generalized Sierpiński gasket graph of  $G$ , denoted by  $S[G, t]$ . Sierpiński, Sierpiński-type, and Sierpiński graphs have many interesting properties and were studied extensively in the literature. The reader is invited to read, for instance, the following recent papers [22], [18], [17], [9], [10], [6] and the references therein. Our purpose in this paper is to find some similar cases for the graph  $S[G, t]$ .

### 2. Generalized Sierpiński Gasket Graph $S[G, t]$

Since one of the most important families of Sierpiński graphs is the Sierpiński gasket  $S_t$ , we generalize this structure for each graph  $G$  by contracting all edges between copies.

**Definition 2.1.** Suppose  $G = (V, E)$  be a graph of order  $n \geq 2$ ,  $t$  be a positive integer, the  $t$ -th generalized Sierpiński gasket graph of  $G$ , is the  $t$ -th Sierpiński graph  $S(G, t)$ , with the difference that two adjacent vertices  $u = v_1v_2 \cdots v_rv_jv_l \cdots v_l$  and  $v = v_1v_2 \cdots v_rv_l v_j \cdots v_j$ ,  $0 \leq r \leq t - 2$  of graph  $S(G, t)$ , are contracted in  $S[G, t]$  when  $v_l v_j \in E(G)$  and this vertex in generalized Sierpiński gasket graph is denoted by  $v_1v_2 \cdots v_r\{v_j, v_l\}_{t-r}$ . See  $S[C_4, 3]$  on Figure 5.

**Remark 2.2.** Note that in  $S[G, t]$  there are two types of vertices; contracted and non-contracted ones. Hereafter, we denote the non-contracted vertices of the form  $v_1v_2 \cdots v_t$ , as type I, and the other vertices

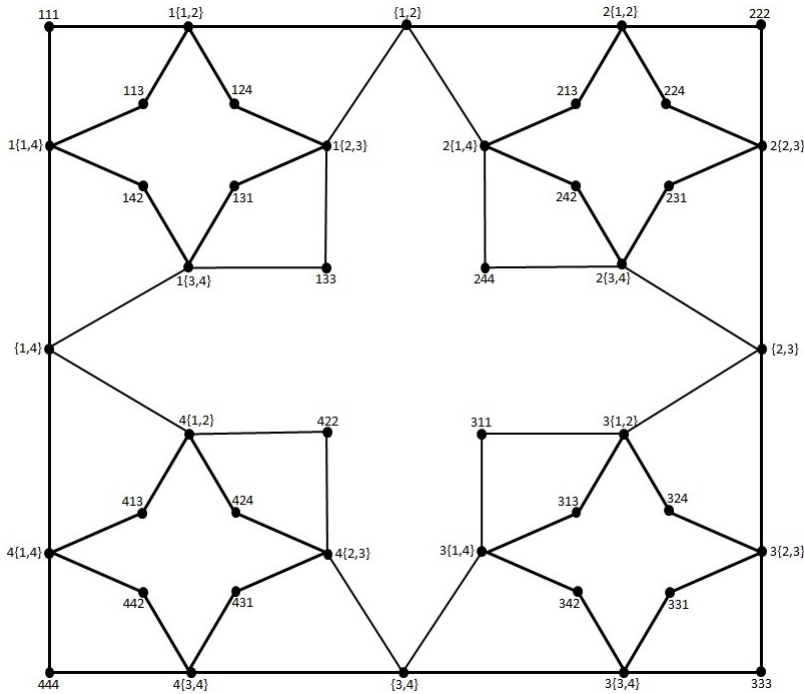


FIGURE 5. The generalized Sierpiński gasket graph  $S[C_4, 3]$ .

of the form  $v_1v_2 \cdots v_r\{v_i, v_j\}_{t-r}$  (of length  $t$ ) as type II. By Definition 2.1, the vertices of  $S[G, t]$  are contracted if and only if they satisfy in the following conditions for some  $i \in \{1, \dots, t\}$ ,

- (i)  $u_j = v_j$  if  $j < i$ ,
- (ii)  $u_i \neq v_i$  and  $u_iv_i \in E(G)$ ,  $i \neq t$ ,
- (iii)  $u_j = v_i$  and  $v_j = u_i$  if  $j > i$ .

**Remark 2.3.** Similar to  $S(G, t)$ , the generalized Sierpiński gasket graph  $S[G, t]$  can be constructed from  $S[G, t - 1]$  with all vertices of  $G$  replaced by copies of  $S[G, t - 1]$ , and then amalgamating vertices  $v_iv_jv_j \cdots v_j$  and  $v_jv_iv_i \cdots v_i$  if and only if  $v_iv_j \in E(G)$ . We identify these two contracted vertices in  $S[G, t]$  with one vertex in the form  $\{v_i, v_j\}_t$ .

Let  $G$  be a graph with order and size  $n$  and  $m$  respectively. We obtain the order and size of the generalized Sierpiński gasket graph as follows.

**Proposition 2.4.** Assume that  $G = (V, E)$  is a graph with  $n$  vertices and  $m$  edges, and let  $t$  be a positive integer. Then,  $S[G, t]$  have  $n^t - m \frac{n^{t-1}-1}{n-1}$  vertices and  $mn^{t-1}$  edges.

*Proof.* By Definition 2.1, for  $t \geq 2$ , we have  $|V(S[G, t])| = n|V(S[G, t-1])| - m$  and since  $|V(S[G, 1])| = n$ , hence  $|V(S[G, t])| = n^t - m \frac{n^{t-1}-1}{n-1}$ . On the other hand,  $|E(S[G, t])| = n|E(S[G, t-1])|$  and since  $|E(S[G, 1])| = m$ , thus  $|E(S[G, t])| = mn^{t-1}$ . □

**Remark 2.5.** The generalized Sierpiński gasket graph  $S[G, t]$  is connected if and only if  $G$  is connected.

**Corollary 2.6.** Let  $G = (V, E)$  be a simple graph with  $n$  vertices and  $m$  edges, and  $t$  be a positive integer. Then the number of contracted vertices in the  $t$ -th generalized Sierpiński gasket graph of  $G$  is  $m \frac{n^{t-1}-1}{n-1}$ .

*Proof.* For  $t = 2$ , the  $S[G, 2]$  consists of  $n$  copy of  $G$  in which  $|E(G)|$  edges between copies are contracted. Hence,  $S[G, 2]$  has  $m$  contracted vertices. According to Remark 2.3, since  $S[G, t]$  consists of  $n$  copy of  $S[G, t-1]$  with  $m$  new contracted vertices, thus by induction, the generalized Sierpiński gasket graph in step  $t$  has  $n(m \frac{n^{t-2}-1}{n-1}) + m = m \frac{n^{t-1}-1}{n-1}$  contracted vertices.  $\square$

**Corollary 2.7.** Let  $T$  be a tree. For any positive integer  $t$ ,  $S[T, t]$  is also a tree.

*Proof.* Let  $T$  be a tree with  $n$  vertices. Since  $T$  is connected,  $S[T, t]$  is connected. On the other hand, by Proposition 2.4  $S[T, t]$  has  $n^{t-1}(n-1) + 1$  vertices and  $n^{t-1}(n-1)$  edges. So,  $S[T, t]$  is a tree.  $\square$

**Remark 2.8.** In the light of Definition 2.1, it is clear that by Theorem 4 in [23] one has

$$\varepsilon(S[T, t]) = \varepsilon(S(T, t)) = \frac{\varepsilon(T)(n^t - 2n^{t-1} + 1)}{n-1},$$

in which,  $\varepsilon(T)$  is the number of leaves of  $T$ .

Adjacency of vertices in  $S[G, t]$  is given in the next theorem.

**Theorem 2.9.** Let  $t \geq 2$ ,  $x = v_1 \cdots v_r \{v_i, v_j\}_{t-r}$  and  $y = v_1 v_2 \cdots v_t$  be vertices in  $S[G, t]$ ,  $v_i, v_j \in V(G)$  and  $v_i \neq v_j$ .

(i) If  $0 \leq r \leq t-3$ , then,

$$\{v_1 v_2 \cdots v_r v_i v_j \cdots v_j \{v_j, v_l\}_2\} \in N_{S[G,t]}(x) \iff v_l \in N_G(v_j)$$

$$\{v_1 v_2 \cdots v_r v_j v_i \cdots v_i \{v_i, v_l\}_2\} \in N_{S[G,t]}(x) \iff v_l \in N_G(v_i).$$

(ii) If  $r = t-2$ , then,

$$\{v_1 v_2 \cdots v_{t-2} v_i v_l\} \in N_{S[G,t]}(x) \iff v_l \in N_G(v_j), v_l \notin N_G(v_i),$$

$$\{v_1 v_2 \cdots v_{t-2} v_j v_l\} \in N_{S[G,t]}(x) \iff v_l \in N_G(v_i), v_l \notin N_G(v_j),$$

$$\{v_1 v_2 \cdots v_{t-2} \{v_i, v_l\}_2\} \in N_{S[G,t]}(x) \iff v_l \in N_G(v_j) \cap N_G(v_i),$$

$$\{v_1 v_2 \cdots v_{t-2} \{v_j, v_l\}_2\} \in N_{S[G,t]}(x) \iff v_l \in N_G(v_j) \cap N_G(v_i).$$

(iii) For  $y = v_1 v_2 \cdots v_t$ , we have,

$$\{v_1 v_2 \cdots v_{t-1} v_l\} \in N_{S[G,t]}(y) \iff v_l \in N_G(v_t), v_l \notin N_G(v_{t-1}),$$

$$\{v_1 v_2 \cdots v_{t-2} \{v_{t-1}, v_l\}_2\} \in N_{S[G,t]}(y) \iff v_l \in N_G(v_t) \cap N_G(v_{t-1}).$$

*Proof.* (i) Let  $r \leq t - 3$ , and  $x = v_1 \cdots v_r \{v_i, v_j\}_{t-r} \in V(S[G, t])$ . By Definition 2.1,  $x$  is obtained from contraction of two vertices  $\bar{x} = v_1 v_2 \cdots v_r v_i v_j \cdots v_j v_j$  and  $\bar{\bar{x}} = v_1 v_2 \cdots v_r v_j v_i \cdots v_i v_i$  of  $S(G, t)$ . Let  $v_l \in N_G(v_j)$ . Then  $\bar{x}$  is adjacent to  $v_1 v_2 \cdots v_r v_i v_j \cdots v_j v_l = v_1 v_2 \cdots v_r v_i v_j \cdots v_j \{v_j, v_l\}_2$ . Similarly, for  $v_l \in N_G(v_i)$ ,  $\bar{\bar{x}}$  is adjacent to  $v_1 v_2 \cdots v_r v_j v_i \cdots v_i \{v_i, v_l\}_2$ .

If  $z = v_1 v_2 \cdots v_r v_i v_j \cdots v_j \{v_j, v_l\}_2 \in N_{S[G,t]}(x)$ , then,  $\bar{z} = v_1 v_2 \cdots v_r v_i v_j \cdots v_j v_l \in N_{S[G,t]}(\bar{x})$ , thus, by Definition 1.2,  $v_l \in N_G(v_j)$ . Also, the argument applies to  $r = 0$ .

(ii) Now, let  $r = t - 2$ . Then, we have  $\bar{x} = v_1 v_2 \cdots v_{t-2} v_i v_j$  and  $\bar{\bar{x}} = v_1 v_2 \cdots v_{t-2} v_j v_i$ . Suppose that  $v_l \in N_G(v_j)$ . If  $v_l$  is not adjacent to  $v_i$  in  $G$ , then  $\bar{x}$  will be adjacent to  $v_1 v_2 \cdots v_{t-2} v_i v_l$  in  $S[G, t]$ . Also, if  $v_l \in N_G(v_i)$ , then according to Definition 2.1,  $\bar{x}$  is adjacent to  $v_1 v_2 \cdots v_{t-2} \{v_i, v_l\}_2$ . Similarly, the argument applies to  $\bar{\bar{x}}$ . If  $z = v_1 v_2 \cdots v_{t-2} v_i v_l \in N_{S[G,t]}(x)$ , then it is obvious that  $z \in N_{S[G,t]}(\bar{x})$ , thus,  $v_l \in N_G(v_j)$  and  $v_l \notin N_G(v_i)$ . Now, if  $z_1 = v_1 v_2 \cdots v_{t-2} \{v_i, v_l\}_2 \in N_{S[G,t]}(x)$ , then,  $\bar{z}_1 = v_1 v_2 \cdots v_{t-2} v_i v_l$  is adjacent to  $\bar{x}$ . Thus,  $v_l \in N_G(v_j) \cap N_G(v_i)$ .

(iii) Let  $v_l \in N_G(v_t)$ . Then,  $y = v_1 v_2 \cdots v_t$  is adjacent to  $v_1 v_2 \cdots v_{t-2} v_{t-1} v_l$ . But, if  $v_l$  and  $v_{t-1}$  are adjacent to each other in  $G$ , then by Definition 2.1,  $y$  is adjacent to  $v_1 \cdots v_{t-2} \{v_{t-1}, v_l\}_2$  in  $S[G, t]$ .  $\square$

**Proposition 2.10.** *Let  $G = (V, E)$  be a simple graph and  $t \geq 1$  be an integer. Then,*

$$\deg_{S[G,t]}(x) = \begin{cases} \deg_G(v_i) + \deg_G(v_j) & \text{if } x = v_1 v_2 \cdots v_r \{v_i, v_j\}_{t-r}, \\ \deg_G(v_t) & \text{if } x = v_1 v_2 \cdots v_t. \end{cases}$$

*Proof.* For  $x = v_1 v_2 \cdots v_r \{v_i, v_j\}_{t-r}$ , according to Theorem 2.9, it is clear that  $\deg_{S[G,t]}(x) = \deg_G(v_i) + \deg_G(v_j)$ . Let  $x = v_1 v_2 \cdots v_t$ . For  $t = 1$ , we have  $\deg_{S[G,1]}(v_i) = \deg_G(v_i)$ . Now, we assume that  $t \geq 2$ . Note that, if  $N_G(v_t) = \emptyset$ , then  $x = v_1 v_2 \cdots v_t$ , is an isolated vertex in  $S[G, t]$ . Therefore, if  $\deg_G(v_t) = 0$ , then  $\deg_{S[G,t]}(v_1 \cdots v_t) = 0$ , and the result follows. Let  $\deg_G(v_t) = l \geq 1$ , and  $N_G(v_t) = \{z_1, \dots, z_l\}$ . For any  $i \in \{1, 2, \dots, l\}$ ,  $x$  is adjacent to  $p_i = v_1 v_2 \cdots v_{t-1} z_i$  in  $S[G, t]$ . Hence,  $\deg_{S[G,t]}(v_1 v_2 \cdots v_t) \geq l = \deg_G(v_t)$ . Now, suppose that  $k = k_1 k_2 \cdots k_t$  be a neighbour of  $x$  in  $S[G, t]$ . Therefore, by part (iii) of previous theorem,  $v_i = k_i$  for  $i \leq t - 1$  and  $v_t \neq k_t$ . Also,  $v_t$  is adjacent to  $k_t$  in  $G$ . Then  $x$  is adjacent to  $v_1 \cdots v_{t-1} k_t$ . Thus,  $k_t \in N_G(v_t) = \{z_1, \dots, z_l\}$ , i.e.  $k \in p_i$ . Hence,

$$\deg_{S[G,t]}(v_1 v_2 \cdots v_t) = \deg_G(v_t).$$

$\square$

**Remark 2.11.** (i) *If  $G$  is an  $r$ -regular, then  $S[G, t]$  is  $(r, 2r)$ -regular.*

(ii) *In general, for every  $e = uv \in E(G)$ , we have  $\deg(e) = \deg(u) + \deg(v)$ . Now, suppose that  $w$  is a vertex of type II. Since  $w$  has been an edge in the generalized Sierpiński graph which is contracted in  $S[G, t]$ , by Proposition 2.10, we have  $\deg(w) = \deg(e) = \deg(u) + \deg(v)$ .*

**Corollary 2.12.** *Let  $G = (V, E)$  be a graph of order  $n \geq 2$  and  $t \geq 1$  be a positive integer. Then,*

- (i)  $\delta(S[G, t]) = \delta(G)$ ,
- (ii)  $\Delta(S[G, t]) = \max\{\deg(x) + \deg(y) \mid xy \in E(G)\}$ .

*Also, we have  $\Delta(S(G, t)) = 1 + \Delta(G) \leq \Delta(S[G, t]) \leq 2\Delta(G)$ , when  $E(G) \neq \emptyset$  and  $t \geq 2$ .*

**Theorem 2.13.** *Suppose that  $\delta(G) \leq k \leq \Delta(G)$ , then the number of vertices with degree  $k$  in  $S[G, t]$  is*

$$|V_k|n^{t-1} - \frac{n^{t-1} - 1}{n - 1}(k|V_k| - a_k).$$

*where  $V_k = \{v \in V : \deg_G(v) = k\}$  and  $a_k = |\{e = xy \in E(G) \mid \deg_G(x) + \deg_G(y) = k\}|$ .*

*Proof.* The result follows for  $t = 1$ . Let  $t \geq 2$  and  $z \in V_k$ . Set  $\Theta_{k,z} = \{v_1 \cdots v_{t-1}z \in V(S[G, t]), v_1, v_2, \dots, v_{t-1} \in V(G)\}$ . Let  $\Theta_k = \cup_{z \in V_k} \Theta_{k,z}$ . By Proposition 2.10, it is clear that the number of vertices of type I with degree  $k$  is  $|\Theta_k| = \sum_{z \in V_k} |\Theta_{k,z}| = n^{t-1} |V_k|$ .

On the other hand, let  $\Gamma$  be the set of all vertices of type II of form  $w = v_1v_2 \cdots v_r\{v_i, v_j\}_{t-r}$  in  $S[G, t]$ , such that  $\deg(v_i) = p$ ,  $\deg(v_j) = q$ . By Remark 2.11 and Proposition 2.10,  $\deg(w) = \deg(v_i) + \deg(v_j) = p + q$ . Let  $a_k$  denote the number of edges of  $G$  in which the sum of degrees of ending vertices is  $k$ , hence, there are  $\sum n^r a_k$  vertices of type II with degree  $k$  in  $S[G, t]$ . By Definition 2.1, contracting the vertices of degree  $k$  yields vertices of degree more than  $k$ .

Therefore, the number of these vertices in generalized Sierpiński gasket graph is equal to  $\sum n^r (k|V_k|)$ .

Hence, the number of vertices of degree  $k$  in  $S[G, t]$  is

$$|V_k|n^{t-1} - \frac{n^{t-1} - 1}{n - 1}(k|V_k| - a_k).$$

□

### 3. Hamiltonicity and Eulerian

A graph  $G$  is called Eulerian if and only if has a spanning closed trail containing all edges of the graph. On the other hand, the graph  $G$  is Eulerian if and only if  $G$  is connected and all its vertices have even degree. Graph  $G$  is called Hamiltonian when it contains a cycle that travels exactly once over each vertex in a graph.

In [27], demonstrated by Tequia and Godbole that generalized Sierpiński gasket graphs are Hamiltonian, and this graph is Eulerian if and only if  $n$  is odd.

Also, in [13], Jakovac proved that generalized Sierpiński gasket graphs for  $t \geq 1$  and  $n \geq 3$  are Hamiltonian. In this section, we obtain two properties of Hamiltonian and Eulerian for generalized Sierpiński gasket graph.

**Proposition 3.1.** *Let  $G$  be a simple graph with  $n \geq 4$  and  $\delta(G) \leq 2$ , then for  $t \geq 2$ ,  $S[G, t]$  is not Hamiltonian.*



*Proof.* First, assume that  $G$  is a graph with a minimum degree of 2. Suppose that  $v_i$  is a vertex of  $G$  that  $\deg(v_i) = 2$ , and let  $v_j$  and  $v_k$  be its neighbors. Consider  $[V'_i]$  that  $V'_i = \{v_i^{t-1}v_l | v_l \in V(G)\} \cup v_i^{t-2}\{v_i, N(v_i)\}$ , where  $v_i^{t-2}\{v_i, N(v_i)\} = \{v_i^{t-2}\{v_i, v_j\}, v_i^{t-2}\{v_i, v_k\}\}$ . It is evident that  $v_i^{t-2}\{v_i, N(v_i)\}$  is a separating set for  $[V'_i]$ . According to Proposition 2.10,  $\deg(v_i^t) = \deg(v_i) = 2$ . On the other hand, by Theorem 2.9, the vertex  $v_i^t$  is only adjacent to  $v_i^{t-2}\{v_i, v_j\}$  and  $v_i^{t-2}\{v_i, v_k\}$ . Since  $G$  has at least 4 vertices, by removing  $v_i^{t-2}\{v_i, N(v_i)\}$ , the induced subgraph  $[V'_i]$  has at least two components, one of them is the isolated vertex  $v_i^t$ . By Definition 2.1, the vertices  $v_i^{t-2}\{v_i, N(v_i)\}$ , connect  $[V'_i]$  to other copies of  $S[G, t]$ . Every path between  $v_i^{t-1}v_l$  and  $v_i^{t-2}\{v_i, N(v_i)\}$ , does not contain a vertex of type II, thus, deleting the vertices  $v_i^{t-2}\{v_i, N(v_i)\}$ , makes at least three components for  $S[G, t]$ . Therefore,  $S[G, t]$  is not Hamiltonian.

Now, suppose that  $G$  has a vertex  $v_i$  with  $\deg(v_i) = 1$ , let  $v_j \in N(v_i)$ . Then,  $v_i^{t-2}\{v_i, v_j\}$  is a cut vertex for  $S[G, t]$ . Hence,  $S[G, t]$  is not Hamiltonian and the proof is complete.  $\square$

In Figure 5, we see that deleting two vertices  $3\{2, 3\}$  and  $3\{3, 4\}$  makes three components for the graph and hence  $S[C_4, 3]$  is not Hamiltonian.

**Remark 3.2.** *If  $G$  is Hamiltonian, then, according to Proposition 3.1, we cannot guarantee that  $S[G, t]$  is Hamiltonian. For example  $C_4$  is Hamiltonian, but  $S[C_4, t]$  for any  $t \geq 2$  is not Hamiltonian.*

**Theorem 3.3.** *Let  $G$  be a simple graph having no isolated vertex, then  $G$  is Eulerian, if and only if  $S[G, t]$  is Eulerian.*

*Proof.* First, let  $G$  be an Eulerian graph. Then  $G$  is connected and every vertex of this graph has an even degree. According to Remark 2.5 and Proposition 2.10,  $S[G, t]$  is connected and even. Hence,  $S[G, t]$  is Eulerian. Now, assume that  $S[G, t]$  is Eulerian. By Remark 2.5,  $G$  is connected. If  $G$  contains a vertex  $v_i \in V(G)$  with odd degree, then by Proposition 2.10 the degree of vertex  $v_i v_i \cdots v_i$  in  $S[G, t]$  is equal to  $\deg_G(v_i)$  which is odd, a contradiction. Thus, each vertex of  $G$  has even degree and hence,  $G$  is Eulerian.  $\square$

#### 4. General first Zagreb index

Chemical applications of graph theory determine a lot of properties such as thermodynamic and physicochemical properties, and chemical and biological activities. We can express these properties as topological indices. Researchers in [12], and [24] determined some of the topological indices of Sierpiński networks and Sierpiński graphs. The general first Zagreb index of a graph  $G$  is defined as follows [20]:

$$Z_\alpha(G) = \sum_{\{u,v\} \in E(G)} ((\deg_G(u))^{\alpha-1} + (\deg_G(v))^{\alpha-1}) = \sum_{u \in V(G)} (\deg_G(u))^\alpha.$$

Where  $\alpha \neq 0, 1$  is a real number. In particular, if  $\alpha = 2$ , then the general first Zagreb index, is called the first Zagreb index ( $Z_2(G) = M_1(G)$ ). Also, if  $\alpha = 3$ , then the general first Zagreb index is called

the forgotten topological index ( $Z_3(G) = F(G)$ ), [1], [3], and [21]. In this section, we obtain a general Zagreb topological index (see [14]) for generalized Sierpiński gasket graphs.

**Theorem 4.1.** *Let  $G$  be a simple graph of order  $n \geq 2$ . For each integer  $\alpha \geq 0$ , the general first Zagreb index of the generalized Sierpiński gasket graph  $S[G, t]$ ,  $t \geq 1$  is given by*

$$Z_\alpha(S[G, t]) = n^{t-1}Z_\alpha(G) - \frac{n^{t-1} - 1}{n - 1}Z_{\alpha+1}(G) + \frac{n^{t-1} - 1}{n - 1} \sum_{k=2}^{2\Delta(G)} a_k \cdot k^\alpha.$$

*Proof.* According to the definition of the general first Zagreb index and by Theorem 2.13,  $Z_\alpha(S[G, t])$  is equal to

$$\begin{aligned} Z_\alpha(S[G, t]) &= \sum_{k=1}^{2\Delta(G)} (|V_k| n^{t-1} - \frac{n^{t-1} - 1}{n - 1}(k |V_k| - a_k)) k^\alpha \\ &= \sum_{k=1}^{2\Delta(G)} |V_k| n^{t-1} k^\alpha - \frac{n^{t-1} - 1}{n - 1} \sum_{k=1}^{2\Delta(G)} k |V_k| k^\alpha \\ &\quad + \frac{n^{t-1} - 1}{n - 1} \sum_{k=1}^{2\Delta(G)} a_k \cdot k^\alpha \\ &= n^{t-1}Z_\alpha(G) - \frac{n^{t-1} - 1}{n - 1}Z_{\alpha+1}(G) + \frac{n^{t-1} - 1}{n - 1} \sum_{k=2}^{2\Delta(G)} a_k \cdot k^\alpha. \end{aligned}$$

□

**Corollary 4.2.** *Let  $G$  be a  $k$ -regular graph. Then we have*

$$Z_\alpha(S[G, t]) = n^{t-1}Z_\alpha(G) - \frac{n^{t-1} - 1}{n - 1}(Z_{\alpha+1}(G) - m(2k)^\alpha).$$

*Especially, if  $G = C_n$  then,*

$$Z_\alpha(S[C_n, t]) = n^{t-1}Z_\alpha(C_n) - \frac{n^{t-1} - 1}{n - 1}(Z_{\alpha+1}(C_n) - m(4)^\alpha).$$

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