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ON METRIC DIMENSION OF EDGE COMB PRODUCT OF VERTEX-TRANSITIVE GRAPHS

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Abstract. Suppose finite graph G is simple, undirected and connected. If W is an ordered set of the vertices such that $|W| = k$, the representation of a vertex v is an ordered k-tuple consisting distances of vertex v with every vertices in W. The set W is defined as resolving vertex of G if the k-tuples of every two vertices are distinct. Metric dimension of G, which is denoted by $dim(G)$, is the lowest size of W. In this paper, we provide a sharp lower bound of metric dimension for edge comb product graphs $G \cong T \rhd_e H$ where T is a tree graph and H is a vertex-transitive graph. Moreover, we determine the exact value of metric dimension for edge comb product graphs $G \cong T \rhd_e Ci_n(1,2)$ where $Ci_n(1,2)$ is a circulant graph.

1. Introduction

All graphs $G = (V, E)$ considered in this paper are simple, finite, undirected, and connected. The length of a shortest path between two vertices $u, v \in V(G)$ is denoted by $d(u, v)$. Let $W = \{w_1, w_2, w_3, \ldots, w_k\}$ be an ordered set of vertices of G. The representation $r(v|W)$ of v with respect to W is defined as k-tuple as follows.

$$r(v|W) = (d(v, w_1), d(v, w_2), d(v, w_3), \ldots, d(v, w_k))$$

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The set W is defined as resolving set of G if $r(u|W) \neq r(v|W)$ for every two distinct vertices u and v. The vertices $w \in W$ resolve the graph, hence it is called resolving vertices. The metric dimension of G is the smallest size of resolving set of G, denoted by $dim(G)$.

Metric dimension was firstly introduced by Slater [16], and Harary and Melter [7] independently. They present some characterization of metric dimension for tree graphs, which is also proven using different approach by Chartrand et al. [4]. Circulant graphs Ci_n are one of the class graph whose metric dimension is already determined by [2], [8], [17] which some Harary graphs are isomorphic to certain circulant graphs. Meanwhile, prism graphs D_n is obtained by cartesian product from cycles C_n and paths with the length of two P_2. Some results of [2] and [3] are presented with equivalent definitions.

Theorem 1.1. [2] *Let $Ci_n(1,2)$ be a circulant graph with $n \geq 5$. Then $dim(Ci_n(1,2)) = 3$ when $n \equiv 0, 2, 3$ mod 4, and $dim(Ci_n(1,2)) = 4$ otherwise.*

Theorem 1.2. [3] *Let $D_n = C_n \square P_2$ be a prism graph with $n \geq 4$. Then $dim(D_n) = 3$ if n is even and $dim(D_n) = 2$ if n is odd.*

Some results of metric dimension has been found for other kind of graphs including amalgamation of cycles [9], block-cactus graphs [10], corona product graphs [18], chain graphs [5], and comb product graphs [14]. Some applications regarding to metric dimension are found in robot navigation [13], chemical structures visualization [11], [12], and strategy of mastermind game [6].

Let G and H be two connected graphs, $|E(G)| = m$, and $e \in E(H)$. The edge comb product between G and H denoted by $G \rhd_e H$ is a graph obtained by taking one copy of G and m copies of H and identify the i-th copy of H at the edge e with p_i, the i-th edge of G. This convention is introduced by Slamin et al. [15]. There is also other convention by Awanis et al. [1] which considers the orientation of the edge. However, our choice of H ensure us that the orientation for the definition of the edge comb product in this case does not matter. An automorphism is a permutation of vertices in $V(G)$ which preserves adjacency. Moreover, a graph is vertex-transitive if for every two distinct vertices $u, v \in V(G)$ there exists an automorphism that maps u to v.

If we choose $H \cong K_n$ the complete graph or $H \cong C_n$ the cycle graph, the edge comb product graph would be a special case of block-cactus graph. According to [10], we can present the metric dimension of graph which is isomorphic to the edge comb product of some graph.

Theorem 1.3. [10] *Let T be any tree whose size is m and K_n be a complete graph with $n \geq 3$, then*

$$dim(T \rhd_e K_n) = m(n-2) + p$$

where p is the number of pendants in T.

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Theorem 1.4. [10] *Let T be any tree and C_n be a cyclic graph with $n \geq 4$ and $d(v)$ denote the degree of vertex v. We have*

$$dim(T \rhd_e C_n) = \begin{cases} \sum_{v \in V(T)}(d(v)-1) & \text{for } n \text{ is odd,} \\ \sum_{v \in V(T)} d(v) - p & \text{for } n \text{ is even.} \end{cases}$$

In this paper, we study metric dimension of edge comb product graphs $G \rhd_e H$, with G is a tree or path graphs and H is a vertex-transitive graph which includes antiprisms, circulant graph, and prisms. Nevertheless, antiprisms are just special case of circulant graph, so that we only consider the case for circulant graphs and prisms. Because of isomorphism, there are few types of edges which we can pick to produce distinct edge comb product graphs in circulant graph and prisms.

2. Edge Comb Product of Trees and Vertex-Transitive Graphs

Suppose T and H are any tree graphs and vertex-transitive graphs respectively whose order is at least three. Let p_i be the i-th edge of T. We define $H_i \cong H$ a subgraph of $T \rhd_e H$ which is identified to an edge p_i. Pendant edge is defined as an edge which contains pendant. We call a subgraph H_i which is identified to a pendant edge p_i as an extreme. It is found that the extremes contain at least one resolving vertex.

Proposition 2.1. *Let T be a tree graph, H be a vertex-transitive graph, and $H_i \cong H$ be a connected vertex-transitive subgraph of $T \rhd_e H$. Suppose W is a resolving set for $T \rhd_e H$ and H_i is the extreme of the graph. Then, there exists $v_{i,j} \in H_i$ which belongs to W for some j.*

Proof. Suppose the contrary that any $v_{i,j}$ does not belong to W, then there exists $v_{k,l}$ which belongs to W for some l while $k \neq i$. Since H_i is vertex-transitive while $H_i \neq K_2$, there exist at least two vertices v_{i,m_1}, v_{i,m_2} which are adjacent to $v_{i,1}$. All shortest path from any resolving vertices to both v_{i,m_1} and v_{i,m_2} contain $v_{i,1}$, hence $r(v_{i,m_1}|W) = r(v_{i,m_2}|W)$. The assumption is false since W supposed to be a resolving set. □

For any $H_i \cong H$, which is a subgraph of $T \rhd_e H$, there would be only one or two vertices v which are adjacent to vertices outside H_i. Consequently, any paths from vertices of H_i to outside of H_i would always contain v. Combining this fact with Proposition 2.1, which ensure the existence of resolving vertex in the extremes, we get the following corollary.

Corollary 2.2. *Let T be a tree graph and H is a vertex-transitive graph. Let $H_i \cong H$ be a connected vertex-transitive subgraph of $T \rhd_e H$ which v is a vertex adjacent to other vertices outside H_i. Then, there exists x, a vertex outside H_i, which is a resolving vertex such that all shortest path from x to any vertices in H_i would contain v.*

Having one $x \in W$, vertex outside of H_i, may not be sufficient to resolve H_i. It is reasonable to add more vertices x to W in order to resolve the subgraph. However, the proceeding proposition tells us that the number of $x \in W$ does not matter.

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Proposition 2.3. *Let T be a tree graph and H is a vertex-transtive graph. Let $H_i \cong H$ be a connected vertex-transitive subgraph of $T \rhd_e H$ which v is a vertex adjacent to other vertices outside H_i. Suppose there exists x_1 and x_2, vertices outside H_i, such that all shortest path from x_1 or x_2 to any vertices in H_i would contain v. If W_1 is a subset of $V(T \rhd_e H)$ such that $W_1 \cup \{x_1\}$ does not resolve H_i, then $W_1 \cup \{x_1, x_2\}$ does not resolve H_i.*

Proof. Suppose $d(x_1, v) = a$. If there exists other resolving vertex x_2 outside H_1 with $d(x_2, v) = b$, the distance would be $d(x_2, v) = d(x_1, v) + (b - a)$. Let v_1, v_2 be vertices in $V(T \rhd_e H)$ such that $d(x_1, v_1) = d(x_1, v_2)$. We have $d(x_2, v_1) = d(x_1, v_1) + (b-a) = d(x_1, v_2) + (b-a) = d(x_2, v_2)$. Hence, if $r(v_1|W_1 \cup \{x_1\}) = r(v_2|W_1 \cup \{x_1\})$ then $r(v_1|W_1 \cup \{x_1, x_2\}) = r(v_2|W_1 \cup \{x_1, x_2\})$ for $v_1, v_2 \in V(H_1)$. □

Next, our concern is shifted to the representation between vertices in H_i and vertices outside H_i, namely in H_j, $i \neq j$. By having some vertices of the extremes contained in W, if $V(H_i) \cap V(H_j) = \varnothing$ (the graph H_i is not adjacent to H_j), then the representation of the vertices may be assured to be different. The illustration of such occurrence is depicted in Figure 1.

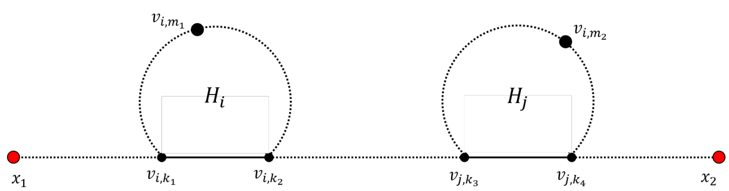

FIGURE 1. Vertices in H_i and H_j are resolved by x_1 and x_2 (resolving vertex denoted by red color).

Lemma 2.4. *Let T be a tree graph and H is a vertex-transitive graph with $q = |V(H)|$. Suppose W is a subset of $V(T \rhd_e H)$ which contains some resolving vertices in the extremes. If $H_i, H_j \cong H$ such that $V(H_i) \cap V(H_j) = \varnothing$, then*

$$r(v_{i,m_1}|W) \neq r(v_{j,m_2}|W)$$

for any $v_{i,m_1} \in H_i$ and $v_{j,m_2} \in H_j$.

Proof. Define p_i and p_j, the edges which H_i and H_j respectively are identified to, and let $v_{i,k_1} v_{i,k_2} = p_i$ and $v_{j,k_3} v_{j,k_4} = p_j$. Without loss of generality, let the sequence $v_{i,k_1}, v_{i,k_2}, \ldots, v_{j,k_3}, v_{j,k_4}$ be a path with $d(v_{i,k_2}, v_{j,k_3}) = r \geq 1$.

By Corollary 2.2, there exists a resolving vertex x_1 such that all shortest path from both $v_{i,l}$ and $v_{j,l}$ to x_1 contain v_{i,k_1} for $1 \leq l \leq q$. Likewise, there exists a resolving vertex x_2 such that all shortest path from both $v_{i,l}$ and $v_{j,l}$ to x_2 contain v_{j,k_4} for $1 \leq l \leq q$. Suppose $d(x_1, v_{i,k_2}) = a$ and $d(x_2, v_{j,k_3}) = b$. Pick any vertex v_{j,m_2}, and let $c = d(v_{j,m_2}, v_{j,k_3})$. Henceforth, we have

$$d(v_{j,m_2}, v_{i,k_2}) = c + r$$

$$d(v_{j,m_2}, x_1) = a + c + r$$

Suppose there exists $d(v_{j,m_2}, x_1) = d(v_{i,m_1}, x_1) = a + c + r$ for some m_1. Since $d(x_1, v_{i,k_2}) = a$, there are two possibilities:

- if the shortest path from v_{i,m_1} to x_1 contain v_{i,k_2} then $d(v_{i,m_1}, v_{i,k_2}) = c + r$,
- otherwise $d(v_{i,m_1}, v_{i,k_2}) > c + r$.

For both cases, $d(v_{i,m_1}, v_{i,k_2}) \geq c + r$ which implies

$$d(v_{i,m_1}, x_2) = d(v_{i,m_1}, v_{i,k_2}) + d(v_{i,k_2}, x_2) \geq b + c + 2r$$

Similarly, we check $d(v_{j,m_2}, x_2)$ which have two possibilities:

- if the shortest path from v_{j,m_2} to x_2 does not contain v_{j,k_3} then $d(v_{j,m_2}, x_2) < b + c$,
- otherwise, $d(v_{j,m_2}, x_2) = b + c$.

For those cases, we have $d(v_{j,m_2}, x_2) \leq b + c$. Therefore, it may be concluded that

$$d(v_{i,m_1}, x_2) \geq b + c + 2r > b + c \geq d(v_{j,m_2}, x_2)$$

which implies $r(v_{i,m_1}|W) \neq r(v_{j,m_2}|W)$. □

What is left to consider is the representations of vertices in H_i and H_j where $V(H_i) \cap V(H_j) \neq \varnothing$. Such process would be presented in the next section.

Next, we would check the necessary conditions for establishing W as a resolving set of edge comb product graphs. We show the necessary condition for subset of vertices W to be a resolving set. The condition is different for extremes and non-extremes. Note that the number of extremes and the number of pendant edges in the graph are equivalent.

Lemma 2.5. *Let T be a tree graph and H is a vertex-transitive graph. For every pendant edge (non-pendant edge) in T, $T \rhd_e H$ would require at least $dim(H) - 1$ ($dim(H) - 2$) resolving vertices.*

Proof. Let $H_i \cong H$ be a subgraph of $T \rhd_e H$ and e be a pendant edge (non-pendant edge). Let v_1 (and v_2) be vertices which are adjacent to vertices outside H_i. Since the graph is vertex-transitive, let W_H be a resolving set of H such that $v_1 \in W_H$. By Corollary 2.2, there exists a resolving vertex x_1 outside H_i such that all shortest path from x_1 to any vertices in H_i would contain v_1. If e is non-pendant edge, there exists a resolving vertex x_2 outside H_i such that all shortest path from x_2 to any vertices in H_1 would contain v_2. For any vertex u in H_i, $d(x_1, u) = d(x_1, v_1) + d(u, v_1)$ (if x_2 exists, $d(x_2, u) = d(x_2, v_2) + d(u, v_2)$).

If e is a pendant edge (or v_2 is not in W_H), assume there exists V_H consists of $dim(H) - 2$ resolving vertices in H_i such that $\{x_1\} \cup V_{H_0}$ is sufficient to resolve H_i. Then, $V_{H_0} \cup \{v_1\}$ is a resolving set of H, but it contradicts the fact that $dim(H)$ is the smallest cardinality of a resolving set for H.

Then, if e is not a pendant edge and v_2 is in W_H, assume there exists V_H consists of $dim(H) - 3$ resolving vertices in H_i such that $\{x_1, x_2\} \cup V_{H_0}$ is sufficient to resolve H_i. Then, $V_{H_0} \cup \{v_1, v_2\}$ is a resolving set of H, but it contradicts the fact that $dim(H)$ is the smallest cardinality of a resolving set for H. □

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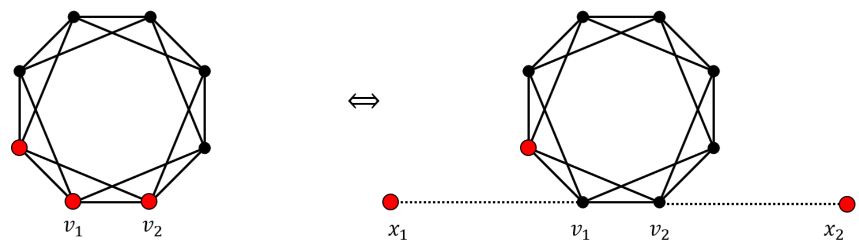

FIGURE 2. The correspondence of resolving sets between graphs H and subgraphs H_i.

Figure 2 illustrates how the Lemma 2.5 works for non-pendant edge. By counting the necessary numbers of resolving vertices in all subgraphs H_i, we have the following theorem.

Theorem 2.6. *Let T be a tree graph whose size is m, H is a vertex-transitive graph, and p is the cardinality of pendants in T. We have*

$$dim(T \rhd_e H) \geq m(dim(H) - 2) + p$$

Proof. The lower bound is assured by applying either Lemma 2.5 for all edges. □

Notice that as stated in Theorem 1.3, $T \rhd_e K_n$ is one of the graph satisfying the equality in Theorem 2.6, while $T \rhd_e C_n$ is not. We determine the metric dimension when H is a circulant graph in the next section.

3. Edge Comb Product of Trees and Circulant Graphs

For this section, let T be any tree graph and $H \cong Ci_n(1, 2)$ for $n \geq 6$. For convenience, pick any pendant in T and call it x. Then, the graph of $G \cong T \rhd_e H$ has the vertex set

$$V(G) = \{v_{i,j} | 1 \leq i \leq q, 1 \leq j \leq n\}$$

with $x = v_{1,1}$ and $v_{a,n} = v_{b,1}$ if e_a, e_b in $E(T)$ are adjacent while the shortest path from x to e_b contain e_a. The graph has the edge set

$$E(G) = \{v_{i,j}v_{i,j+1}, v_{i,j}v_{i,j+2} | 1 \leq i \leq q, 1 \leq j \leq n\}$$

for j is taken modulo n.

For $T \rhd_e H$, picking $e = v_1v_2$ or $e = v_1v_3$ will produce different graph, but any other choice of e would produce a graph which isomorphic to either of them. For $1 \leq i \leq n-1$, choosing $e = v_iv_{i+1}$ or $e = v_nv_1$ would produce a same graph, hence it would only be necessary to prove $e = v_1v_2$. For $1 \leq i \leq n-2$, choosing $e = v_iv_{i+2}$, $e = v_{n-1}v_1$ or $e = v_nv_2$ would produce a same graph, hence it would only be necessary to prove $e = v_1v_3$.

Our goal is to prove the proceeding main theorem.

Theorem 3.1. *Let T be a tree graph whose size is m. For $n \geq 6$,*

$$dim(T \rhd_e Ci_n(1,2)) = \begin{cases} m+p & if\ n \not\equiv 1 \ (mod\ 4), \\ 2m+p & if\ n \equiv 1 \ (mod\ 4). \end{cases}$$

with p is the cardinality of pendants.

In order to prove this, we will show a lemma which accumulates all resolving sets of subgraphs of $T \rhd_e Ci_n(1,2)$.

Lemma 3.2. *Let $H \cong Ci_n(1,2)$. For every subgraph $K_{1,2}$ of T, l vertices are sufficient to resolve $K_{1,2} \rhd_e Ci_n(1,2)$ with*

$$l = \begin{cases} 4+\alpha, & \text{if the subgraph } K_{1,2} \text{ contains two pendant edge,} \\ 3+\alpha, & \text{if the subgraph } K_{1,2} \text{ contains exactly one pendant edge,} \\ 2+\alpha, & \text{otherwise.} \end{cases}$$

where

$$\alpha = \begin{cases} 0, & if\ n \not\equiv 1 \ (mod\ 4), \\ 2, & if\ n \equiv 1 \ (mod\ 4). \end{cases}$$

The proof will consists of two parts, general preparations for all cases and determining the tuples of every vertices in the graph by cases according to n.

Proof. Without loss of generality, suppose G_1 is the subgraph of $T \rhd_e H$ which has the vertex set

$$V(G_1) = \{v_{1,j}, v_{2,j} | 1 \leq j \leq n\}$$

with $v_{1,n} = v_{2,1}$ and the edge set

$$E(G_1) = \{v_{i,j}v_{i,j+1}, v_{i,j}v_{i,j+2} | i \in \{1,2\}, 1 \leq j \leq n\}$$

for j is taken modulo n. Notice that for every $e \in E(G)$, there exists an automorphism which maps e to either v_1v_2 or v_1v_3. Hence, we only need to prove the choice of e either $e = v_1v_2$ or $e = v_1v_3$.

Case 1. The subgraph $K_{1,2}$ contains two pendant edges or $e = v_1v_2$.

Since $Ci_n(1,2)$ is vertex-transitive, without loss of generality we can set the following.

(1) If the subgraph $K_{1,2}$ contains a pendant edge, let $v_{2,n}$ be a vertex adjacent to vertices outside of G_1. By Corollary 2.2, there exists a resolving vertex x_1 outside G_1 such that all shortest path from x_1 to any vertices in G_1 would contain $v_{2,n}$.

(2) In addition, if the subgraph $K_{1,2}$ contains no pendant edges, let both $v_{1,1}$ be another vertex adjacent to vertices outside of G_1. Similarly by Corollary 2.2, there exists a resolving vertex x_2 outside G_1 such that all shortest path from x_2 to any vertices in G_1 would contain $v_{1,1}$.

Let W_k be a subset of vertex set of $K_{1,2} \rhd_e Ci_n(1,2)$. An additional vertex set U_k is needed if $n \equiv 1 \pmod 4$, since the metric dimension of $Ci_n(1,2)$ is larger than other choice of n. We set

- If $n \not\equiv 1 \pmod 4$, let

$$W_k = \begin{cases} \{v_{1,1}, v_{1,n-1}, v_{2,2}, v_{2,n}\} & \text{if the subgraph } K_{1,2} \text{ contains two pendant edges,} \\ \{v_{1,1}, v_{1,n-1}, v_{2,2}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains exactly one pendant edge,} \\ \{v_{1,2}, v_{2,2}, x_2, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains no pendant edges.} \end{cases}$$

- If $n \equiv 1 \pmod 4$, let

$$W_k = \begin{cases} \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,3}, v_{2,n}\} & \text{if the subgraph } K_{1,2} \text{ contains two pendant edges,} \\ \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,n-1}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains exactly one pendant edge,} \\ \{v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,n-1}, x_2, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains no pendant edges.} \end{cases}$$

We will prove W_k is a resolving set after the proceeding case.

Case 2. The subgraph $K_{1,2}$ consists no more than one pendant edge and $e = v_1v_3$.
Since $Ci_n(1,2)$ is vertex-transitive, without loss of generality we can set the following.

(1) Let $v_{2,n-1}$ be a vertex adjacent to vertices outside of G_1. By Corollary 2.2, there exists a resolving vertex x_1 outside G_1 such that all shortest path from x_1 to any vertices in G_1 would contain $v_{2,n-1}$.

(2) Moreover, if the subgraph $K_{1,2}$ contains no pendant edges, let both $v_{1,2}$ be another vertex adjacent to vertices outside of G_1. Again by Corollary 2.2, there exists a resolving vertex x_2 outside G_1 such that all shortest path from x_2 to any vertices in G_1 would contain $v_{1,2}$.

In the proof of Theorem 3.1, we will pick an initial pendant edge in order to accumulate all resolving sets of the subgraph. This resolving set is different with other pendant edges. We will elaborate this later in the proof of the theorem.

Let W_k be a subset of vertex set of $K_{1,2} \rhd_e Ci_n(1,2)$. We set W_k as follows.

- If $n \equiv 0 (\text{mod } 2)$, let

$$W_k = \begin{cases} \{v_{1,1}, v_{1,n-1}, v_{2,3}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains an initial pendant edge,} \\ \{v_{1,1}, v_{1,n-1}, v_{2,n-3}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains other pendant edge,} \\ \{v_{1,4}, v_{2,3}, x_2, x_1\}, & \text{otherwise.} \end{cases}$$

- If $n \equiv 3 (\text{mod } 4)$, let

$$W_k = \begin{cases} \{v_{1,1}, v_{1,n-1}, v_{2,2}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains an initial edge,} \\ \{v_{1,1}, v_{1,n-1}, v_{2,n-2}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains other pendant edge,} \\ \{v_{1,3}, v_{2,2}, x_2, x_1\}, & \text{otherwise.} \end{cases}$$

- If $n \equiv 1 \pmod{4}$, let

$$W_k = \begin{cases} \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,n-2}, x_1\}, & \text{if the subgraph } K_{1,2} \text{ contains a pendant edge,} \\ \{v_{1,3}, v_{1,n-1}, v_{2,2}, v_{2,n-2}, x_2, x_1\}, & \text{otherwise.} \end{cases}$$

After everything has set, we will show that W_k is indeed a resolving set. Define a, b, c, d as follows.

$$a = d(x_2, v_{1,1}) \qquad b = d(x_1, v_{2,n})$$
$$c = d(x_2, v_{1,2}) \qquad d = d(x_1, v_{2,n-1})$$

We split the case as follows.

Case 1. $n \equiv 0 (\mathrm{mod}\ 2)$.

Subcase 1.1. The subgraph $K_{1,2}$ contains two pendant edge.

Let $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,2}, v_{2,n}\}$. We have

$$r(v_{1,i}|W_k)\& = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \leq i \leq \frac{n-2}{2}, \\ (\lceil \frac{n-2}{4} \rceil, \lceil \frac{n-2}{4} \rceil, \lceil \frac{n+4}{4} \rceil, \lceil \frac{n+4}{4} \rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+2}{2} \leq i \leq n-1, \\ (1,1,1,1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, \lceil \frac{i}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n}{2}, \\ (\lceil \frac{n+4}{4} \rceil, \lceil \frac{n+4}{4} \rceil, \lceil \frac{n-2}{4} \rceil, \lceil \frac{n-2}{4} \rceil) & \text{for} \quad i = \frac{n+2}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i}{2} \rceil) & \text{for} \quad \frac{n+4}{2} \leq i \leq n. \end{cases}$$

Subcase 1.2. The subgraph $K_{1,2}$ contains exactly one pendant edge and $e = v_1v_2$.

Let $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,2}, x_1\}$. We have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, b + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \leq i \leq \frac{n-2}{2}, \\ (\lceil \frac{n-2}{4} \rceil, \lceil \frac{n-2}{4} \rceil, \lceil \frac{n+4}{4} \rceil, b + \lceil \frac{n+4}{4} \rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad b + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+2}{2} \leq i \leq n-1, \\ (1,1,1,b+1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, b + \lceil \frac{i}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n}{2}, \\ (\lceil \frac{n+4}{4} \rceil, \lceil \frac{n+4}{4} \rceil, \lceil \frac{n-2}{4} \rceil, b + \lceil \frac{n-2}{4} \rceil) & \text{for} \quad i = \frac{n+2}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad b + \lceil \frac{n-i}{2} \rceil) & \text{for} \quad \frac{n+4}{2} \leq i \leq n. \end{cases}$$

Subcase 1.3. The subgraph $K_{1,2}$ contains no pendant edges and $e = v_1v_2$. Recall the W_k used in this case is $W_k\{v_{1,2}, v_{2,2}, x_2, x_1\}$ to yield

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil\frac{|i-2|}{2}\rceil, \lceil\frac{i+2}{2}\rceil, a+\lceil\frac{i-1}{2}\rceil, b+\lceil\frac{i+2}{2}\rceil) & \text{for} \quad 2 \le i \le \frac{n}{2}, \\ (\lceil\frac{n-2}{4}\rceil, \lceil\frac{n+2}{4}\rceil, a+\lceil\frac{n}{4}\rceil, b+\lceil\frac{n+2}{4}\rceil) & \text{for} \quad i = \frac{n+2}{2}, \\ (\lceil\frac{n-i+2}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, a+\lceil\frac{n-i+1}{2}\rceil, & \\ \qquad b+\lceil\frac{n-i+2}{2}\rceil) & \text{for} \quad \frac{n+4}{2} \le i \le n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil\frac{i+1}{2}\rceil, \lceil\frac{i-2}{2}\rceil, a+\lceil\frac{i+1}{2}\rceil, b+\lceil\frac{i}{2}\rceil) & \text{for} \quad 2 \le i \le \frac{n}{2}, \\ (\lceil\frac{n+4}{4}\rceil, \lceil\frac{n-2}{4}\rceil, a+\lceil\frac{n+4}{4}\rceil, b+\lceil\frac{n-2}{4}\rceil) & \text{for} \quad i = \frac{n+2}{2}, \\ (\lceil\frac{n-i+3}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, a+\lceil\frac{n-i+3}{2}\rceil, & \\ \qquad b+\lceil\frac{n-i}{2}\rceil) & \text{for} \quad \frac{n+4}{2} \le i \le n. \end{cases}$$

Subcase 1.4. The subgraph $K_{1,2}$ contains exactly one pendant edge and $e = v_1v_3$. If e is an initial pendant edge, we use $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,3}, x_1\}$ which yields

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil\frac{i-1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i+2}{2}\rceil, d+\lceil\frac{i+2}{2}\rceil) & \text{for} \quad 1 \le i \le \frac{n-2}{2}, \\ (\lceil\frac{n-2}{4}\rceil, \lceil\frac{n-2}{4}\rceil, \lceil\frac{n+4}{4}\rceil, d+\lceil\frac{n+4}{4}\rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil\frac{n-i+1}{2}\rceil, \lceil\frac{n-i-1}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, & \\ \qquad d+\lceil\frac{n-i+2}{2}\rceil) & \text{for} \quad \frac{n+2}{2} \le i \le n-1, \\ (1, 1, 1, d+1) & \text{for} \quad i = n. \end{cases}$$

In this subcase, it would be convenient to exclude the resulting tuples only for small order, which is when $n = 6$. Here, we have

$$r(v_{2,i}|W_k) = \begin{cases} (2, 2, 1, d+2) & \text{for} \quad i = 2, \\ (2, 2, 0, d+1) & \text{for} \quad i = 3, \\ (3, 3, 1, d+1) & \text{for} \quad i = 4, \\ (2, 2, 1, d) & \text{for} \quad i = 5, \\ (2, 2, 2, d+1) & \text{for} \quad i = 6, \end{cases}$$

Else, if $n \ge 8$, we have

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil\frac{i+1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{|i-3|}{2}\rceil, d+\lceil\frac{i+1}{2}\rceil) & \text{for} \quad 2 \le i \le \frac{n-2}{2}, \\ (\lceil\frac{n+2}{4}\rceil, \lceil\frac{n+2}{4}\rceil, \lceil\frac{n-6}{4}\rceil, d+\lceil\frac{n-2}{4}\rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil\frac{n-i+4}{2}\rceil, \lceil\frac{n-i+4}{2}\rceil, \lceil\frac{i-3}{2}\rceil, d+\lceil\frac{n-i-1}{2}\rceil) & \text{for} \quad i \in \{\frac{n+2}{2}, \frac{n+4}{2}\}, \\ (\lceil\frac{n-i+4}{2}\rceil, \lceil\frac{n-i+4}{2}\rceil, \lceil\frac{n-i+3}{2}\rceil, & \\ \qquad d+\lceil\frac{n-i-1}{2}\rceil) & \text{for} \quad \frac{n+6}{2} \le i \le n-1, \\ (2, 2, 2, d+1) & \text{for} \quad i = n. \end{cases}$$

Next, if e is not an initial pendant edge, we use $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,n-3}, x_1\}$ to have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil\frac{i-1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i+4}{2}\rceil, d+\lceil\frac{i+2}{2}\rceil) & \text{for} \quad 1 \le i \le \frac{n-2}{2}, \\ (\lceil\frac{n-2}{4}\rceil, \lceil\frac{n-2}{4}\rceil, \lceil\frac{n+8}{4}\rceil, d+\lceil\frac{n+4}{4}\rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil\frac{n-i+1}{2}\rceil, \lceil\frac{n-i-1}{2}\rceil, \lceil\frac{n-i+4}{2}\rceil, & \\ \qquad d+\lceil\frac{n-i+2}{2}\rceil) & \text{for} \quad \frac{n+2}{2} \le i \le n-1. \end{cases}$$

Again, if $n = 6$ we have

$$r(v_{2,i}|W_k) = \begin{cases} (1,1,2,d+1) & \text{for} \quad i = 1, \\ (2,2,1,d+2) & \text{for} \quad i = 2, \\ (2,2,0,d+1) & \text{for} \quad i = 3, \\ (3,3,1,d+1) & \text{for} \quad i = 4, \\ (2,2,1,d) & \text{for} \quad i = 5, \\ (2,2,2,d+1) & \text{for} \quad i = 6, \end{cases}$$

Else, if $n \ge 8$, we have

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil\frac{i+1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i+3}{2}\rceil, d+\lceil\frac{i+1}{2}\rceil) & \text{for} \quad 1 \le i \le \frac{n-6}{2}, \\ (\lceil\frac{i+1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{n-i-3}{2}\rceil, d+\lceil\frac{n-2}{4}\rceil) & \text{for} \quad i \in \{\frac{n-4}{2}, \frac{n-2}{2}, \} \\ (\lceil\frac{n+2}{4}\rceil, \lceil\frac{n+2}{4}\rceil, \lceil\frac{n-6}{4}\rceil, d+\lceil\frac{n-2}{4}\rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil\frac{n-i+4}{2}\rceil, \lceil\frac{n-i+4}{2}\rceil, \lceil\frac{|n-i-3|}{2}\rceil, & \\ \qquad d+\lceil\frac{n-i-1}{2}\rceil) & \text{for} \quad \frac{n+2}{2} \le i \le n-1, \\ (2,2,2,d+1) & \text{for} \quad i = n. \end{cases}$$

Subcase 1.5. The subgraph $K_{1,2}$ does not contain pendant edge and $e = v_1v_3$. Let $W_k = \{v_{1,4}, v_{2,3}, x_2, x_1\}$. Similar with previous subcase, if $n = 6$ we have

$$r(v_{1,i}|W_k) = \begin{cases} (2,2,c+1,d+2), & \text{for} \quad i = 1, \\ (1,2,c,d+2), & \text{for} \quad i = 2, \\ (1,3,c+1,d+3), & \text{for} \quad i = 3, \\ (0,2,c+1,d+2), & \text{for} \quad i = 4, \\ (1,2,c+2,d+2), & \text{for} \quad i = 5, \\ (1,1,c+1,d+1), & \text{for} \quad i = 6. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (2,1,c+2,d+2), & \text{for} \quad i = 2, \\ (2,0,c+2,d+1), & \text{for} \quad i = 3, \\ (3,1,c+3,d+1), & \text{for} \quad i = 4, \\ (2,1,c+2,d), & \text{for} \quad i = 5, \\ (2,2,c+2,d+1), & \text{for} \quad i = 6. \end{cases}$$

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Else, if $n \geq 8$, we have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{|i-4|}{2} \rceil, \lceil \frac{i+2}{2} \rceil, c + \lceil \frac{|i-2|}{2} \rceil, d + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \leq i \leq \frac{n}{2}, \\ (\lceil \frac{n-6}{4} \rceil, \lceil \frac{n+2}{4} \rceil, c + \lceil \frac{n-2}{4} \rceil, d + \lceil \frac{n+2}{4} \rceil) & \text{for} \quad i = \frac{n+2}{2}, \\ (\lceil \frac{i-4}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, c + \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad i \in \{\frac{n+4}{2}, \frac{n+6}{2}\}, \\ (\lceil \frac{n-i+4}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, c + \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+8}{2} \leq i \leq n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+3}{2} \rceil, \lceil \frac{|i-3|}{2} \rceil, c + \lceil \frac{i+1}{2} \rceil, d + \lceil \frac{i+1}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n-2}{2}, \\ (\lceil \frac{n+6}{4} \rceil, \lceil \frac{n-6}{4} \rceil, c + \lceil \frac{n+2}{4} \rceil, d + \lceil \frac{n-2}{4} \rceil) & \text{for} \quad i = \frac{n}{2}, \\ (\lceil \frac{n-i+6}{2} \rceil, \lceil \frac{i-3}{2} \rceil, c + \lceil \frac{n-i+4}{2} \rceil, & \\ \qquad d + \lceil \frac{n-i-1}{2} \rceil) & \text{for} \quad i \in \{\frac{n+2}{2}, \frac{n+4}{2}\}, \\ (\lceil \frac{n-i+6}{2} \rceil, \lceil \frac{n-i+4}{2} \rceil, c + \lceil \frac{n-i+4}{2} \rceil, & \\ \qquad d + \lceil \frac{|n-i-1|}{2} \rceil) & \text{for} \quad \frac{n+6}{2} \leq i \leq n. \end{cases}$$

Case 2. $n \equiv 3 (\text{mod } 4)$.

Subcase 2.1. The subgraph $K_{1,2}$ contains two pendant edge.
We use $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,2}, v_{2,n}\}$ which yields

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \leq i \leq \frac{n-1}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+1}{2} \leq i \leq n-1, \\ (1, 1, 1, 1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, \lceil \frac{i}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n+1}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \leq i \leq n. \end{cases}$$

Subcase 2.2. The subgraph $K_{1,2}$ contains exactly one pendant edge and $e = v_1 v_2$.
Let $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,2}, x_1\}$. We have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, b + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \leq i \leq \frac{n-1}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad b + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+1}{2} \leq i \leq n-1, \\ (1, 1, 1, b+1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, b + \lceil \frac{i}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n+1}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad b + \lceil \frac{n-i}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \leq i \leq n. \end{cases}$$

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Subcase 2.3. The subgraph $K_{1,2}$ contains no pendant edges and $e = v_1v_2$.

Recall the W_k used in this case is $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,2}, x_1\}$. Therefore, we have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{|i-2|}{2} \rceil, \lceil \frac{i+2}{2} \rceil, a + \lceil \frac{i-1}{2} \rceil, b + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \le i \le \frac{n+1}{2}, \\ (\lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, a + \lceil \frac{n-i+1}{2} \rceil, & \\ \qquad b + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \le i \le n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, a + \lceil \frac{i+1}{2} \rceil, b + \lceil \frac{i}{2} \rceil) & \text{for} \quad 2 \le i \le \frac{n+1}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, a + \lceil \frac{n-i+3}{2} \rceil, & \\ \qquad b + \lceil \frac{n-i}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \le i \le n. \end{cases}$$

Subcase 2.4. The subgraph $K_{1,2}$ contains exactly one pendant edge and $e = v_1v_3$.

If e is an initial edge, we use $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,2}, x_1\}$ yielding

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, d + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \le i \le \frac{n-1}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+1}{2} \le i \le n-1, \\ (1, 1, 1, d+1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, d + \lceil \frac{i+1}{2} \rceil) & \text{for} \quad 2 \le i \le \frac{n-1}{2}, \\ (\frac{n+5}{4}, \frac{n+5}{4}, \frac{n-3}{4}, d + \frac{n-3}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\lceil \frac{n-i+4}{2} \rceil, \lceil \frac{n-i+4}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, d + \lceil \frac{|n-i-1|}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \le i \le n. \end{cases}$$

Meanwhile, if e is not an initial edge, use $W_k = \{v_{1,1}, v_{1,n-1}, v_{2,n-2}, x_1\}$ to yield

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+4}{2} \rceil, d + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \le i \le \frac{n-1}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+4}{2} \rceil, & \\ \qquad d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+1}{2} \le i \le n-1, \\ (1, 1, 2, d+1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, d + \lceil \frac{i+1}{2} \rceil) & \text{for} \quad 2 \le i \le \frac{n-3}{2}, \\ (\frac{n+1}{4}, \frac{n+1}{4}, \frac{n-3}{4}, d + \frac{n+1}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\lceil \frac{n-i+4}{2} \rceil, \lceil \frac{n-i+4}{2} \rceil, \lceil \frac{|n-i-2|}{2} \rceil, & \\ \qquad d + \lceil \frac{|n-i-1|}{2} \rceil) & \text{for} \quad \frac{n+1}{2} \le i \le n. \end{cases}$$

Subcase 2.5. The subgraph $K_{1,2}$ does not contain pendant edge and $e = v_1v_3$.

Recall $W_k = \{v_{1,3}, v_{2,2}, x_2, x_1\}$. We have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{|i-3|}{2} \rceil, \lceil \frac{i+2}{2} \rceil, c + \lceil \frac{|i-2|}{2} \rceil, d + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \le i \le \frac{n-1}{2}, \\ (\frac{n-3}{4}, \frac{n+1}{4}, c + \frac{n+1}{4}, d + \frac{n+1}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, c + \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \le i \le n. \end{cases}$$

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$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+3}{2} \rceil, \lceil \frac{i-2}{2} \rceil, c + \lceil \frac{i+1}{2} \rceil, d + \lceil \frac{i+1}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n-1}{2}, \\ (\frac{n+9}{4}, \frac{n-3}{4}, c + \frac{n+5}{4}, d + \frac{n-3}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\lceil \frac{n-i+6}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, c + \lceil \frac{n-i+4}{2} \rceil, & \\ \qquad d + \lceil \frac{|n-i-1|}{2} \rceil) & \text{for} \quad \frac{n+3}{2} \leq i \leq n. \end{cases}$$

Case 3. $n \equiv 1 \pmod 4$.

Subcase 3.1. The subgraph $K_{1,2}$ contains two pendant edge.

Recall in this case $W_k = \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,3}, v_{2,n}\}$ which yields

$$r(v_{1,i}|W_k) = \begin{cases} (0,1,1,2,2,2) & \text{for} \quad i = 1, \\ (\lceil \frac{i-1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, & \\ \qquad \lceil \frac{i+2}{2} \rceil, \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 2 \leq i \leq \frac{n-3}{2}, \\ (\frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+5}{2} \leq i \leq n-1, \\ (1,1,1,1,1,1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (2,2,2,0,1,1) & \text{for} \quad i = 2, \\ (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, & \\ \qquad \lceil \frac{i-3}{2} \rceil, \lceil \frac{i}{2} \rceil) & \text{for} \quad 3 \leq i \leq \frac{n-1}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-5}{4}, \frac{n-1}{4}, \frac{n-5}{4}) & \text{for} \quad i = \frac{n+5}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i}{2} \rceil) & \text{for} \quad \frac{n+7}{2} \leq i \leq n. \end{cases}$$

Subcase 3.2. The subgraph $K_{1,2}$ contains exactly one pendant edge and $e = v_1v_2$.

Using $W_k = \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,n-1}, x\}$, we have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{i-1}{2} \rceil, \lceil \frac{|i-2|}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, & \\ \qquad \lceil \frac{i+2}{2} \rceil, b + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \leq i \leq \frac{n-3}{2}, \\ (\frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+3}{4}, b + \frac{n+3}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+3}{4}, b + \frac{n+3}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, \frac{n+3}{4}, \frac{n+3}{4}, b + \frac{n+3}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil \frac{n-i+1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, & \\ \qquad \lceil \frac{n-i+2}{2} \rceil, b + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+5}{2} \leq i \leq n-1, \\ (1,1,1,1,1,b+1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil\frac{i+1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i-2}{2}\rceil, \\ \lceil\frac{i+1}{2}\rceil, b+\lceil\frac{i}{2}\rceil) & \text{for} \quad 2 \le i \le \frac{n-3}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-5}{4}, \frac{n-1}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-1}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil\frac{n-i+1}{2}\rceil, \lceil\frac{n-i+1}{2}\rceil, \lceil\frac{n-i+1}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, \\ \lceil\frac{n-i+1}{2}\rceil, b+\lceil\frac{n-i}{2}\rceil) & \text{for} \quad \frac{n+5}{2} \le i \le n-1, \\ (2,2,2,1,1,b) & \text{for} \quad i = n. \end{cases}$$

Subcase 3.3. The subgraph $K_{1,2}$ contains no pendant edges and $e = v_1v_2$.
Recall the W_k used in this case is $W_k = \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,n-2}, x_1\}$. Henceforth, we have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil\frac{|i-2|}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i+2}{2}\rceil, \lceil\frac{i+2}{2}\rceil, \\ a+\lceil\frac{i-1}{2}\rceil, b+\lceil\frac{i}{2}\rceil) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ (\frac{n-5}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+3}{4}, a+\frac{n-1}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\frac{n-1}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+3}{2}, a+\frac{n-1}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n-1}{4}, \frac{n-5}{4}, \frac{n+3}{4}, \frac{n+3}{4}, a+\frac{n-1}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil\frac{n-i+2}{2}\rceil, \lceil\frac{|n-i-1|}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, \\ a+\lceil\frac{n-i+1}{2}\rceil, b+\lceil\frac{n-i+2}{2}\rceil) & \text{for} \quad \frac{n+5}{2} \le i \le n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil\frac{i+1}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i-2}{2}\rceil, \lceil\frac{i+2}{2}\rceil, \\ a+\lceil\frac{i+1}{2}\rceil, b+\lceil\frac{i}{2}\rceil) & \text{for} \quad 2 \le i \le \frac{n-3}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n-5}{4}, \frac{n-1}{4}, a+\frac{n+3}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-1}{2}, a+\frac{n+3}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, a+\frac{n+3}{4}, b+\frac{n-1}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil\frac{n-i+3}{2}\rceil, \lceil\frac{n-i+3}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, \lceil\frac{|n-i-1|}{2}\rceil, \\ a+\lceil\frac{n-i+1}{2}\rceil, b+\lceil\frac{n-i}{2}\rceil) & \text{for} \quad \frac{n+5}{2} \le i \le n. \end{cases}$$

Subcase 3.4. The subgraph $K_{1,2}$ contains exactly one pendant edge and $e = v_1v_3$.
In this case, we only use $W_k = \{v_{1,1}, v_{1,2}, v_{1,n-1}, v_{2,2}, v_{2,n-2}, x_1\}$ to yield

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil\frac{i-1}{2}\rceil, \lceil\frac{|i-2|}{2}\rceil, \lceil\frac{i+1}{2}\rceil, \lceil\frac{i+2}{2}\rceil, \\ \lceil\frac{i+4}{2}\rceil, d+\lceil\frac{i+2}{2}\rceil) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ (\frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+7}{4}, d+\frac{n+3}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+7}{4}, d+\frac{n+3}{4}) & \text{for} \quad i = \frac{n+1}{2}, \\ (\frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, \frac{n+3}{4}, \frac{n+7}{4}, d+\frac{n+3}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil\frac{n-i+1}{2}\rceil, \lceil\frac{n-i+2}{2}\rceil, \lceil\frac{n-i-1}{2}\rceil, \lceil\frac{n-i+4}{2}\rceil, \\ \lceil\frac{n-i+2}{2}\rceil, d+\lceil\frac{n-i+2}{2}\rceil) & \text{for} \quad \frac{n+5}{2} \le i \le n-1, \\ (1,1,1,1,2,d+1) & \text{for} \quad i = n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, \\ \quad \lceil \frac{i+2}{2} \rceil, d + \lceil \frac{i+1}{2} \rceil) & \text{for} \quad 2 \le i \le \frac{n-5}{2}, \\ (\frac{n-1}{4}, \frac{n-1}{4}, \frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}, d + \frac{n-1}{4}) & \text{for} \quad i = \frac{n-3}{2}, \\ (\lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{n-i+1}{2} \rceil, \\ \quad \lceil \frac{n-i-2}{2} \rceil, d + \lceil \frac{n-i-1}{2} \rceil) & \text{for} \quad i \in \{\frac{n-1}{2}, \frac{n+1}{2}\}, \\ (\frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n+3}{4}, \frac{n-5}{4}, d + \frac{n-5}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+1}{2} \rceil, \\ \quad \lceil \frac{|n-i-2|}{2} \rceil, d + \lceil \frac{|n-i-1|}{2} \rceil) & \text{for} \quad \frac{n+5}{2} \le i \le n. \end{cases}$$

Subcase 3.5. The subgraph $K_{1,2}$ does not contain pendant edge and $e = v_1v_3$. Let $W_k = \{v_{1,3}, v_{1,n-1}, v_{2,2}, v_{2,n-2}, x_2, x_1\}$. We have

$$r(v_{1,i}|W_k) = \begin{cases} (\lceil \frac{|i-3|}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i+2}{2} \rceil, \lceil \frac{i+4}{2} \rceil, \\ \quad c + \lceil \frac{|i-2|}{2} \rceil, d + \lceil \frac{i+2}{2} \rceil) & \text{for} \quad 1 \le i \le \frac{n-3}{2}, \\ (\frac{n-5}{4}, \frac{n-1}{4}, \frac{n+3}{4}, \frac{n+7}{4}, c + \frac{n-5}{4}, d + \frac{n+3}{4}) & \text{for} \quad i = \frac{n-1}{2}, \\ (\lceil \frac{i-3}{2} \rceil, \lceil \frac{n-i-1}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i+4}{2} \rceil, \\ \quad c + \lceil \frac{i-2}{2} \rceil, d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad i \in \{\frac{n+1}{2}, \frac{n+3}{2}\}, \\ (\frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}, \frac{n+3}{4}, c + \frac{n-1}{4}, d + \frac{n-1}{4}) & \text{for} \quad i = \frac{n+5}{2}, \\ (\lceil \frac{n-i+2}{2} \rceil, \lceil \frac{|n-i-1|}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{n-i+4}{2} \rceil, \\ \quad c + \lceil \frac{n-i+2}{2} \rceil, d + \lceil \frac{n-i+2}{2} \rceil) & \text{for} \quad \frac{n+7}{2} \le i \le n. \end{cases}$$

$$r(v_{2,i}|W_k) = \begin{cases} (\lceil \frac{i+3}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, \lceil \frac{i+2}{2} \rceil, \\ \quad c + \lceil \frac{i+1}{2} \rceil, d + \lceil \frac{i+1}{2} \rceil) & \text{for} \quad 2 \le i \le \frac{n-5}{2}, \\ (\frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, \frac{n-1}{4}, c + \frac{n-1}{4}, d + \frac{n-1}{4}) & \text{for} \quad i = \frac{n-3}{2}, \\ (\lceil \frac{i+3}{2} \rceil, \lceil \frac{i+1}{2} \rceil, \lceil \frac{i-2}{2} \rceil, \lceil \frac{n-i-2}{2} \rceil, \\ \quad c + \lceil \frac{i+1}{2} \rceil, d + \lceil \frac{n-i-1}{2} \rceil) & \text{for} \quad i \in \{\frac{n-1}{2}, \frac{n+1}{2}\}, \\ (\frac{n+7}{4}, \frac{n+3}{4}, \frac{n-1}{4}, \frac{n-5}{4}, c + \frac{n+3}{4}, d + \frac{n-5}{4}) & \text{for} \quad i = \frac{n+3}{2}, \\ (\lceil \frac{n-i+5}{2} \rceil, \lceil \frac{n-i+3}{2} \rceil, \lceil \frac{n-i+2}{2} \rceil, \lceil \frac{|n-i-2|}{2} \rceil, \\ \quad c + \lceil \frac{n-i+3}{2} \rceil, d + \lceil \frac{|n-i-1|}{2} \rceil) & \text{for} \quad \frac{n+5}{2} \le i \le n. \end{cases}$$

It is a routine to check that all these representation in all cases are different. We conclude that the lemma holds. □

In Figure 3, two vertices inside $K_{1,2} \rhd_e Ci_n(1,2)$ and two vertices outside $K_{1,2} \rhd_e Ci_n(1,2)$ resolves certain subgraph $K_{1,2} \rhd_e Ci_n(1,2)$. Since for every pendant edge there would be two resolving vertices and for every edge which is not a pendant edge there would be one resolving vertex, by Lemma 2.4 and Lemma 3.2 we are ready to prove Theorem 3.1.

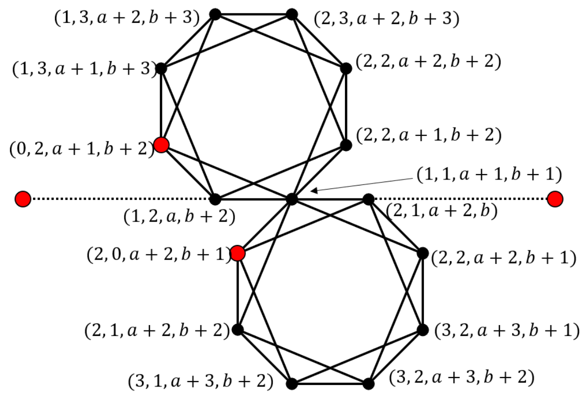

FIGURE 3. There are two vertices in the subgraph which belongs to the resolving set.

Proof of Theorem 3.1. According to Theorem 1.1,

$$dim(Ci_n(1,2)) = \begin{cases} 3, & \text{if } n \not\equiv 1 \pmod 4, \\ 4, & \text{if } n \equiv 1 \pmod 4, \end{cases}$$

Therefore, by Theorem 2.6,

$$dim(T \rhd_e Ci_n(1,2)) \geq \begin{cases} m+p, & \text{if } n \not\equiv 1 \pmod 4, \\ 2m+p, & \text{if } n \equiv 1 \pmod 4, \end{cases}$$

Let K be the number of subgraphs H_i in the graph. For the upper bound, start by applying Lemma 3.2 for subgraph $H_1 \cong H$ which contains the vertex $v_{1,1}$ and set an edge incident to $v_{1,1}$ as initial pendant edge to get W_1. Then, continue the process for H_2 which contains vertices of W_1, H_3 which contains vertices W_2, and so on. The process eventually terminates, by Lemma 2.4 and Lemma 3.2 we achieved

$$W = \bigcup_{k \in K} W_k$$

as a resolving set for the graph since all tuples of vertices are distinct. Hence, the upper bounds are equal to the lower bounds, implying

$$dim(T \rhd_e Ci_n(1,2)) = \begin{cases} m+p, & \text{if } n \not\equiv 1 \pmod 4, \\ 2m+p, & \text{if } n \equiv 1 \pmod 4, \end{cases}$$

□

In Figure 4, one of the example of $T \rhd_e Ci_6(1,2)$ is presented. For several graphs H_j which have a fixed $u_j \in V(H_j)$, $Amal(H_j, u_j)$ is a graph obtained from the union of all H_j with identifying every u_j with each other. If we choose $H_j = Ci_n(1,2)$ for some n, then $Amal(H_j, u_j) \cong S_m \rhd_e Ci_n(1,2)$ for any chosen $u_j \in V(H_j)$. Using Theorem 3.1, some implications may occur as written in corollaries below.

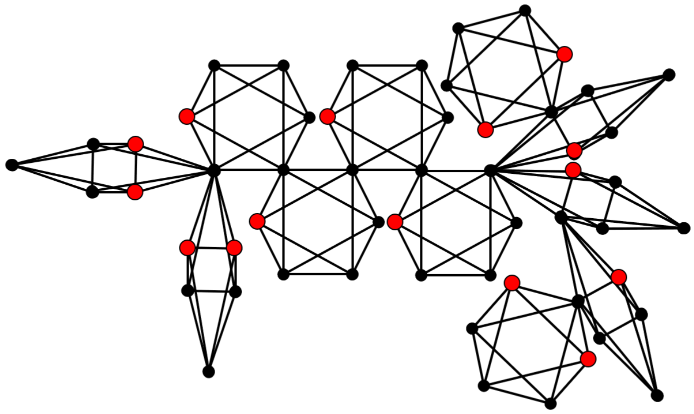

FIGURE 4. Graph $G \cong T \rhd_e Ci_6(1,2)$ with $dim(G) = 15$.

Corollary 3.3. *Let P_m be a path graph whose size is $m-1$. For $n \geq 6$, we have*

$$dim(P_m \rhd_e Ci_n(1,2)) = \begin{cases} m+1 & \text{if } n \not\equiv 1 \pmod 4, \\ 2m & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Corollary 3.4. *Let m be a positive integer, $H_j = Ci_n(1,2)$ and $u_j \in V(H_j)$ for $j \in [1,m]$. For $n \geq 6$, we have*

$$dim(Amal(H_j, u_j)) = \begin{cases} 2m & \text{if } n \not\equiv 1 \pmod 4, \\ 3m & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

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