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FORBIDDEN SUBGRAPHS OF CO-PRIME GRAPHS OF FINITE GROUPS

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Abstract. For a finite group G the co-prime graph $\Gamma(G)$ is defined as a graph with vertex set G in which two distinct vertices x and y are adjacent if and only if $gcd(o(x), o(y)) = 1$ where $o(x)$ and $o(y)$ denote the orders of the elements x and y respectively. In this paper we find properties of groups whose co-prime graphs forbid graphs such as $C_4, K_{1,3}, P_4$ and asteroidal triples.

1. Introduction

For a finite group G, several graphs can be defined using different group properties. Power graphs, Enhanced power graphs, commuting graphs, Intersection graphs and co-prime graphs are some of the examples of graphs defined on groups. There are many useful applications for graphs defined on groups and are related to automata theory [3].

Ma et.al [13] introduced the co-prime graph $\Gamma(G)$ of a finite group G in the year 2014. They defined the co-prime graph as a simple undirected graph with vertex set G and the edge set consists of the unordered pair of vertices $\{x, y\}$ such that $gcd(o(x), o(y)) = 1$, where $o(x)$ and $o(y)$ denote the orders of the elements x and y respectively. The authors studied some graph theoretic properties such as diameter, clique number, planarity and automorphism group of co-prime graphs. The number of edges in co-prime graphs of cyclic groups and dihedral groups are computed in [4]. In [5] the authors proved that the clique number and chromatic number of co-prime graphs are same for any

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finite group G. Also they classified all finite groups whose co-prime graphs are complete r-partite or planar. The authors in [11] studied co-prime graphs of cyclic groups. The clique numbers and chromatic numbers of co-prime graphs of dihedral groups are determined in [2]. The co-prime graph of generalized quaternion groups are studied in [7]. Refer [12] and [6] for some related graphs.

A forbidden graph characterization can be used to describe numerous significant families of graphs. Graph theorists have been searching for such characterization since in 1930, Kuratowski characterized planar graphs as those graphs which does not contain induced subgraphs isomorphic to K_5 or $K_{3,3}$. Cographs and split graphs are some of the important graph classes which can be defined using forbidden induced subgraphs. Forbidden subgraphs of power graphs of groups have been studied by Doostabadi et al. [1] and Cameron et al. [10].

In this paper we classify finite groups whose co-prime graphs are AT-free, C_4-free, Claw-free, co-graphs and split-graphs.

2. **Preliminaries**

A graph Γ is a pair (V, E) of two sets, where V is a finite nonempty set of objects called vertices and E is a set of 2-element subsets of V called edges. Two vertices x and y are said to be adjacent in Γ if $\{x, y\}$ is an edge of Γ. We denote $x \sim y$ if the vertices x and y are adjacent in Γ. The degree of a vertex x is the number of vertices adjacent to x, which is denoted by $deg(x)$. The set of all vertices adjacent to x is called the neighborhood of x, which is denoted by $N(x)$.

Let $\mathcal{G}$ be a family of graphs. $\mathcal{G}$ is called H-free if no $G \in \mathcal{G}$ contains H as an induced subgraph, also H is called the forbidden subgraph of $\mathcal{G}$. A graph is called a co-graph if it is P_4-free. A split graph is a graph in which the vertices can be partitioned into an independent set and a clique. A graph is said to be Claw-free if it is $K_{1,3}$-free. An Asteroidal Triple or briefly AT in a graph is an independent set of three vertices such that there is a path between each pair of those three vertices which does not contain any neighbours of the third. The graph given in Figure 1 is a cograph and a split graph that is not claw-free and not AT-free. The subgraph induced by the vertices u, v, w and r is a claw, and the vertices u, w and z form an AT.

The order of a group G is the number of elements in G, which is denoted by $|G|$. The identity element in G is denoted by e. The order of an element $g \in G$ is the smallest positive integer n such that $g^n = e$, which is denoted by $o(g)$. We denote set of all prime divisors of $|G|$ by $\pi(G)$ and that of $o(g)$ by $\pi(g)$. The center of G is the set of all elements in G which commute with every other element in G, and is denoted $Z(G)$. An EPPO-group is a group in which every element has some prime power order.

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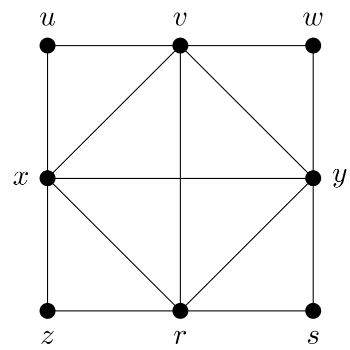

Figure 1

3. Forbidden subgraphs

In this section we characterize all groups whose co-prime graphs are C_4-free, claw-free, cographs and split graphs. We begin with a proposition which we use throughout this paper.

Proposition 3.1. *Let g_1 and g_2 be two elements of a group G. Then the following holds,*

(a) *$g_1 \sim g_2$ in $\Gamma(G)$ if and only if $\pi(g_1) \cap \pi(g_2) = \phi$.*
(b) *$\pi(g_1) \subseteq \pi(g_2)$ if and only if $N(g_2) \subseteq N(g_1)$.*

Proof. (a) The statement holds from the definition of the co-prime graph.
(b) Suppose $\pi(g_1) \subseteq \pi(g_2)$. If $x \in N(g_2)$ then $\pi(g_2) \cap \pi(x) = \phi$, which implies $\pi(g_1) \cap \pi(x) = \phi$, and hence $x \in N(g_1)$.

Conversely suppose $N(g_2) \subseteq N(g_1)$ and let $p \in \pi(g_1)$. If $p \notin \pi(g_2)$, then $g_2 \sim x$ for some $x \in G$ with $o(x) = p$. Since $N(g_2) \subseteq N(g_1)$, $x \in N(g_1)$, which is not possible. □

Next we prove that co-prime graphs of finite groups form a universal graph, which gives some context to the results about groups defined by forbidden subgraphs.

Theorem 3.2. *Every finite graph is an induced subgraph of the co-prime graph of some group.*

Proof. We prove this theorem using the empty intersection graphs. If X is a set then $P(X)$ with the intersection operator is a semigroup with $\emptyset$ as a zero. The empty intersection graph of the set X is the zero divisor graph of this semigroup. Erdos et al. [8] proved that every simple graph is isomorphic to an induced subgraph of an empty intersection graph of some set. So it suffices to prove that the empty intersection graph is isomorphic to an induced subgraph of a co-prime graph. Without loss of generality assume that $X = \{p_1, \ldots, p_n\}$, where $p_1, \ldots, p_n$ are prime numbers. Let $k = p_1 \cdots p_n$ and $G = \mathbb{Z}_k$. For any subset A of X choose an element $g_A \in G$ such that $\pi(g_A) = A$. Then the function f defined by $f(A) = g_A$ is an isomorphism from empty intersection graph of X to its image as an induced subgraph of co-prime graph of G. Since if A, B are adjacent then $A \cap B = \emptyset$.

So $\pi(g_A) \cap \pi(g_B) = \emptyset$. So $f(A) = g_A$ and $f(B) = g_B$ are adjacent in the co-prime graph of G by Proposition 3.1. □

The following theorem gives a characterization to the co-prime graphs which are C_4-free.

Theorem 3.3. *Let G be a finite group. $\Gamma(G)$ is C_4-free if and only if G is a nilpotent group of order p^n or $2p^n$, p is a prime.*

Proof. If G is a p-group, then $\Gamma(G)$ is a star graph, which is C_4-free. Let G be a nilpotent group of order $2p^n$ where p is an odd prime. If $\Gamma(G)$ contains an induced C_4, say $g_1 \sim g_2 \sim g_3 \sim g_4$, then the orders of g_1 and g_3 must be some powers of p and orders of g_2 and g_4 must be 2. which is not possible since G contains unique element of order 2 as it is a nilpotent group.

Now, suppose $|G|$ has two odd prime divisors, say p and q. Take $x_1, x_2 \in G$ having order p and y_1, y_2 having order q, then $x_1 \sim y_1 \sim x_2 \sim y_2$ is an induced C_4.

Conversely assume $\Gamma(G)$ is C_4-free and G is not a p-group. Let P, Q be two Sylow subgroups of G such that $|Q| < |P|$. Since $|P| \geq 3$, there are two nontrivial elements $p_1, p_2 \in P$. If $3 \leq |Q|$ then there are two nontrivial elements $q_1, q_2 \in Q$. Then $p_1 \sim q_1 \sim p_2 \sim q_2 \sim p_1$ is a C_4 which is a contradiction. So $|Q| = 2$ and $|G| = 2p^n$. If G contains two elements h, k of order two then $p_1 \sim g \sim p_2 \sim h \sim p_1$ is a C_4 which is a contradiction. So G has a unique element of order two. Hence $Q \lhd G$. Also $P \lhd G$. So G is nilpotent of order $2p^n$. □

Corollary 3.4. *For a finite group G, $\Gamma(G)$ is a split graph if and only if G is a nilpotent group of order p^n or $2p^n$ where p is a prime.*

Proof. We prove this using the characterization of split graphs which states that: a graph is a split graph if and only if it has no induced subgraphs isomorphic to C_4, C_5 and $2K_2$.

First we show that if $\Gamma(G)$ contains an induced $2K_2$, then $|\pi(G)| \geq 4$. Suppose x_1, x_2 and y_1, y_2 forms $2K_2$ in $\Gamma(G)$. Then there exists primes p_1, p_2, p_3 and p_4 in $\pi(G)$ such that $p_1 \in \pi(x_1) \cap \pi(y_1)$, $p_2 \in \pi(x_1) \cap \pi(y_2)$, $p_3 \in \pi(x_2) \cap \pi(y_1)$ and $p_4 \in \pi(x_2) \cap \pi(y_2)$ and $p_1 \neq p_2 \neq p_3 \neq p_4$ since $\pi(x_1) \cap \pi(x_2) = \emptyset$ and $\pi(y_1) \cap \pi(y_2) = \emptyset$. Similarly we can show that if $\Gamma(G)$ contains an induced C_5, then $|\pi(G)| \geq 5$.

Conversely if $\Gamma(G)$ is a split graph then it is C_4-free. So G is a nilpotent group of order p^n or $2p^n$ by Theorem 3. □

The next theorem characterises the co-prime graphs which are claw-free.

Theorem 3.5. *Let G be a finite group. $\Gamma(G)$ is Claw-free if and only if $|G| \leq 3$.*

Proof. If $p \mid |G|$ where p is a prime greater than 3, then there are at least three elements each of order p, and any three of these elements together with the identity form a claw.

Suppose $|G| = 2^i 3^j$, $i > 1$ or $j > 1$. If $i > 1$ then three elements each of order a power of 2, or if $j > 1$ then three elements each of order a power of 3, together with the identity induce a claw.

If $|G| = 6$, then elements of orders 6, 2 and identity in $\mathbb{Z}_6$ and three elements of order 2 and identity in S_3 induce claws.

Clearly $\Gamma(G)$ is claw-free if $|G| \leq 3$. □

Using similar proofs we can show that $\Gamma(G)$ is $K_{1,4}$-free if and only if $|G| \leq 4$.

Let G be a finite group of order n. The *Gruenberg-Kegel graph*[9] or *prime graph* of G is a graph with vertex set $\pi(n)$ in which two distinct vertices p_1 and p_2 are adjacent if and only if G contains an element of order p_1p_2.

A graph forbidding path of order 4 is called a cograph. Cographs have many key properties such as they form the smallest class of graphs containing the 1-vertex graph and closed under disjoint union and complementation.

Theorem 3.6. *Let G be a finite group. $\Gamma(G)$ is a cograph if and only if Gruenberg-kegel graph of G doesn't contain P_3 as a subgraph.*

Proof. First we prove that if $\Gamma(G)$ is a cograph then the Gruenberg-Kegel graph of G contains no induced P_3. Suppose that *Gruenberg-Kegel graph* of G contains an induced P_3, say, p_1, p_2, p_3. Then G contains elements of order p_1p_2 and p_2p_3 and does not contain an element of order p_1p_3. Take $x, y, z, w \in G$ such that $o(x) = p_1$, $o(y) = p_3$, $o(z) = p_1p_2$ and $o(w) = p_2p_3$. Then $z \sim y \sim x \sim w$ is a P_4 in $\Gamma(G)$.

Conversely suppose $\Gamma(G)$ is not a cograph. Let $x \sim y \sim z \sim w$ be an induced P_4. Then by Proposition 1, $\pi(x) \cap \pi(y) = \emptyset$, $\pi(y) \cap \pi(z) = \emptyset$, $\pi(z) \cap \pi(w) = \emptyset$, $\pi(x) \cap \pi(z) \neq \emptyset$, $\pi(y) \cap \pi(w) \neq \emptyset$, and $\pi(x) \cap \pi(w) \neq \emptyset$. Let $p_1 \in \pi(x) \cap \pi(z)$, $p_2 \in \pi(y) \cap \pi(w)$ and $p_3 \in \pi(x) \cap \pi(w)$, then $p_1 \neq p_2 \neq p_3$ since $\pi(x) \cap \pi(y) = \emptyset$. Also $p_1p_3 \mid o(x)$ and $p_2p_3 \mid o(w)$. Therefore there are elements in G of orders p_1p_3 and p_2p_3, and hence the Gruenberg-Kegel graph of G contains P_3 as subgraph. □

In the converse of Theorem 3.6, the P_3 in the Gruenberg-Kegel graph of G need not be an induced one. $\mathbb{Z}_{30}$ is not a cograph and its *Gruenberg-Kegal graph* is K_3 which contains P_3 as a subgraph but not as an induced subgraph.

The following corollary is immediate from Theorem 3.6.

Corollary 3.7. *Let G be a finite nilpotent group. Then, $\Gamma(G)$ is a cograph if and only if $|\pi(G)| < 3$.*

4. Asteroidal triples

Asteroidal triples highlight the resilience of a graph to the removal of individual vertices. The fact that there is always a path between any two vertices in the triple, even after the removal of one vertex, indicates a certain level of connectivity and robustness. In this section we investigate groups whose co-prime graphs does not contain asteroidal triples.

Theorem 4.1. *Let G be a finite group. Then $\Gamma(G)$ is AT-free if and only if the Gruenberg-Kegel graph of G is K_3-free.*

Proof. If the Gruenberg-Kegel graph of G contains K_3 then G contains elements of order pq, pr, qr for three distinct prime numbers p, q, r. Select elements $u, v, w, x, y, z \in G$ such that $o(u) = p_1, o(v) = p_2, o(w) = p_3, o(x) = p_1p_2, o(z) = p_1p_3, o(y) = p_2p_3$. Then $\{x, y, z\}$ is an asteroidal triple and the paths between them are $x \sim w \sim u \sim y$, $y \sim u \sim v \sim z$ and $x \sim w \sim v \sim z$.

Conversely assume the Gruenberg-Kegel graph is K_3-free. Then for each element $g \in G$ we have $|\pi(G)| \leq 2$. If $\Gamma(G)$ contains an AT then there exist elements g_1, g_2, g_3 such that $\pi(g_i) \cap \pi(g_j) \neq \emptyset$ and $\pi(g_i) \nsubseteq \pi(g_j)$ for any $i \neq j$. Hence there exist three prime elements p, q, r such that $\pi(g_1) = \{p, q\}, \pi(g_2) = \{p, r\}, \pi(g_3) = \{q, r\}$. So G contains elements of order pq, pr, qr. Hence these elements form a K_3 in Gruenberg-Kegel graph of G which is a contradiction. □

Corollary 4.2. *Let G be a finite group such that $\Gamma(G)$ contains an AT, then $|\pi(G)| \geq 3$.*

Corollary 4.3. *Let G be a finite nilpotent group. Then, $\Gamma(G)$ is AT-free if and only if $|\pi(G)| < 3$.*

Proof. Let $|\pi(G)| \geq 3$ and let p_1, p_2, p_3 are distinct primes which divide $|G|$. Since G is nilpotent there are elements in G of orders p_1p_2, p_2p_3 and p_1p_3. □

The following example shows that this characterization does not hold if G is not nilpotent.

Example 4.4. *The group $S_3 \times \mathbb{Z}_5$ is not nilpotent and $|\pi(S_3 \times \mathbb{Z}_5)| = 3$, which does not contain an AT, whereas D_{30}, which is also not nilpotent and $|\pi(D_{30})| = 3$, contains asteroidal tripples.*

Proposition 4.5. *Let G be a finite group such that $\Gamma(G)$ is AT-free and $|\pi(G)| \geq 4$, then $Z(G)$ is trivial.*

Proof. Suppose $Z(G)$ is not trivial and let $x \in Z(G)$, $x \neq e$. Without loss of generality assume that $o(x)$ is a prime, say, p. Since $\pi(G) \geq 4$ there are elements y, z, w in G of different prime orders other than p. Since x commutes with all these elements $\{xy, xz, xw\}$ will form an AT in $\Gamma(G)$. □

If G is a group such that $\Gamma(G)$ is AT-free and $|\pi(G)| = 3$, then $Z(G)$ need not be trivial, $S_3 \times \mathbb{Z}_5$ is an example for that.

5. Conclusion and open problems

We have completely characterized finite groups whose co-prime graphs are C_4-free, Claw-free, Cographs and Split graphs, and we have chararacterized nilpotent groups which are AT-free. We have been unable to solve the following problem.

problem 5.1. *Find a characterization for the finite groups whose co-prime graphs are AT-free.*

The problem of finding asteroidal triples can be extended to other families of graphs defined on groups such as Power graphs, Enhanced power graphs, Commuting graphs, etc.

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