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# INDUCED GEODETIC SEQUENCE OF A GRAPH

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Abstract. A vertex subset $S$ of a graph $G=(V,E)$ is said to be a geodetic set if every vertex in $G$ is in some $u-v$ geodesic for any $u,v \in S$. The minimum cardinality of such a set is the geodetic number, which is denoted as $g(G)$. In this paper, we introduce the concepts of induced geodetic number and induced geodetic sequence of a graph. We discuss this concept in some graph classes. Also, established the characterization of induced geodetic sequences for trees, unicyclic graphs and cacti.

## 1. Introduction

The graphs discussed in this study are simple, connected, and non-directional. For the basic graph definitions and notations, we refer to [1, 6]. The cardinality of the vertex set of a graph $G$ is its order and the edge set's cardinality is the graph's size. For any vertex $v$ in a graph $G$, $N(v)$ [6] denotes the set of all vertices which are adjacent to $v$. The set $N[v]$ [6] is the union of sets $N(v)$ and $\{v\}$. Let $S$ be a vertex subset of graph $G$. Then $< S >$ denotes the subgraph induced [1, 10] by $S$. A vertex $v$ in a graph $G$ is said to be extreme or link-complete [4], if $< N(v) >$ is a complete graph. The number of extreme vertices in a graph $G$ is its extreme order [4], which is denoted as $ex(G)$. A vertex with degree one is the pendant vertex.

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A $u-v$ geodesic [8] in a graph $G$ is the shortest path between the vertices $u$ and $v$ in $G$. The distance [6] between the vertices $u$ and $v$ is the number of edges in the $u-v$ geodesic and is denoted by $d(u,v)$. For any vertex subset $S$, $I[S]$ [5] represents the set of all vertices in the $u-v$ geodesic including $u$ and $v$, for any $u, v \in S$. The eccentricity [1] of a vertex $v$ in a graph $G$ is the farthest distance between $v$ and any other vertex in $G$. The maximum eccentricity among the vertices in a graph is its diameter [1] and the minimum is the radius [1] of the graph, which are denoted by $diam(G)$ and $rad(G)$ respectively. The study on geodetic numbers was initiated from the shortest path problems and it is defined as follows.

**Definition 1.1.** [11] *A pineapple graph $K_p^q$ is obtained by attaching $q$ pendant vertices to a vertex in the complete graph $K_p$.*

**Definition 1.2.** [3, 7, 8] *A vertex subset $S$ of a graph $G=(V,E)$ is said to be a geodetic set, if $I[S]=V(G)$. The minimum cardinality of such a set is the geodetic number, which is represented as $g(G)$.*

**Definition 1.3.** [2] *A sequence is a function whose domain set is $\mathbb{N}$ and the range set can be any set $S$.*

**Theorem 1.4.** [7, 8] *The pendant and extreme vertices set in a graph $G$ are always a subset of any geodetic set of $G$.*

**Theorem 1.5.** [4] *The extreme order of a graph $G$ is always a lower bound of the geodetic number of a graph*

**Theorem 1.6.** [2, 4, 6, 7] *Let $G$ be a graph of order $n$. Then,*

*(i)* $2 \leq g(G) \leq n$

*(ii)* $g(G) \leq n - diam(G) + 1$.

*(iii)* $g(G) = n$ *if and only if* $G \cong K_n$.

*(iv)* $g(G) = n-1$ *if and only if* $G \cong K_1 + K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$, *where* $1 + n_1 + n_2 + \cdots + n_r = n$.

Consider the road network in a city as shown in Figure 1. The vertices in the figure represent the main junctions and the edges represent the road between them. The vertex $C$ represents the company location. Let the company want to provide taxi-cab services for their employees who are staying in that city. The company wants to locate a halting location for taxis so that they can start from there cover the entire network and finally reach the company. To reduce the cost, the number of such locations should be a minimum. Then the company requires halting locations only at $S$ and $C$ as shown in the figure below. The employees in the company are working on different shift times. So, $C$ is also chosen as a halting location. The number of halting stations in this network is two, this is nothing but the geodetic number of the network.

Now consider a situation in which some parts of the road network or junctions are not accessible due to some natural calamities like floods. In that situation, the company wants to arrange some temporary

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halting locations, so that they can ensure the transportation of their employees who are not affected by flood. Then this number of temporary halting locations is the geodetic number of corresponding induced subgraphs. This type of location-allocation problem motivated us for this study. In this problem, there may arise a situation in which only given $k$ junctions and the network connecting these junctions are accessible. In such situations, the company has to identify suitable halting locations among these $k$ junctions so that the service runs smoothly. This can happen to any number of $k$ junctions in the network. In all such situations, the company has to identify the suitable locations and the minimum number of locations required. Thus we propose the induced geodetic sequence.

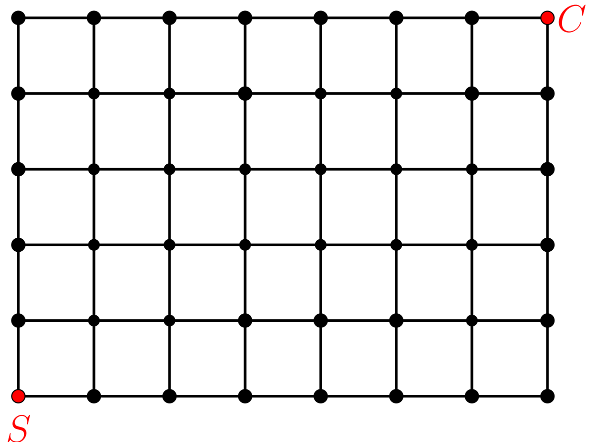


FIGURE 1. An example of a road network.

## 2. Induced Geodetic Sequence of a Graph

In this section, we will define the induced geodetic number and induced geodetic sequence of a graph and its basic properties.

**Definition 2.1.** *Let $G = (V, E)$ be a graph. For any set $S \subseteq V$ or $S \subseteq E$, the induced geodetic number of the graph $G$ associated with set $S$ is the geodetic number of $< S >$, denoted by $g(< S >)$.*

**Definition 2.2.** *Let $G = (V, E)$ be a graph of order $n$. For any set $S \subseteq V$ of order $i$ where $2 \leq i \leq n$, the induced geodetic number of order $i$ is the minimum induced geodetic number of among all subgraphs of order $i$, denoted by $g^i(G)$.i.e,*

$$g^i(G) = min\{g(< S >) : S \subseteq V, |S| = i\}$$

**Definition 2.3.** *Let $G = (V, E)$ be a graph of order $n$ and size $m$. For any set $R \subseteq E$ of cardinality $j$ where $1 \leq j \leq m$, the induced geodetic number of order $j$ is the minimum induced geodetic number of all subgraphs of order $j$, denoted by $g^{j_e}(G)$.i.e,*

$$g^{J_e}(G) = min\{g(< R >) : R \subseteq E, |R| = j\}$$

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**Definition 2.4.** *The geodetic sequence associated with vertex set of a graph $G$ of order $n$ denoted by $(g^i)$ is*

$$(g^i) = (g^2(G), \ldots, g^n(G))$$

*where $g^n(G) = g(G)$*

**Definition 2.5.** *The geodetic sequence associated with edge set of a graph $G$ of size $m$ denoted by $(g^{i_e})$ is*

$$(g^{i_e}) = (g^{1_e}(G), g^{2_e}(G), \ldots, g^{m_e}(G))$$

For example, consider the graph $G = K_4 - e$ which is shown in Figure 2. The vertices $b$ and $d$ are extreme in $G$ and form the minimum geodetic set of $G$. Hence, $g(G) = g^4(G) = 2$. Let $S \subseteq V(G)$. Now consider $|S| = 3$. Then $< S >$ is either $K_3$ or the star $S_3$. But $g(K_3) = 3$ and $g(S_3) = 2$. Therefore, $g^3(G) = 2$. Let $|S| = 2$. Then $< S >$ is $P_2$. Here we are not considering the graph $\overline{K}_2$. Hence $g^2(G) = 2$. Therefore, the geodetic sequence associated with the vertex set of $G$ is $(2, 2, 2)$.

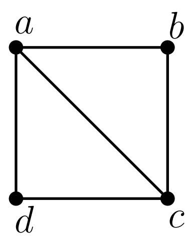


FIGURE 2. The graph $K_4 - e$.

**Theorem 2.6.** *For a path $P_n$ and cycle $C_n$,*

*(a.)* $g^i(P_n) = 2$ *for all* $2 \leq i \leq n$.

*(b.)* $g^i(C_n) = \begin{cases} 3, & \text{if } n \text{ is odd and } i=n \\ 2, & \text{Otherwise} \end{cases}$

*Proof.* Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $S \subset V(P_n)$ be of cardinality $i$. If $S = \{v_1, v_2, \ldots, v_i\}$ or $S = \{v_{n-i}, v_{n-i+1}, \ldots, v_n\}$ then $< S >$ must be a path, hence $g(< S >) = 2$. In other case $< S >$ must be disconnected graph, hence $g(< S >) > 2$. Thus $g^i = 2$ for $i = 2, 3, \ldots, n$.

Clearly $g(C_n) = 3$ if $n$ is odd and $g(C_n) = 2$ if $n$ is even. Also if $S \subset V(C_n)$ is of cardiniality $n-1$, then $< S >$ is a path of order $n-1$, hence $g(< S >) = 2$. The rest follows from the previous paragraph. □

**Theorem 2.7.** *Let $G$ be a graph of order $n$. Then $G$ is complete if and only if $g^i(G) = i$ for all $i$, where $2 \leq i \leq n$.*

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*Proof.* Let $G = K_n = (V, E)$. Then for any $S \subseteq V$ of order $i$ , the induced subgraph $< S >= K_i$, thus $g^i(G) = i$ for all $2 \leq i \leq n$. Conversely assume that $g^i(G) = i$ for all $2 \leq i \leq n$. Then it's enough to prove that every pair of vertices in $G$ are adjacent. If possible let $u, v$ be vertices such that $u$ is not adjacent to $v$ in $G$. Let $S \subseteq V$ be of order $i$, and $u, v \in S$, then the shortest $u - v$ path $P$ in $< S >$ must be of length at least 2. Thus, the geodesic $P$ must contain another vertex $w$ of $< S >$. Hence $g^i(G) \leq i - 1$, which is a contradiction. Therefore $G$ is complete. □

**Corollary 2.8.** *Let $G$ be a graph of order $n$. Then $G$ is complete if and only if it's induced geodetic sequence $(g^i) = (2, 3, \ldots, n)$*

**Theorem 2.9.** *For a complete bipartite graph $K_{m,n}$ $(m \leq n)$. If $m, n \geq 4$, then $g^i = (2, 2, \ldots, 2, 3, \ldots, 4)$ where 2 repeats $n + 1$ times and 4 repeats $m - 3$ times. In particular, $g^i = (2, 2, 2, \ldots, 2, 3)$ for $m = 3$ and $g^i = (2, 2, 2, \ldots, 2, 2)$ for $m = 2$.*

*Proof.* Let $X$ and $Y$ be the bipartition of the vertex set in $K_{m,n}$ with $|X| = m$ and $|Y| = n$. When $m, n \geq 4$, $g^{m+n} = 4$, since any two vertexes from $X$ and any two vertexes from $Y$ can be considered as the geodetic set of $K_{m,n}$. Let $S \subset V(K_{m,n})$ is of order $i$. Then $< S >$ must be a complete bipartite graph with bipartition $X_i$ and $Y_i$. When $n + 4 \leq i \leq m + n$, we must have $|X_i| \geq 4$ and $|Y_i| \geq 4$. Thus $g^i = 4$. If $i = n + 3$ and $|X_i| \geq 4$ and $|Y_i| \geq 4$, then $g(< S >) = 4$. But if $|X_i| = 3$ and $|Y_i| \geq 4$, then $g(< S >) = 3$. Therefore, $g^i = 3$. When $2 \leq i \leq n + 2$ and $|X_i| \geq 3$ and $|Y_i| \geq 4$. Then $g(< S >) = 3$ or 4. If $|X_i| = 2$ and $|Y_i| \geq 2$, then $g(< S >) = 2$. Also when $|X_i| = 1$ and $|Y_i| = 2$ then $< S >$ is $P_3$ and for $|X_i| = 1$ and $|Y_i| = 1$, $< S >$ is $P_2$. Thus $g(< S >) = 2$. Therefore, $g^i(G) = 2$ when $2 \leq i \leq n + 2$. □

**Remark 2.10.** *For a complete $r-$partite graph $K_{n_1,n_2,\ldots,n_r}$ where $n_1 \leq, n_2, \leq \ldots, \leq n_r$. If $n_i \geq 4$, then $g^i = (2, 2, \ldots, 2, 3, \ldots, 4)$ where 4 repeats $n_1 - 3$ times 3 occurs exactly one time and 2 occurs for the rest of the values. In particular, $g^i = (2, 2, 2, \ldots, 2, 3)$ for $n_1 = 3$ and $g^i = (2, 2, 2, \ldots, 2, 2)$ for $n_1 = 2$.*

**Theorem 2.11.** *Let $G$ be a graph of order $n$. Then, $g^i = (2, 2, 3, 4, \ldots, n - 2, n - 1)$ if and only if $G = K_1 + \bigcup_{i=1}^{k} K_{n_i}$.*

*Proof.* Let $V(K_{n_i}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$ for $i = 1, 2, \ldots, k$ and $V(K_1) = \{u\}$. Clearly $g^n = n - 1$. Let $S \subset V(G)$ be of order $i$. For $i = n - 1$, if $u \in S$, then $< S >= K_1 + \bigcup_{\substack{i=1 \\ i \neq j}}^{k} K_{n_i} \cup K_{n_j - 1}$ for some $j, 1 \leq j \leq k$. Hence, $g(< S >) = n - 2$. If $u \notin S$, then $< S >= \bigcup_{i=1}^{k} K_{n_i}$, hence $g(< S >) = n - 1$. Thus $g^{n-1}(G) = n - 2$.Proceeding like this, for $n - n_1 + 1 \leq i \leq n - 2$ $g^i = i - 1$. For $i = n - n_1$ and If $u \notin S$, $< S >$ is the union of complete graphs of order $n - n_1$, thus $g(< S >) = n - n_1 - 1$. If $u \in S$, then $< S >$ is the join of $K_1$ and the union of some complete graphs, where the union of complete graphs is of order $n - n_1$. Clearly, $g(< S >) = n - n_1 - 2$. Using the same arguments, $g^i = i - 1$ where $4 \leq i \leq n - n_1 - 1$. Now, when $i = 3$, $< S >$ can be any one among $P_3, \overline{K}_3, P_2 \cup K_1$. Thus, $g^3(G) = 2$. Now when $i = 2$, $< S >$ is either $P_2$ or $\overline{K}_2$. Hence $g^2 = 2$.

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Conversely let $g^i = (2, 2, 3, 4, \ldots, n-2, n-1)$, then $g^n(G) = n-1$. Then by Theorem 1.6, $G = K_1 + \bigcup_{i=1}^{k} K_{n_i}$. □

Let $\mathcal{T}_n, \mathcal{U}_n, \mathcal{B}_n$ denote the collection of all trees, unicyclic graphs and bicyclic graphs of order $n$.

**Theorem 2.12.** *For any tree $T \in \mathcal{T}_n$, the induced geodetic sequence of $T$ is constant if and only if $G \cong P_n$.*

*Proof.* For any connected graph $G$, $g^2(G) = 2$. Thus $(g^i)$ is constant if and only if $g^i = (2, 2, \ldots, 2)$. By Theorem 2.6, $g^i(P_n) = 2$ for all $i$. Let $T \in \mathcal{T}_n$ be an arbitrary tree. If $T \neq P_n$, Clearly the number of pendant vertices in $T$ is greater than that of $P_n$, hence $g^n(T) > g^n(P_n)$. Also, for every $i = 2, \ldots n-1$, $g^i(T) \geq 2 = g^i(P_n)$. Thus, $g^i(G) \neq 2$ for some $i$. Hence $g^i$ is not a constant sequence, a contradiction. Therefore, $T = P_n$. □

Let $C_{r,m}$ denote a unicyclic graph with a cycle $C_r$ along with the pendant vertex of a path of length $m$ attached at some vertex of $C_r$(See Figure 3). $C_{r,m,k}$ (where $r$ is even) is a graph with cycle $C_r$ along with two paths of lengths $m, k$ respectively attached at diametrically opposite vertices of the cycle $C_r$(See Figure 4).

**Theorem 2.13.** *For any graph $G \in \mathcal{U}_n$, the induced geodetic sequence $(g^i)$ is constant if and only if $G$ is any one of the following.*

*(a.) $G = C_n$, where $n$ is even and $G$ does not have any pendant vertices*

*(b.) $G = C_{r,m}$, where $n = r + m$ and $r$ is even, $m \geq 1$.*

*(b.) $G = C_{r,m,k}$ where $n = r + m + k$ and $r$ even, $m, k \geq 1$.*

*Proof.* Let $G \in \mathcal{U}_n$ be such that $G$ is not one of the graphs mentioned above. Then, $G$ must have an odd cycle or $G$ must have more than two pendant vertices. If $G$ has an odd cycle, then $g^n(G) > 3$, and $g^i(G) \geq 2$ for $i = 2, 3, \ldots, n-1$. Therefore, $(g^i)$ is not a constant sequence. If $G$ has more than two pendant vertices, each of these pendant vertices must be in the minimum geodesic set, thus $g^n(G) > 2$. Also, $g^i(G) \geq 2$ for $i = 2, 3, \ldots, n-1$. Therefore, $(g^i)$ is not a constant sequence. Thus for any graph $G$ other than the graph mentioned above, $(g^i)$ is not a constant sequence.

Now consider the case that $(g^i)$ is a constant sequence. Then $g^i(G) = 2$ for $i = 2, 3, \ldots, n$. Then $G$ cannot have more than two pendant vertices, as each of these pendant vertices must contribute in $g^n(G)$, hence $g^n(G) > 2$, impossible. Now we divide the problem into three different cases,

**Case I: $G$ has no pendant vertices.** Then $G = C_n$, by Theorem 2.6 $n$ must be even.

**Case II: $G$ has exactly one pendant vertex.** Then $G$ must be of the form $C_{r,m}$ where $r + k = n$ as shown in Figure 4. The cycle $C_r$ in $G$ must be of even length, since if it is odd length then $g^n(G) > 2$. Thus $r$ is even.

**Case III: $G$ has exactly two pendant vertices.** Then $G$ must have a cycle $C_r$ along with two paths of lengths $m, k$ attached at some vertices of $C_r$. In $G$ the cycle must be of even length, since if $r$ is odd,

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then $g^n(G) > 2$, is impossible. Also, the two paths cannot be attached at the same vertex, since if they are attached at the same vertex, then the minimum geodesic set must have a vertex from $C_r$ as well as two pendant vertices from two paths, thus $g^n(G) > 2$, impossible. Now, the two paths must be attached at the diametrically opposite vertices of the cycle $C_r$ as shown in Figure 3. If possible, let the paths be attached at the vertices $c_i, c_j$ where $d(c_i, c_j) < \frac{r}{2}$, then the minimum geodesic set must have a vertex from $C_r$ as well as two pendant vertices from two paths, thus $g^n(G) > 2$, impossible. Therefore $G$ must be of the form $C_{r,m,k}$.

Conversely assume that $G = C_n$ or $G = C_{r,m}$ or $G = C_{r,m,k}$, $r$ is even. If $G = C_n$, by Theorem 2.6, $(g^i)$ is a constant sequence. Now let $G = C_{r,m}$ and $\{c_1, c_2, \ldots, c_r\}$ be the vertices of the cycle and $\{v_1, v_2, \ldots, v_m\}$ be the vertices of the path, clearly, $g^n(G) = 2$. For $i = r+1, r+2, \ldots, n-1$, the graph induced by the set $S = \{c_1, c_2, \ldots, c_r, v_1, v_2, \ldots, v_{i-r}$ must be of the form $C_{r,i-r}$, clearly $g^i(G) = 2$. When $i = r$, then $< S >= C_r$ and $g^r(G) = 2$. When $i = 2, 3, \ldots, r-1$ the graph $< S >$ must be a path of order $i$, hence $g^i(G) = 2$. Therefore,$(g^i)$ is a constant sequence. Now let $G = C_{r,m,k}$ and $\{c_1, c_2, \ldots, c_r\}$ be the vertices of the cycle and $\{v_1, v_2, \ldots, v_k\}$, $\{u_1, u_2, \ldots, u_m\}$ be the vertices of the paths of length $k, m$ respectively. But, $g^n(G) = 2$, since the two pendant vertices of the paths are the minimum geodesic set. When $i = r+m+1, \ldots, n-1$, the graph induced by the set $S = \{c_1, c_2, \ldots, c_r, u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_{i-(r+m)}$ must be of the form $C_{r,m,i-r}$, clearly $g^i(G) = 2$. When $i = r+1, \ldots, r+m$, the $< S >$ can be taken of the form $C_{r,i-r}$, by previous argument, $g^i(G) = 2$. When $i = 2, 3, \ldots, r$, by previous argument, $g^i(G) = 2$, Therefore, $(g^i)$ is a constant sequence.

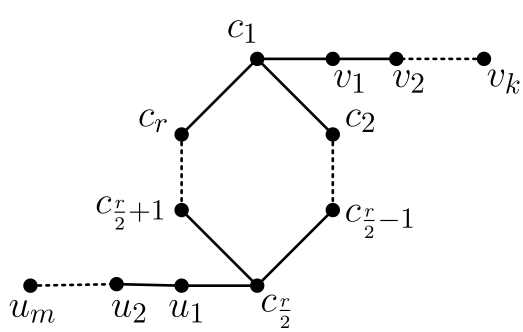


FIGURE 3. The graph $C_{r,m,k}$ mentioned in the proof of Theorem 2.13.

□

Cacti [9] are connected graphs in which every pair of distinct cycles has at most one vertex common. The collection of all cacti of order $n$ having $r$ cycles is denoted by $\mathcal{C}[n, r]$. Let $\mathcal{C}[a_1, a_2, \ldots, a_r, p_1, p_2, \ldots, p_{r+1}]$ denote a cacti of order $n$ having cycles of order $a_i$ for $i = 1, 2, \ldots, r$. The pendant vertex of a path of length $p_1$ is identified at some vertex $v_1$ of $C_{a_1}$ and a pendant vertex of a path of length $p_{r+1}$ is identified

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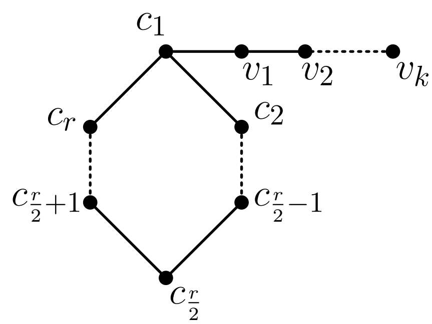


FIGURE 4. The graph $C_{r,m}$ mentioned in the proof of Theorem 2.13.

at some vertex $v_{2r}$ of $C_{a_r}$ along with paths of length $p_j$ is connects some vertex of$v_{2j}$ of $C_{a_{j-1}}$ and $v_{2j+1}$ of $C_{a_j}$ $j = 1, 2, \ldots, r-1$. Let $H$ be the subgraph of $G$ and $v \in V(G)$. The set $N_v(H)$ denotes the set of all neighbors of $v$ in $H$.

**Theorem 2.14.** *Let $G$ be a cacti of order $n$ with $r$ cycles. Then $g^i$ is constant if and only if $G$ is of the form $G = \mathcal{C}[a_1, a_2, \ldots, a_r, p_1, p_2, \ldots, p_{r+1}]$ where $a_i$ is even even for all $i = 1, 2, \ldots, r$ and $p_j \geq 0$ for all $j = 1, 2, \ldots, r+1$ with $d(v_{2i-1}, v_{2i}) = \frac{a_i}{2}, i = 1, 2, \ldots, r$ (see Figure 5).*

*Proof.* Assume that $G$ is not one among the graph mentioned above. Then $G$ must satisfy any one of the following cases.

**Case I: $G$ has more than two pendant vertices.** If $G$ has more than two pendant vertices, then each must be part of the minimum geodetic set of $G$. Thus $g^n(G) \geq 3$, hence $g^i$ is not a constant sequence.

**Case II: $G$ contains at least one odd cycle.** In order for cacti graph $G$ to have its geodetic number 2. It must have at least two different shortest paths of equal lengths connecting the vertices of the minimum geodetic set such that every vertex of $G$ must be part of one among the paths. Thus, if $G$ has at least one odd cycle, say $C_{a_j}$. Then every path passing through $C_{a_j}$ must contain either a part of $C_{a_j}$ of length greater than $\frac{a_j}{2}$ or must contain a part of $C_{a_j}$ of length less than $\frac{a_j}{2}$. Since the graph is a cactus, every path connecting two cycles must start and end at the same vertex within the cycle. Thus there does not possibly exist two paths of the same length containing all the vertices of the graphs. Thus, $g^n(G) \geq 3$. The sequence is not constant.

**Case III: $d(v_{2i-1}, v_{2i}) \neq \frac{a_i}{2}$ for some $i$.** Then either $d(v_{2i-1}, v_{2i}) < \frac{a_i}{2}$ or $d(v_{2i-1}, v_{2i}) > \frac{a_i}{2}$ for some $i$. Using the arguments in the previous case, there do not exist two shortest paths of the same length such that the paths contain all the vertices of the graph $G$. Thus, $g^n(G) \geq 3$, hence the sequence is not constant.

Thus if $(g^i)$ is a constant sequence, then $G$ must be of the given form.

Conversely, let $G$ be of the given form. Let the vertices be denoted by $V(P_1) = \{u_{1,1}, u_{1,2}, \ldots, u_{1,p_1}\}$

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and $V(P_{r+1}) = \{u_{r+1,1}, u_{r+1,2}, \ldots, u_{r+1,p_{r+1}}\}$ and $V(P_j) = \{u_{j,1}, u_{j,2}, \ldots, u_{j,p_j-1}\}$ for $j = 2, 3, \ldots, r$. Also let $N_{v_j}(C_{a_i}) = \{v_{i,1}, v_{i,2}\}$ if $i$ is odd and $N_{v_j}(C_{a_i}) = \{v_{i,3}, v_{i,4}\}$ if $i$ is even $i = 1, 2, \ldots, 2r$. When $p_1 \geq 1, p_{r+1} \geq 1$ then the pendant vertices $u_{1,1}$ and $u_{r+1,p_{r+1}}$ are the minimum geodetic set. Hence $g^n(G) = 2$. Now for convenience, let $|V(P_1)| \geq |V(P_{r+1})|$, then for $n - p_{r+1} \leq i \leq n-1$, if $u_{j,k} \notin S$ where $j = 2, 3, \ldots, r$ then $< S >$ must be disconnected, hence $g(< S >) \geq 4$. If $v_{j,k} \notin S$ where $v_{j,k} \in V(C_{a_j}), j = 1, 2, \ldots, r$ then $< S >$ must be either disconnected or contains at least three pendant vertices, hence $g(< S >) \geq 3$. Now if $(\cup_{j=2}^{r} V(P_j)) \cup (\cup_{i=1}^{r} V(C_{a_i})) \in S$, then $< S >$ must be of the form $\mathcal{C}[a_1, a_2, \ldots, a_r, p'_1, p_2, \ldots, p'_{r+1}]$ where either $p'_1 \geq 1$ and $p'_{r+1} \geq 1$ or $p'_1 \geq 1$ and $p'_{r+1} = 0$. When $p'_1 \geq 1$ and $p'_{r+1} \geq 1$ the two pendant vertices must be the minimum geodetic set and when $p'_1 \geq 1$ and $p'_{r+1} = 0$ then the one pendant vertex together with $v_{2r}$ must be the minimum geodetic set. Therefore, $g(< S >) = 2$, hence $g^i(G) = 2$ when $n - p_{r+1} \leq i \leq n-1$. When $n - p_1 - p_{r+1} \leq i \leq n - p_{r+1} - 1$, using the arguments in the previous case $g(< S >) \geq 4$ when $u_{j,k} \notin S$ where $j = 2, 3, \ldots, r$. Now if $(\cup_{j=2}^{r} V(P_j)) \cup (\cup_{i=1}^{r} V(C_{a_i})) \in S$, then $< S >$ must be of the form $\mathcal{C}[a_1, a_2, \ldots, a_r, p'_1, p_2, \ldots, p'_{r+1}]$ where $p'_1 = p'_{r+1} = 0$, then $v_1, v_{2r}$ must be the minimum geodetic set, hence $g(< S >) = 2$. If $v_{j,k} \notin S$ where $v_{j,k} \in V(C_{a_j}), j = 2 \ldots r-1$ then $g(< S >) \geq 3$, since at least one vertex from $V(C_{a_1}), V(C_{a_r}), V(C_{a_j})$ must be part of the minimum geodetic set. If $p'_{r+1} = 0$ (similarly for $p'_1 = 0$) and $v_{r,k} \notin S$ with $k \neq 1, 2$ then $< S >$ must have at least two pendant vertex, thus $g(< S >) \geq 3$. If $p'_{r+1} = 0$ and $v_{r,k} \notin S$ with $k = 1$ or $2$ then the cycle $C_{a_r}$ is transformed into a path, thus the pendant vertex of the new path together with some vertex of $p'_1$ of $V(C_{a_1})$ is the minimum geodetic set of $< S >$, thus $g(< S >) = 2$. Therefore, $g^i(G) = 2$ when $n - p_1 - p_{r+1} \leq i \leq n - p_{r+1} - 1$. Now, when $i = n - p_1 - p_{r+1} - 1$, as in the previous case, $g(< S >) \geq 3$ when $u_{j,k} \notin S$ where $j = 2, 3, \ldots, r$ or $v_{j,k} \notin S$ where $v_{j,k} \in V(C_{a_j}), j = 2 \cdots r-1$. If $(\cup_{j=2}^{r} V(P_j)) \cup (\cup_{i=1}^{r} V(C_{a_i}) - \{v_{1,k}\}) \in S, k = 3, 4$ (or $(\cup_{j=2}^{r} V(P_j)) \cup (\cup_{i=1}^{r} V(C_{a_i}) - \{v_{r,k}\}) \in S, k = 1, 2$), then the cycle $C_{a_1}$ ($C_{a_r}$) is transformed into a path in $< S >$. Hence, the minimum geodetic set must be the pendant vertex together with $v_{2r}$ ($v_1$), therefore $g(< S >) = 2$. Thus $g^i(G) = 2$. Proceeding like this we get $g^i(G) = 2$ for all $i$.

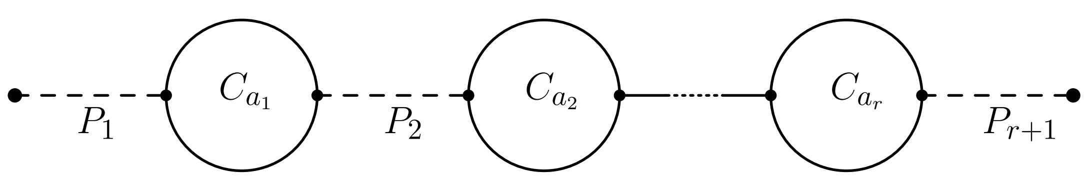


FIGURE 5. The figure $\mathcal{C}[a_1, a_2, \ldots, a_r, p_1, p_2, \ldots, p_{r+1}]$ mentioned in Theorem 2.14

.

□

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Let $\Phi_{a,b}$ denote the collection of all graphs that have two cycles, and the two cycles do not share any common edges and $\Phi_n$ denote the collection of all such bicyclic graphs of order $n$. Let $\phi_{a,b,p,q,r} \in \Phi_n$ ($a, b$ even) denote the bicyclic graph in which two paths of length $p, r$ are attached at vertices $u_1 \in C_a$ and $v_1 \in C_b$ respectively and a path of length $q$ which connects some vertex $u_2 \in C_a$ to $v_2 \in C_b$. Also, $d(u_1, u_2) = \frac{a}{2}$ and $d(v_1, v_2) = \frac{b}{2}$.

**Corollary 2.15.** *For any graph $G \in \Phi_n$, the induced geodetic sequence $(g^i)$ is constant if and only $G = \phi_{a,b,p,q,r}$ where $p \geq 0, q \geq 0, r \geq 0$, $a, b$ are even.*

*Proof.* Using Theorem 2.14. □

## 3. Induced geodetic index and Normalized induced geodetic index

In this section, we define the two topological indices associated with the induced geodetic sequence of a graph and study its basic properties.

**Definition 3.1.** *Le $G$ be a graph of order $n$, then the induced geodetic index of $G$ is denoted by $Ig(G)$ and is defined as*

$$Ig(G) = \sum_{i=2}^{n} g^i(G) \tag{3.1}$$

**Definition 3.2.** *Le $G$ be a graph of order $n$, then the normalized induced geodetic index of $G$, denoted by $Ig_N(G)$ and is defined as*

$$Ig_N(G) = \frac{\sum_{i=2}^{n} g^i(G)}{n} \tag{3.2}$$

**Theorem 3.3.** *(a.) $Ig(P_n) = 2(n-1)$ and $Ig_N(P_n) = 2 - \frac{2}{n}$*

*(b.)* $Ig(C_n) = \begin{cases} 2n-1 & \text{if } n \text{ is odd} \\ 2n-2 & \text{if } n \text{ is even} \end{cases}$ *and* $Ig_N(C_n) = \begin{cases} 2 - \frac{1}{n} & \text{if } n \text{ is odd} \\ 2 - \frac{2}{n} & \text{if } n \text{ is even} \end{cases}$

*Proof.* By Theorem 2.6. □

**Theorem 3.4.** *(a.) A graph $G$ is complete if and only if $Ig(G) = \frac{(n-1)(n+2)}{2}$.*

*(b.) A graph $G$ is complete if and only if $Ig_N(G) = \frac{(n-1)(n+2)}{2n}$.*

*Proof.* By Theorem 2.7. □

**Theorem 3.5.** *For a complete bipartite graph $K_{m,n}, m \leq n$,*

*(a.)* $Ig(G) = \begin{cases} 4(m+n)-8 & \text{if } m, n \geq 4 \\ 4(m+n)-7 & \text{if } m = 3, n \geq 3 \\ 2(m+n)-2 & \text{if } m = 2, n \geq 3 \end{cases}$.

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$$(b.)\ Ig_N(G) = \begin{cases} 4 - \frac{8}{m+n} & \text{if } m, n \geq 4 \\ 4 - \frac{7}{m+n} & \text{if } m = 3, n \geq 3 \\ 2 - \frac{2}{m+n} & \text{if } m = 2, n \geq 3 \end{cases}.$$

*Proof.* By Theorem 2.9. □

**Theorem 3.6.** *For any tree $T \in \mathcal{T}_n$, $2n - 2 \leq Ig(T) \leq \dfrac{n^2 - n + 2}{2}$ and the equality holds if and only if $G \cong P_n$ and $G \cong S_n$.*

*Proof.* By Theorem 2.6, $Ig(P_n) = 2n-2$. Let $T \in \mathcal{T}_n$ be an arbitrary tree. If $T \neq P_n$, Clearly the number of pendant vertices in $T$ is greater than that of $P_n$, hence $g^n(T) > g^n(P_n)$. Also, for every $i = 2, \ldots, n-1$, $g^i(T) \geq 2 = g^i(P_n)$. Thus, $Ig(T) > Ig(P_n)$. Now when $Ig(T) = 2n - 2$, $g^i(T) = 2$ for all $i = 2, \ldots, n$. Since $g^n(T) = 2$, $T$ has exactly two pendant vertices, therefore $T = P_n$.

By Theorem 2.11, $Ig(S_n) = \dfrac{n^2 - n + 2}{2}$. Let $T \in \mathcal{T}_n$ and $T \neq S_n$. Then the number of pendant vertices of $T$ is less than that of $S_n$, hence $g^n(T) < g^n(S_n)$. Also, for every $i = 2, \ldots n-1$, $g^i(T) \geq i - 1 = g^i(S_n)$. Thus, $Ig(T) < Ig(S_n)$. Now when $Ig(T) = \dfrac{n^2 - n + 2}{2}$, $g^i(T) = i - 1$ for $i = 3, \ldots, n$ and $g^2(T) = 2$. Since $g^n(T) = n - 1$, $T$ has exactly $n - 1$ pendant vertices, therefore $T = S_n$. □

**Remark 3.7.** *For any tree $T \in \mathcal{T}_n$, $2 - \frac{2}{n} \leq Ig_N(T) \leq \dfrac{n^2 - n + 2}{2n}$ and the equality holds if and only if $G \cong P_n$ and $G \cong S_n$.*

**Theorem 3.8.** *For any graph $G \in \mathcal{U}_n$, $Ig(G) \leq \dfrac{n^2 - n + 2}{2}$ and the equality holds if and only if $G$ is a pineapple graph $K_{3,n-3}$.*

*Proof.* Since $K_{3,n-3} = K_1 + (K_2 \cup_{i=1}^{n-3} K_1)$, by Theorem 2.11, $Ig(G) = \dfrac{n^2 - n + 2}{2}$. Let $G \neq K_{3,n-3}$ and since $G \neq K_n$ then by Theorem 2.7, $g^n(G) < n$. Also, by Theorem 2.11, $g^n(G) < n - 1$. Therefore $g^n(G) < n - 1 = g^n(K_{3,n-3})$. For any $i = 4, \ldots n - 1$, $g^i(G) \leq i - 1$, since $g^i(G) = i$ implies there exists a complete graph of order $i$ in the induced graph, which is impossible. Thus $g^i(G) \leq i - 1 = g^i(K_{3,n-3})$. Now when $i = 3, 2$, $g^i(G) = 2 = g^i(K_{3,n-3})$. Therefore, $g^i(G) \leq g^i(K_{3,n-3})$ for $i = 2, 3, \ldots n - 1$ and $g^n(G) < g^n(K_{3,n-3})$. Hence the theorem. □

**Theorem 3.9.** *For any graph $G \in \mathcal{U}_n$, $Ig(G) \geq 2n - 2$ and the equality holds if and only $G$ is any of the following.*

*(a.) $G = C_n$, where $n$ is even and $G$ does not have any pendant vertices*

*(b.) $G = C_{r,m}$, where $n = r + m$ and $r$ is even, $m \geq 1$.*

*(b.) $G = C_{r,m,k}$ where $n = r + m + k$ and $r$ even, $m, k \geq 1$.*

*Proof.* By Theorem 2.13 for any unicyclic graph other than the ones mentioned above, $Ig(G) > 2n - 2$. Now consider the case that $Ig(G) = 2n - 2$, then $g^i(G) = 2$ for all $i$. By Theorem 2.13, the result follows. □

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**Remark 3.10.** *For any graph* $G \in \mathcal{U}_n$*,* $2 - \frac{2}{n} \leq Ig_N(G) \leq \dfrac{n^2-n+2}{2n}$

**Theorem 3.11.** *For any connected graph* $G$ *of order* $n$*,* $2n-2 \leq Ig(G) \leq \dfrac{n^2+n-2}{2}$ *and the right-hand equality holds if and only if* $G = K_n$*.*

*Proof.* For any connected graph $G$ of order $n$, for $2 \leq i \leq n$, $2 \leq g^i \leq i$ and the right hand equality holds if and only if $G = K_n$ (By Theorem 2.9). Therefore the result. □

**Remark 3.12.** *For any connected graph* $G$ *of order* $n$*,* $2 - \frac{2}{n} \leq Ig(G) \leq \dfrac{n^2+n-2}{2n}$.

Now we propose some problems for further study.

**Problem 3.13.** *Every connected graph* $G$ *has an induced geodetic sequence. What is the condition in which given a sequence* $g^i = (2, a_1, a_2, \dots, a_{n-2})$ *where* $a_i \leq a_{i+1}$ *for all* $i = 2, 3, \dots, n-2$ *there exist a connected graph* $G$ *with* $g^i$ *as its induced geodetic sequence.*

**Problem 3.14.** *Characterize bicyclic graphs for which the induced geodetic sequence is constant.*

## 4. Conclusion

In this study, we have introduced a new sequence associated with the geodetic number of a graph. We studied its properties for various classes of graphs and discussed some characterization results. These concepts can be extended to other variants of geodetic numbers.

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