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A REMARK ON SEQUENTIALLY COHEN-MACAULAY MONOMIAL IDEALS

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ABSTRACT. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K. We show that if G is a connected graph with a basic 5-cycle C, then G is a sequentially Cohen-Macaulay graph if and only if there exists a shedding vertex x of C such that $G \setminus x$ and $G \setminus N[x]$ are sequentially Cohen-Macaulay graphs. Furthermore, we study the sequentially Cohen-Macaulay and Castelnuovo-Mumford regularity of square-free monomial ideals in some special cases.

1. Introduction

Throughout this paper, we assume that $R = K[x_1, \dots, x_n]$ is the polynomial ring in n variables over a field K and I is a monomial ideal of R. If I is the square-free monomial ideal of R, we may consider the simplicial complex Δ over vertex set $V = \{x_1, \dots, x_n\}$ for which $I = I_\Delta$ is the Stanley-Reisner ideal of Δ and $K[\Delta] = R/I_\Delta$ is the Stanley-Reisner ring. Note that the simplicial complex Δ on V is a collection of subsets of V such that: (1) $\{x_i\} \in \Delta$ for $i = 1, \dots, n$, and (2) if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$. If x is a vertex of the simplicial complex Δ, then the *deletion* of x from Δ, denoted by $del_\Delta(x)$, is the simplicial complex over the vertex set $V \setminus \{x\}$ with faces $\{F : F \in \Delta, x \notin F\}$. The *link* of x in Δ, denoted by $link_\Delta(x)$, is the subcomplex of $del_\Delta(x)$ with faces $\{F : F \in_\Delta (x), F \cup \{x\} \in \Delta\}$. It is clear that $I_{del_\Delta(x)} = (I_\Delta, x)$ and $I_{link_\Delta(x)} = ((I_\Delta : x), x)$.

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We say a monomial ideal I is Cohen-Macaulay (sequentially Cohen-Macaulay) when R/I is Cohen-Macaulay (sequentially Cohen-Macaulay). Stanley [21] defined that a graded R-module M is to be sequentially Cohen-Macaulay (i.e., SCM) if there exists a finite filtration of graded R-modules $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each M_i/M_{i-1} is Cohen-Macaulay (i.e., CM) and the Krull dimension of the quotients are increasing: $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1})$. Note that every CM R-module is a SCM R-module. Moreover, it is known that M is a CM R-module if and only if M is an unmixed and a SCM R-module.

Let G be a simple graph (no loops or multiple edges) on the vertex set $V = \{x_1, \ldots, x_n\}$ and the edge set E. The *edge ideal* of the graph G is the quadratic square-free monomial ideal $I(G) = (x_ix_j \mid \{x_i, x_j\} \in E)$ and it was first introduced by Villarreal [25]. One can associated to G the simplicial complex Δ_G called the *independence complex*, whose faces are the independent sets of the graph G. Note that the independent set in G is the set with no two of its vertices are adjacent. The independence complex is the simplicial complex associated to $I(G)$ via the Stanley-Reisner correspondence. Hence we may consider the simplicial complex Δ_G for which $I(G)$ is the Stanley-Reisner ideal of Δ_G. The graph G is called SCM if $R/I(G)$ is SCM. The SCM of simplicial complexes and graphs are studied in [1, 6, 8, 13, 16, 23, 24, 27].

In this paper we show that if G is a connected graph with a basic 5-cycle C, then G is a SCM graph if and only if there exists a shedding vertex x of C such that $G \setminus x$ and $G \setminus N[x]$ are SCM graphs, where $N[x] = N(x) \cup \{x\}$ such that $N(x)$ is the neighborhoods set of x and $G \setminus x$ is the induced subgraph of G over the vertex set $V \setminus \{x\}$. Moreover, we study SCM and Castelnuovo-Mumford regularity of square-free monomial ideals in some special cases.

For any unexplained notion or terminology, we refer the reader to [14] and [26].

2. The results

We start this section by recalling the following definition and theorem. Suppose that Δ be a simplicial complex. The pure i-skeleton of Δ is defined as: $\Delta^{[i]} = \langle\{F \in \Delta : \dim(F) = i\}\rangle; -1 \leq i \leq \dim(\Delta)$.

Theorem 2.1. [6, Theorem 3.3] *A simplicial complex Δ is SCM if and only if the pure i-skeleton $\Delta^{[i]}$ is CM for all $-1 \leq i \leq \dim(\Delta)$.*

Proposition 2.2. *Let I be a monomial ideal and u be a monomial element of R. If I is SCM, then $(I : u)$ is SCM.*

Proof. By via polarization [7, Proposition 4.11] we may assume that I and u are square-free. Now we assume that Δ is a simplicial complex of I. Since I is SCM, by Theorem 2.1 we have $\Delta^{[i]}$ is CM for all i and so by [26, Proposition 6.3.15] $link_{\Delta^{[i]}}(u)$ is CM. By using [26, Proposition 6.3.17] we have $link_{\Delta^{[i]}}(u) = (link_\Delta)^{[i-1]}(u)$ and it follows that $(link_\Delta)^{[i-1]}(u)$ is CM for all $i > 1$. Hence by Theorem 2.1 $link_\Delta(u)$ is SCM. Since $link_\Delta(u) = ((I : u), u)$, it follows that $(I : u)$ is SCM. □

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Recall that an ideal I is called unmixed if all prime ideal of $Ass(I)$ have the same height. A vertex x of a simplicial complex Δ is called a shedding vertex when no face of $link_\Delta(x)$ is a facet (maximal face) of $del_\Delta(x)$.

Proposition 2.3. *Let Δ be a simplicial complex and x be a shedding vertex. If $I_\Delta = I$ is unmixed, then $(I : x)$ and (I, x) are unmixed. In particular, $I_{link_\Delta(x)}$ and $I_{del_\Delta(x)}$ are unmixed.*

Proof. From the exact sequence

$$0 \longrightarrow R/(I : x) \xrightarrow{x} R/I \longrightarrow R/(I, x) \longrightarrow 0,$$

we conclude that $Ass(I : x) \subseteq Ass(I) \subseteq Ass(I : x) \cup Ass(I, x)$. Since x is a shedding vertex, by using [19, Proposition 2.1] we have $Ass(I, x) \subseteq Ass(I)$ and hence $Ass(I) = Ass(I : x) \cup Ass(I, x)$. Now, since I is unmixed it therefore follows that $(I : x)$ and (I, x) are unmixed. Since $I_{del_\Delta(x)} = (I_\Delta, x)$ and $I_{link_\Delta(x)} = ((I_\Delta : x), x)$, the result is clear. □

In the next result we use the following definition; a 5-cycle C of G is called basic if C does not contain two adjacent vertices of degree three or more in G, see [3].

Lemma 2.4. [3, Lemma 38] *Let G be a connected graph. Then every vertex of degree at least* 3 *in a basic* 5*-cycle is a shedding vertex.*

Suppose that x is a vertex of graph G. Then it is clear that $link_\Delta(x) = \Delta_{G \setminus N[x]}$ and $del_\Delta(x) = \Delta_{G \setminus x}$. The following theorem is a generalization of [3, Theorem 40].

Theorem 2.5. *Let G be a connected graph with a basic* 5*-cycle C. Then G is a SCM graph if and only if there exists a shedding vertex $x \in V(C)$ such that $G \setminus x$ and $G \setminus N[x]$ are SCM graphs.*

Proof. $(\Longrightarrow)$: We may assume that $C = (x_1, x_2, x_3, x_4, x_5)$. If $G = C$, then by [8, Proposition 4.1] it follows that G is SCM. Since each vertex is a shedding vertex, $G \setminus x_i$ and $G \setminus N[x_i]$ are path and edge for each $1 \leq i \leq 5$, by [27, Corollary 7] it follows that $G \setminus x_i$ and $G \setminus N[x_i]$ are SCM graphs. Now suppose that $G \neq C$. We may assume that $\deg(x_1) \geq 3$. Since C is a basic 5-cycle, then $\deg(x_2) = 2 = \deg(x_5)$. Also, we may assume that $\deg(x_3) = 2$ and $\deg(x_4) \geq 2$. By Lemma 2.4, x_1 is a shedding vertex. By [24, Theorem 3.3], we conclude that $G \setminus N[x_1]$ is SCM. Now, we will prove that $G_1 = G \setminus x_1$ is SCM. Since G is SCM, then $G_2 = G \setminus N[x_2]$ and $G_3 = G \setminus N[x_3, x_5]$ are SCM. Suppose that $F_1, \ldots, F_r$ and $H_1, \ldots, H_t$ are facets of Δ_{G_2} and Δ_{G_3}, respectively. Take $F \in \mathcal{F}(\Delta_{G_1})$. If $x_2 \in F$, then $F \setminus x_2 \in \mathcal{F}(\Delta_{G_2})$ and there exists F_i such that $F = F_i \cup \{x_2\}$, where $1 \leq i \leq r$. If $x_2 \notin F$, then $x_3 \in F$ and $x_4 \notin F$. Therefore $x_5 \in F$. Thus, $F \setminus \{x_3, x_5\} \in \mathcal{F}(\Delta_{G_3})$ and so there exists H_j such that $F = H_j \cup \{x_3, x_5\}$, where $1 \leq j \leq t$. This implies that $\mathcal{F}(\Delta_{G_1}) = \{F_i \cup \{x_2\}, H_j \cup \{x_3, x_5\} : 1 \leq i \leq r, 1 \leq j \leq t\}$. Set $\Delta_1 = \langle F_i \cup \{x_2\} : 1 \leq i \leq r\rangle$ and $\Delta_2 = \langle H_j \cup \{x_3, x_5\} : 1 \leq j \leq t\rangle$. Consider the exact sequence

$$0 \longrightarrow K[\Delta_1 \cup \Delta_2] \longrightarrow K[\Delta_1] \oplus K[\Delta_2] \longrightarrow K[\Delta_1 \cap \Delta_2] \longrightarrow 0.$$

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By using Auslander-Buchsbaum Theorem and [12, Corollary 3.2] $depth(K[\Delta_1 \cap \Delta_2])^{[i]} \geq depth(K[\Delta_1] \oplus K[\Delta_2])^{[i]} - 1$ and since $(K[\Delta_1] \oplus K[\Delta_2])^{[i]}$ is CM, by Depth Formula [26, Lemma 2.3.9] we conclude that $K[\Delta_1 \cup \Delta_2]^{[i]}$ is CM. Therefore, by Theorem 2.1, $K[\Delta_1 \cup \Delta_2]$ is SCM. Since $I_{\Delta_1 \cup \Delta_2} = I_{\Delta_{G_1}}$ and $I_{\Delta_{G_1}} = (I, x_1)$, it therefore follows that (I, x_1) is SCM, as required.
$(\Longleftarrow)$: It follows by [19, Theorem 2.2]. □

Corollary 2.6. *Let G be a connected unmixed graph with a basic 5-cycle C. Then G is a CM graph if and only if there exists a shedding vertex $x \in V(C)$ such that $G \setminus x$ and $G \setminus N[x]$ are CM graphs.*

Proof. It is known that if G is a connected unmixed graph, then G is CM if and only if G is SCM. Therefore the result follows by Theorem 2.5 and Proposition 2.3. □

It is known that n-cycle graphs are SCM if and only if $n = 3, 5$. So we give the following examples which show that having a basic 5-cycle C in G is essential. By using [1, Lemma 2.3] and Macaulay2 [9] the following examples easily follow.

Example 2.7. *Let $I = (x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_1x_5)$ be an edge ideal of a bipartite graph G. Then G is SCM, but $G \setminus x_5$ is not SCM.*

Example 2.8. *Let $I = (x_1x_2, x_1x_4, x_1x_5, x_2x_3, x_2x_7, x_3x_4, x_4x_5, x_4x_7, x_6x_7)$ be an edge ideal of a graph G. Then G is SCM and x_5 is a shedding vertex, but $G \setminus x_5$ is not SCM.*

The following lemma easily follows by [26, Theorem 6.4.23] and Auslander-Buchsbaum formula.

Lemma 2.9. *Let I be a monomial ideal of R. If I is SCM, then $depth(R/I) = \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in Ass(I)\}$.*

The following result is a generalization of [4, Lemma 4.1(i)].

Corollary 2.10. *Let I, J be two monomial ideals of R such that $I : J$ is SCM. Then $depth(R/I) \leq depth(R/I : J)$.*

Proof. It is clear that $Ass(I : J) \subseteq Ass(I)$. Therefore by Lemma 2.9 we have

$$depth(R/I : J) = \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in Ass(I : J)\}$$
$$\geq \min\{\dim R/\mathfrak{p} : \mathfrak{p} \in Ass(I)\}$$
$$\geq depth(R/I).$$

□

The following example shows that the condition on $I : J$ in Corollary 2.10 is essential. For the computation of the following example we use Macaulay2 [9].

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Example 2.11. *Let $R = K[x_1, x_2, x_3, x_4]$, $I = (x_1x_3, x_2x_4)$ and $J = (x_2x_3, x_1x_4)$. Then $I : J = (x_1, x_3)(x_2, x_4)$ such that $depth(R/I) = 2 \nleq 1 = depth(R/I : J)$.*

In the following result, we use the concept of Castelnuovo-Mumford regularity of a graded R-module M which is defined as $reg(M) = \max\{j - i | \ \beta_{i,j}(M) \neq 0\}$.

Proposition 2.12. *Let I be a square-free monomial ideal and x is a shedding vertex such that (I, x) is SCM. Then we have the following:*

(i) $depth(R/I) = \min\{depth(R/(I : x)), depth(R/(I, x))\}$;
(ii) $pd(R/I) = \max\{pd(R/(I : x)), pd(R/(I, x))\}$;
(iii) $reg(R/I) = \max\{reg(R/(I : x)), reg(R/(I, x)) + 1\}$.

Proof. (i) Since x is a shedding vertex, we have $Ass(I) = Ass(I : x) \cup Ass(I, x)$ and since (I, x) is SCM, by Lemma 2.9 we have

$$depth(R/(I, x)) = \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in Ass(I, x)\}$$

$$\geq \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in Ass(I)\} \geq depth(R/I).$$

On the other hand by using [4, Lemma 4.1] we have $depth(R/I) \leq depth(R/(I : x))$. It therefore follows that $depth(R/I) \leq \min\{depth(R/(I : x)), depth(R/(I, x))\}$. Conversely by using the Depth Formula [26, Lemma 2.3.9] on the exact sequence

$$0 \longrightarrow R/(I : x) \xrightarrow{x} R/I \longrightarrow R/(I, x) \longrightarrow 0,$$

we have $depth(R/I) \geq \min\{depth(R/(I : x)), depth(R/(I, x))\}$. This completes the proof of case (i).
(ii) It follows by (i) and the Auslander-Buchsbaum Theorem.
(iii) It follows by [11, Theorem 4.2]. □

A vertex x of G is called codominated if there exists a vertex $y \in V \setminus \{x\}$ such that $N[y] \subseteq N[x]$.
The following lemma immediately follows from [27, Lemma 6] and [3, Theorem 5].

Lemma 2.13. *Let G be a C_5-free graph. Then a vertex x of G is a shedding vertex if and only if it is codominated. In particular, if G is a bipartite graph then a vertex x of G is a shedding vertex if and only if it is codominated.*

Two edges $\{x, y\}$ and $\{z, u\}$ of G is called 3-*disjoint* if the induced subgraph of G on $\{x, y, z, u\}$ is disconnected or equivalently in the complement of G the induced graph on $\{x, y, z, u\}$ is four-cycle. A subset A of edges of G is called a pairwise 3-disjoint set of edges in G if each pair of edges of A is 3-disjoint. The maximum cardinality of all pairwise 3-disjoint sets of edges in G is denoted by $a(G)$.

Lemma 2.14. [2, Lemma 23] *If x is a codominated vertex of a graph G, then $a(G \setminus x) \leq a(G)$ and $a(G \setminus N[x]) + 1 \leq a(G)$.*

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Lemma 2.15. [5, Lemma 2.10] *Let x be a vertex of a graph G. Then $reg(R/I(G)) \leq \max\{reg(R/I(G \setminus x)), reg(R/I(G \setminus N[x])) + 1\}$. Moreover, $reg(R/I(G))$ always equals to one of $reg(R/I(G \setminus x))$ or $reg(R/I(G \setminus N[x])) + 1$.*

Theorem 2.16. *Let $\mathcal{F}$ be a family of graphs such that every graph has a codominated vertex. If $G \setminus x$ and $G \setminus N[x]$ are in $\mathcal{F}$ for $G \in \mathcal{F}$ and a codominated vertex x of G, then $reg(R/I(G)) = a(G)$ for all $G \in \mathcal{F}$.*

Proof. Let G be an arbitrary element of $\mathcal{F}$. By [17, Lemma 2.2], we have $reg(R/I(G)) \geq a(G)$. Now by induction on $|V|$, we prove that $reg(R/I(G)) \leq a(G)$. If $|V| = 2$, then $reg(R/I(G)) = 1$ and also $a(G) = 1$. Thus the result follows in this case. Suppose that $|V| \geq 2$. There exists a codominated vertex $x \in V$ such that $G \setminus x$ and $G \setminus N[x]$ are in $\mathcal{F}$. By Lemma 2.15 we have $reg(R/I(G)) \leq \max\{reg(R/I(G \setminus x)), reg(R/I(G \setminus N[x])) + 1\}$ and by using induction hypothesis it follows $reg(R/I(G)) \leq \max\{a(G \setminus x), a(G \setminus N[x]) + 1\}$. Now by Lemma 2.14 we have $reg(R/I(G)) \leq a(G)$. This completes the proof, as required. $\square$

By using Theorem 2.16, we readily conclude the following known result.

Corollary 2.17. *Let G be a graph such that one of the following conditions satisfies:*

(i) [10, Corollary 6.9] *G is a chordal graph;*
(ii) [23, Theorem 3.2] *G is a SCM bipartite graph;*
(iii) [18, Theorem 2.4] *G is a C_5-free vertex decomposable;*
(iv) [22, Theorem 11] *G is a Cameron-Walker graph.*
(v) [20, Lemma 3.4] *G is a very well-covered Cohen-Macaulay graph.*

Then $reg(R/I(G)) = a(G)$.

Proof. If G satisfies in $(i), (ii), (iii)$ and (iv), then it has a codominated vertex and also in case (v) G has a codominated vertex by proof of [20, Theorem 3.2]. Thus by Theorem 2.16 the result follows. $\square$

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